The kernel of the Dirac operator

B. Ammann[1] M. Dahl[2] E. Humbert[3]

[1]Universität Regensburg
Germany

[2]Institutionen för Matematik
Kungliga Tekniska Högskolan, Stockholm
Sweden

[3]Laboratoire de Mathématiques et Physique Théorique
Université de Tours
France

SFB-Talk Berlin 2011

Outline

Preliminaries
- The Dirac operator
- Atiyah-Singer index theorem
- Comparison to Gauss-Bonnet-Chern operator
- Is dim ker D a topological invariant?

Large kernel conjecture

Generic metrics are D-minimal
- D-minimality theorem
- History of partial solutions

The surgery method
- Surgery
- D-minimality and surgery
- Local D-minimality theorem
- D-surgery thm implies local D-minimality thm

The Dirac operator

Let M be a (fixed) compact manifold with spin structure, $n = \dim M$.

For any metric g on M one defines

- the *spinor bundle* $\Sigma_g M$: a vector bundle with a metric, a connection and Clifford multiplication $TM \otimes \Sigma_g M \to \Sigma_g M$. Sections $M \to \Sigma_g M$ are called *spinors*.
- the *Dirac operator* $D\!\!\!/_g : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_g M)$: a self-adjoint elliptic differential operator of first order.

$\Longrightarrow \quad \dim\ker D\!\!\!/_g$ is finite-dimensional.

The elements of $\ker D\!\!\!/_g$ are called *harmonic spinors*.

Why are harmonic spinors interesting?

- $n = \dim M = 2$, M a Riemann surface,
  $D\!\!\!/_g \phi = 0 \Leftrightarrow \phi$ is a holomorphic section of $\sqrt{TM} \oplus \sqrt{TM}$.
  Such a section corresponds to a conformal immersion of $\widetilde{M}$ into $\mathbb{R}^3$ with vanishing mean curvature.
  (Weierstrass representation 1866) ▸ Example
- Physics: Stationary state of a fermion with mass 0 or energy 0.
- On asymptotically euclidean manifold with scal≥ 0, i.e. a model of a star or black hole:
  $D\!\!\!/_g \phi = 0 \Leftrightarrow \phi$ is a Witten spinor
  $\rightsquigarrow$ Central ingredient in Witten's proof of the positive mass theorem.
- $D\!\!\!/_g \phi = \psi$ is solvable in ϕ if $\psi \perp \ker D\!\!\!/_g$.
  The solution is unique up to $\ker D\!\!\!/_g$.

Atiyah-Singer Index Theorem for $n = 4k$

Let $n = 4k$. $\Sigma_g M = \Sigma_g^+ M \oplus \Sigma_g^- M$. $\not{D}_g = \begin{pmatrix} 0 & \not{D}_g^- \\ \not{D}_g^+ & 0 \end{pmatrix}$

$$\mathrm{ind}\ \not{D}_g^+ = \dim\ker \not{D}_g^+ - \mathrm{codim\ im}\ \not{D}_g^+ = \dim\ker \not{D}_g^+ - \dim\ker \not{D}_g^-$$

Theorem (Atiyah-Singer 1968)

$$\mathrm{ind}\ \not{D}_g^+ = \int_M \widehat{A}(TM)$$

Hence: $\dim\ker \not{D}_g \geq |\int \widehat{A}(TM)|$

Index Theorem for $n = 8k + 1$ and $8k + 2$

$n = 8k + 1$:

$$\alpha(M) := \dim\ker \not{D}_g \mod 2$$

$n = 8k + 2$:

$$\alpha(M) := \frac{\dim\ker \not{D}_g}{2} \mod 2$$

$\alpha(M) \in \mathbb{Z}/2\mathbb{Z}$ is independent of g.
However, $\alpha(M)$ depends on the choice of spin structure.

Consequence

$$\dim \ker D\!\!\!/^{g} \geq \begin{cases} |\int \widehat{A}(TM)|, & \text{if } n = 4k; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Comparison to Gauss-Bonnet-Chern operator

The operator $d + d^* : \Gamma(\Lambda^\bullet T^*M) \to \Gamma(\Lambda^\bullet T^*M)$ is a “twisted” Dirac operator, called *Gauss-Bonnet-Chern operator*.
Index theorem

$$(d + d^*)_+ : \Gamma(\Lambda^{\mathrm{even}} T^*M) \to \Gamma(\Lambda^{\mathrm{odd}} T^*M)$$

$$(d + d^*)_- : \Gamma(\Lambda^{\mathrm{odd}} T^*M) \to \Gamma(\Lambda^{\mathrm{even}} T^*M)$$

$$\chi(M) = \dim\ker(d + d^*)_+ - \dim\ker(d + d^*)_-$$

On the other hand

$$\ker(d + d^*) = \bigoplus_{i=0}^{n} \ker(d + d^*)|_{\Gamma(\Lambda^i T^*M)}$$

$$b_i(M) := \dim\ker(d + d^*)|_{\Gamma(\Lambda^i T^*M)} \qquad i\text{-th Betti number}$$

$$\dim\ker(d + d^*) = \sum_{i=0}^{n} b_i(M)$$

$$\dim \ker(d + d^*) = \sum_{i=0}^{n} b_i(M)$$

The dimension of the kernel is a topological invariant!
The lower bound given by the index theorem

$$\dim \ker(d + d^*) \geq |\chi(M)| = |\sum_{i=0}^{n}(-1)^i b_i(M)|$$

is attained iff n is even and $b_1 = b_2 = \ldots = b_{n-1} = 0$.

- Analogous results for the signature operator.
- On Kähler manifolds $\dim \ker(\bar{\partial} + \bar{\partial}^*)$ is invariant under deformation of the complex structure.
- Kotschick 1996: On some Kähler surfaces the kernel of $\not{D}_g$ never attains the Atiyah-Singer bound.

Is dim ker $\not{D}$ a topological invariant?

No.

Definition

A metric g on a **connected** spin manifold is called $\not{D}$-minimal if

$$\dim \ker \not{D}^g = \text{Bound given by Atiyah-Singer}$$

Theorem

Generic (=most) metrics are $\not{D}$-minimal.

Conjecture (Large kernel conjecture (Bär-Dahl 2002))

Let $\dim M \geq 3$. *For any* $k \in \mathbb{N}$ *there is a metric* g_k *with* $\dim \ker \not{D}^{g_k} \geq k$.

Large kernel conjecture

Conjecture

Let $\dim M \geq 3$. *For any* $k \in \mathbb{N}$ *there is a metric* g_k *with* $\dim \ker D \geq k$.

This conjecture has been proved by

- *Hitchin 1974* on $M = S^3$ for any $k \in \mathbb{N}$,
- *Hitchin 1974* in dimensions $n \equiv 0, 1, 7 \mod 8$ for $k = 1$,
- *Bär 1996* in dimensions $n \equiv 3, 7 \mod 8$ for $k = 1$,
- *Kotschick 1996* $n = 4$: For any $k \in \mathbb{N}$ there is a Kähler surfaces M_k with $\dim \ker \not{D} \geq |\int \hat{A}| + k$.
- *Seeger 2000* on S^{2m}, $m \geq 2$, for $k = 1$,
- *Dahl 2006* on S^n, $n \geq 5$, for $k = 1$.
- *Dahl and Große, current research*: Systematic approach: Bordism theory. Index on Bordisms.

Many open cases!

$\not{D}$-minimality theorem

Theorem ($\not{D}$-minimality theorem, ADH, 2009)

Generic metrics on connected compact spin manifolds are $\not{D}$-minimal.

Generic = dense in C^∞-topology and open in C^1-topology.

The investigations for this result were initiated by Hitchin (1974). The theorem was explicitly conjectured by Bär-Dahl (2002).

To prove the $\not{D}$-minimality theorem it is sufficient to show that there is *one* $\not{D}$-minimal metric.

History of partial solutions

In order to show that generic metrics are $\not{D}$-minimal, if suffices to show that one $\not{D}$-minimal metric exists.

- Hitchin (1974): $\dim \ker \not{D}_g$ depends on g.
- Maier (1996) proved the theorem if

$$n = \dim M \leq 4.$$

- Bär-Dahl (2002) proved the theorem when

$$n \geq 5 \text{ and } \pi_1(M) = \{e\}.$$

  They used the surgery method (Gromov-Lawson 1980, Stolz 1992).
- We (ADH, Adv. Math. 2009) also use the surgery method. It works under no restriction on n or π_1.
- A new proof (ADH, Math. Res. Lett., to appear) proves a local, stronger version.

Surgery

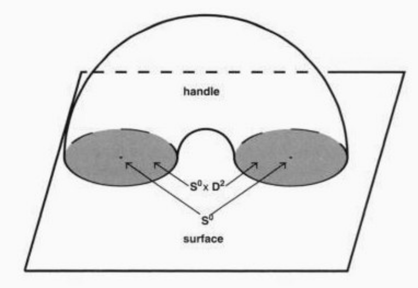
Let $f : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We define

$$M^{\#} := M \setminus f(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1})/\sim$$

where $/\sim$ means gluing the boundaries via

$$M \ni f(x,y) \sim (x,y) \in S^k \times S^{n-k-1}.$$

We say that $M^{\#}$ is obtained from M by surgery of dimension k.



Example: 0-dimensional surgery on a surface.

$D\!\!\!/$-minimality and surgery

Theorem ($D\!\!\!/$-Surgery Theorem, ADH 2009)

Let $k \leq n - 2$.
If M carries a $D\!\!\!/$-minimal metric, then $M^{\#}$ carries a $D\!\!\!/$-minimal metric as well.

Bär-Dahl (2002) proved the theorem with other methods for $k \leq n - 3$.

ADH proved, Math. Res. Lett., to appear

Theorem (Local $\not{D}$-Minimality Theorem)

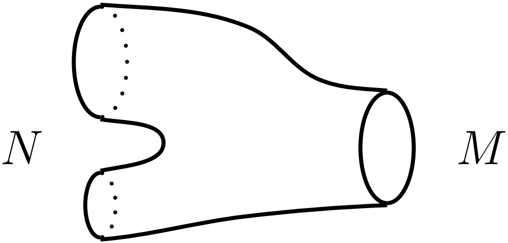
Let M be a compact connected spin manifold with a riemannian metric g. Let U be a non-empty open subset of M. Then there is metric $\tilde{g}$ on M which is $\not{D}$-minimal and which coincides with g on $M \setminus U$.

This theorem implies that generic metrics are $\not{D}$-minimal.

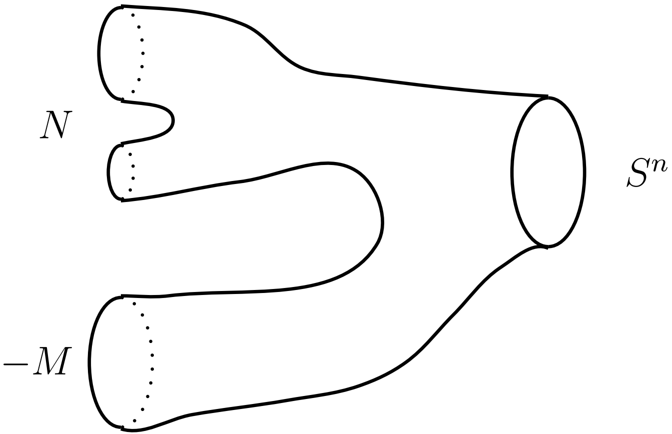
Proof of "$\not{D}$-surgery Thm $\Longrightarrow$ Local $\not{D}$-minimality Thm"

We use a theorem from Stolz 1992.
The given spin manifold M is spin bordant to $N = N_0 \cup P$, where

- P carries a metric of positive scalar curvature,
- N_0 is a disjoint union of products of S^1, a $K3$-surface and a Bott manifold, and carries a $\not{D}$-minimal metric.

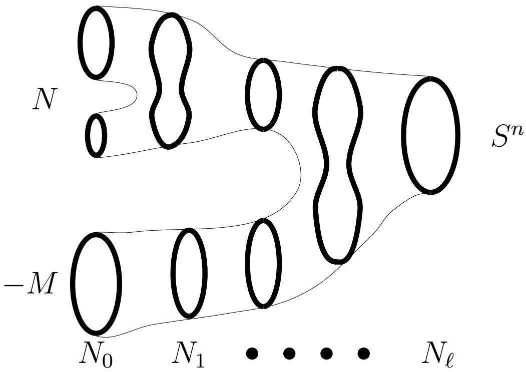


Assume now $n \geq 5$. Remove a ball and move M to the other side.



Modify the bordism W such that W is connected and $\pi_1(W) = \pi_2(W) = 0$.
As $\pi_1(S^n) = 0$, the bordism W can be decomposed into pieces corresponding to surgeries of dimension

$$k \in \{0, 1, \ldots, n-3\}$$

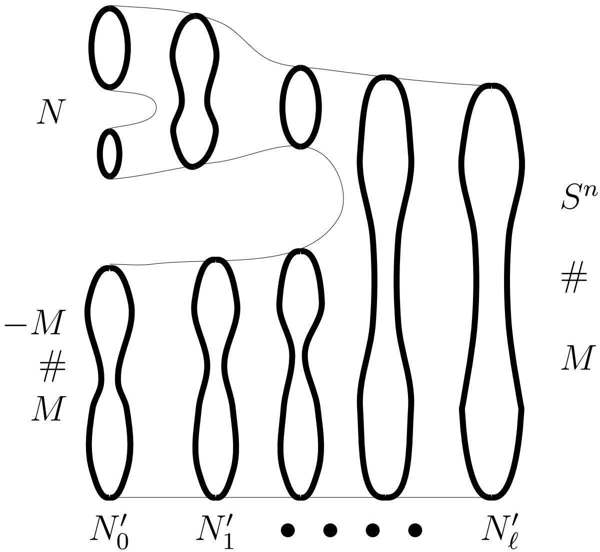


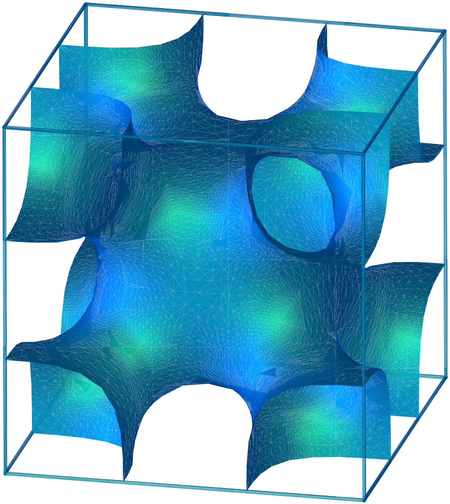
Invertible Double

Proposition

Let M be compact, connected and spin. Then there is a metric g on $M\#(-M)$ with invertible $\not{D}_g$.

See e.g. the book by Booß-Bavnbek and Wojchiekowski. Uses the unique continuation property of $\not{D}^g$.


N
−M
#
M
S^n
#
M
N'_0
N'_1
N'_ℓ



Copyright: KGB, Darmstadt