

A surgery formula for the smooth Yamabe invariant

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Einstein-Hilbert functional

History

Surgery

Main results

Topological conclusions

Comments on the proofs

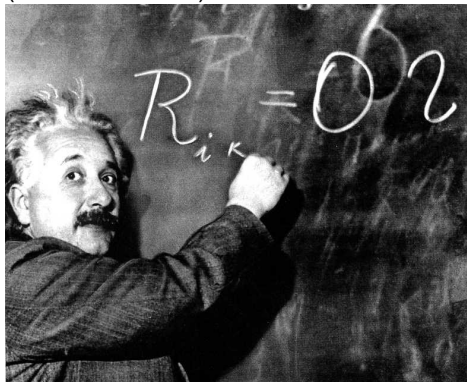
The numbers $L(n, k)$



Einstein's equation

1915: Einstein's General Relativity.

What is the equation that describes evolution of spacetime (without matter)?



Answer: Einstein metrics

$$\text{Ric} = \kappa g,$$

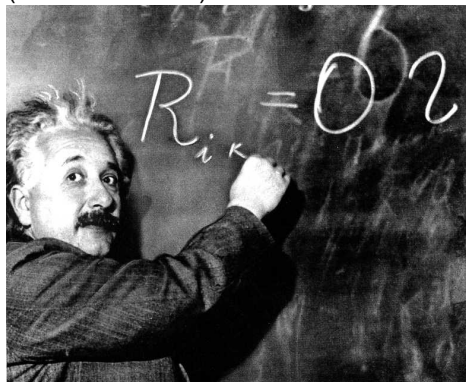
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Lagrangian formulation

What is the associated Lagrange functional?



David Hilbert, 1915

Let \mathcal{M} be the set of all semi-Riemannian metrics on M^n . The Einstein equations are stationary points of the Einstein-Hilbert-functional

$$\begin{aligned} \mathcal{E} : \mathcal{M} &\rightarrow \mathbb{R}, \\ g &\mapsto \frac{\int_M \text{scal}^g \, dv^g}{\left(\int_M dv^g\right)^{(n-2)/n}} \end{aligned}$$

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The Setting

Now: Let M be n -dimensional, $n \geq 3$.

Search for an Einstein metric on M .

This are the stationary points of

$$\mathcal{E} : \mathcal{M}_1 \rightarrow \mathbb{R}, \quad g \mapsto \int_M \text{scal}^g dv^g$$

$\mathcal{M}_1 := \{\text{metrics on } M \text{ of volume } 1\}$.

$[g] := \{u^{4/(n-2)}g \mid \text{vol}(u^{4/(n-2)}g) = 1\}$.



Observation

Inside a conformal class

$$Y(M, [g]) := \inf_{\tilde{g} \in [g]} \mathcal{E}(\tilde{g}) > -\infty.$$

The conformal Yamabe invariant.

$$Y(M, [g]) \leq Y(\mathbb{S}^n)$$

where \mathbb{S}^n is the sphere with the standard structure.

On the set of conformal classes

$$\sigma(M) := \sup_{[g] \in \mathcal{M}_1} Y(M, [g]) \in (-\infty, Y(\mathbb{S}^n)]$$

The smooth Yamabe invariant, also called Schoen's σ -constant.

Remark $\sigma(M) > 0$ if and only if M carries a metric of positive scalar curvature.



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Yamabe's idea

To find stationary points of \mathcal{E} :

- ▶ Find a minimizer in each conformal class, i.e. $\tilde{g} \in [g]_1$ with $\mathcal{E}(\tilde{g}) = Y(M, [g])$. Such metrics always exist, and are called Yamabe metrics. They are unique if $Y(M, [g_0]) < 0$.
- ▶ Find a maximizing conformal class $[g_{\max}]$, i.e. $Y(M, [g_{\max}]) = \sigma(M)$.

Hope: The minimaximizer $\widetilde{g_{\max}}$ is Einstein.



Example $\mathbb{C}P^2$

The Fubini-Study g_{FS} metric is Einstein and

$$53, 31\dots = \mathcal{E}(g_{\text{FS}}) = Y(\mathbb{C}P^2, [g_{\text{FS}}]) = \sigma(\mathbb{C}P^2).$$

Claude LeBrun '97

Similar examples

- ▶ $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- ▶ Gromov & Lawson, Schoen & Yau \approx ' 83: Tori $\mathbb{R}^n/\mathbb{Z}^n$.
 $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$.
- ▶ LeBrun '99: All Kähler-Einstein surfaces with non-positive scalar curvature.
- ▶ Bray & Neves '04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$.
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Dimension $n = 3$

- ▶ Einstein 3-manifolds with $\kappa = 0$ are flat.
- ▶ Einstein 3-manifolds with $\kappa > 0$ are quotients of spheres.
- ▶ Einstein 3-manifolds with $\kappa < 0$ are quotients of hyperbolic spaces.

Examples where Yamabe's idea fails

There is no Einstein metric on non-prime 3-manifolds and $S^2 \times S^1$.

Schoen: $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$. The supremum is **not attained**.

Other problem In the case $\sigma(M) < 0$ the minimizers are unique. Hence, if a maximizing conformal class exists, then **the unique** minimizing metric in that class is Einstein.

However, in the case $\sigma(M) > 0$, a minimizing metric in a maximizing class may be non-Einstein!



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Some known values of σ

- ▶ All examples above.
- ▶ Akutagawa & Neves '07: Some non-prime 3-manifolds, e.g.

$$\sigma(\mathbb{R}P^3 \# (S^2 \times S^1)) = \sigma(\mathbb{R}P^3).$$

- ▶ Compact quotients of nilpotent Lie groups: $\sigma(M) = 0$.

Unknown cases

- ▶ Nontrivial quotients of spheres, except $\mathbb{R}P^3$.
- ▶ $S^k \times S^m$, with $k, m \geq 2$.
- ▶ No example of dimension ≥ 5 known with $\sigma(M) \neq 0$ and $\sigma(M) \neq \sigma(S^n)$.



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Relations to similar invariants...

... to the spectrum of the conformal Laplacian

$$\inf_{\tilde{g} \in [g]} \lambda_1 \left(4 \frac{n-1}{n-2} \Delta^{\tilde{g}} + \text{scal } \tilde{g} \right) = \begin{cases} Y(M, [g]) & \text{if } Y(M, [g]) \geq 0 \\ -\infty & \text{if } Y(M, [g]) < 0 \end{cases}$$

... to the Perelman-invariant (Akutagawa, Ishida, LeBrun '06)

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⇒ Hence $\min\{\sigma(M), 0\}$ is determined by an infimum.



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Surgery

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.

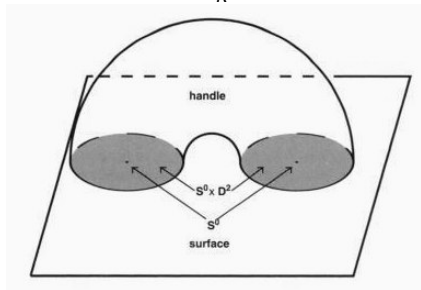
We define

$$M_k^\Phi := M \setminus \Phi(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1}) / \sim$$

where $/ \sim$ means gluing the boundaries via

$$M \ni \Phi(x, y) \sim (x, y) \in S^k \times S^{n-k-1}.$$

We say that M_k^Φ is obtained from M by surgery of dimension k .



Example: 0-dimensional surgery on a surface.

Known surgery formulas

Theorem (Gromov & Lawson '80, Schoen & Yau '79)

If $0 \leq k \leq n - 3$, then

$$\sigma(M) > 0 \implies \sigma(M_k^\Phi) > 0.$$

Theorem (Kobayashi '87)

If $k = 0$, then

$$\sigma(M_0^\Phi) \geq \sigma(M).$$

Theorem (Petean & Yun '99)

If $0 \leq k \leq n - 3$, then

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Main results

Theorem (ADH, # 1)

Let $0 \leq k \leq n - 3$. There is a positive constant $L(n, k)$ depending only on n and k such that

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), L(n, k)\}$$

if M_k^Φ is obtained from M by k -dimensional surgery.
Furthermore $L(n, 0) = Y(\mathbb{S}^n)$.

This theorem implies all three previously known surgery formulas.

Thm # 1 follows directly from Thm # 2.

Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k -dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M, [g]), L(n, k)\} \leq \liminf_{i \rightarrow \infty} Y(M_k^\Phi, [g_i]) \leq Y(M, [g]).$$



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Topological conclusions

$$L(n) := \min\{L(n, 1), L(n, 2), \dots, L(n, n-3)\},$$

$$\bar{\sigma}(M) := \min\{\sigma(M), L(n)\}.$$

Let M_k^Φ be obtained from M by surgery of dimension $k \in \{2, 3, \dots, n-3\}$, then $\bar{\sigma}(M) = \bar{\sigma}(M_k^\Phi)$.

Goal: Find a bordism invariant!



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Let Γ be a finitely presented group.

Let $\Omega_n^{\text{spin}}(B\Gamma)$ the spin cobordism group over $B\Gamma$.

Any class in $\Omega_n^{\text{spin}}(B\Gamma)$ has a π_1 -bijective representative, i.e. it is represented by (M, f) , where M is a connected compact spin manifold, and where $f : M \rightarrow B\Gamma$ induces an isomorphism $\pi_1(M) \rightarrow \pi_1(B\Gamma) = \Gamma$.

Lemma

$n \geq 5$. Let (M_1, f_1) and (M_2, f_2) be spin cobordant over $B\Gamma$ and let (M_2, f_2) be π_1 -bijective. Then

$$\bar{\sigma}(M_2) \geq \bar{\sigma}(M_1).$$

We define

$$s_\Gamma([M, f]) := \max \bar{\sigma}(N)$$

where the maximum runs over all (N, h) with

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$$\bar{\sigma} := \min\{\sigma(M), L(n)\}$$

$s_\Gamma([M, f]) := \bar{\sigma}(M)$ if (M, f) is π_1 -bijective

$$s_\Gamma : \Omega_n^{\text{spin}}(B\Gamma) \rightarrow (-\infty, L(n)]$$

Corollary

Let $n \geq 5$ and $\epsilon \in (0, L(n))$. The groups

$\{x \in \Omega_n^{\text{spin}}(B\Gamma) \mid s_\Gamma(x) > \epsilon\}$ and $\{x \in \Omega_n^{\text{spin}}(B\Gamma) \mid s_\Gamma(x) \geq \epsilon\}$ are subgroups.

Application

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.

Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\#\cdots\#M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\#\cdots\#M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M ?

Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$
for $\#\Gamma$ large.



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Proof of Theorem # 2

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Easy part:

$$\liminf_{i \rightarrow \infty} Y(M_k^\Phi, [g_i]) \leq Y(M, [g])$$

Difficult and important part

$$\min \{ Y(M, [g]), L(n, k) \} \leq \liminf_{i \rightarrow \infty} Y(M_k^\Phi, [g_i])$$

For this, we construct a sequence explicitly and study the behavior of the minimizers in $[g_i]$ for $i \rightarrow \infty$.



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Suppose that M_k^Φ is obtained from M by k -dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min \{ Y(M, [g]), L(n, k) \} \leq \liminf_{i \rightarrow \infty} Y(M_k^\Phi, [g_i]) \leq Y(M, [g]).$$

Easy part:

$$\liminf_{i \rightarrow \infty} Y(M_k^\Phi, [g_i]) \leq Y(M, [g])$$

Difficult and important part

$$\min \{ Y(M, [g]), L(n, k) \} \leq \liminf_{i \rightarrow \infty} Y(M_k^\Phi, [g_i])$$

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Construction of the metrics

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.

We write close to $S := \Phi(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{\text{round}}^{n-k-1}$$

where g_{round}^{n-k-1} is the round metric on S^{n-k-1} .

$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{\text{round}}^{n-k-1}$$

We define a metric

$$g_i = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{\text{round}}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

that extends to a metric on M_k^Φ .



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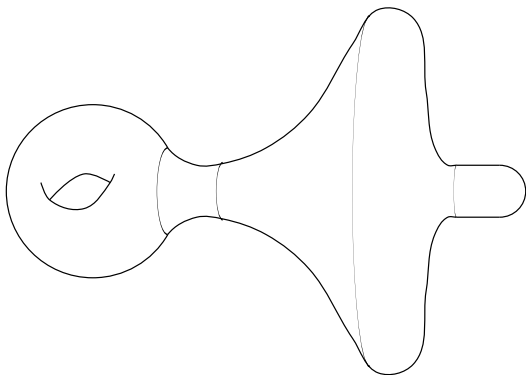
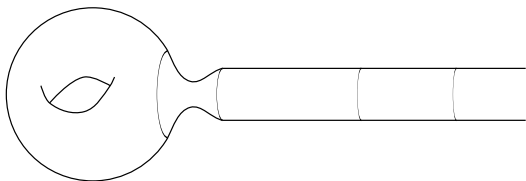
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$\xleftarrow{g_i = g}$
 $\xrightarrow{g_i = F^2 g}$

$\xleftarrow{S^{n-k-1} \text{ has constant length}}$



Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.

We obtain a PDE:

$$4 \frac{n-1}{n-2} \Delta^{g_i} u_i + \text{scal}^{g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \quad \int u_i^{2n/(n-2)} dV^{g_i} = 1, \quad \lambda_i = Y([g_i])$$

This sequence might:

- ▶ Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- ▶ Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- ▶ Concentrate on the new part.

Then study pointed Gromov-Hausdorff limits.

Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

\mathbb{H}_c^{k+1} : simply connected, complete, $K = -c^2$



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The numbers $L(n, k)$

$$L(n, k) := \inf_{c \in [0, 1]} Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$$

Note: $\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1} \cong \mathbb{S}^n \setminus \mathbb{S}^k$.

$k = 0$: $L(n, 0) = Y(\mathbb{R} \times \mathbb{S}^{n-1}) = Y(\mathbb{S}^n)$

$k = 1, \dots, n-3$: $L(n, k) > 0$

Conjecture #1: $L(n, k) = Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$

Conjecture #2: For determining $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ it suffices to consider $O(k+1) \times O(n-k)$ invariant functions.



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If both conjectures are true, then

$$L(4, 1) = 59, 4\dots$$

$$Y(S^4) = 61, 5\dots$$

$$Y(S^2 \times S^2) = 50, 2\dots$$

One obtains $S^2 \times S^2$ via 1-dimensional surgery from S^4 .
Hence

$$\sigma(S^2 \times S^2) \geq 59, 4 > 50, 2\dots = Y(S^2 \times S^2).$$

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