The spectral zeta function

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Abstract

In this talk we introduce spectral zeta functions. The spectral zeta function of the Laplace-Beltrami operator was already introduced by Minakshisundaram and Pleijel in 1948, see [6]. The concept was generalized in several directions, e.g. for large classes of elliptic pseudodifferential operators and it provides an important tool for defining determinants for elliptic operators. We give an introduction in the form of an overview into the subject, using heat kernel methods and the Mellin transform. We include reference for many details that we omitted and for some further reading.

1 The spectrum of generalized Laplacians

Let M be a compact manifold. All manifolds in this talk are manifolds without boundary. Let $E \to M$ be a real vector bundle with a fiberwise scalar product and connection ∇. $\Gamma(E)$ will denote the smooth sections of E. We equip M with a smooth volume element $d\mu$ on M. Then for sections $s_1, s_2 \in \Gamma(E)$ one defines the (global) scalar product as

$$(s_1, s_2) = \int_M \langle s_1(x), s_2(x)\rangle \, d\mu(x).$$

The completion of $\Gamma(E)$ with respect to this scalar product is the space of L^2-sections $\Gamma_{L^2}(E)$, and for other types of regularity, e.g. $R \in \{H^1, H^2, C^2, \ldots\}$ we write $\Gamma_R(E)$ for the associated space of sections. In particular $\Gamma(E) = \Gamma_{C^\infty}(E)$. We have $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$, and $\nabla^* : \Gamma(T^*M \otimes E) \to \Gamma(E)$.

Definition 1.1. *A* generalized Laplacian *on $E \to M$ is an operator*

$$\Delta := \nabla^*\nabla + \mathcal{K} : \Gamma(E) \to \Gamma(E)$$

where $\mathcal{K} \in \Gamma(E * \otimes E)$ is selfadjoint, i.e. for all $x \in M$, $\mathcal{K}(x) \in \mathrm{End}(T_xM)$ in a smooth way, and we further assume that $\mathcal{K}$ is selfadjoint. i.e. $\mathcal{K}(x)^* = \mathcal{K}(x)$. Here $\mathcal{K}$ acts fiberwise, i.e. for $s \in \Gamma(E)$ and $x \in M$ we define $(\mathcal{K}s)(x) := (\mathcal{K}(x))(s(x))$.

Example 1.2. *Let $E = M \times \mathbb{R} \to M$ be the trivial bundle of rank 1 with the trivial connection, $\mathcal{K} = 0$. Then $\Gamma(E) = C^\infty(M, \mathbb{R})$. Further $\nabla = d : C^\infty(M, \mathbb{R}) \to \Gamma(T^*M)$ is the exterior differential, and $\Delta = \nabla^*\nabla = d^*d :$*

$C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ *is the Laplace-Beltrami operator. In the case* $M = \mathbb{R}^n/\mathbb{Z}^n$ *we have the standard Laplacian*

$$\Delta = \sum_{i=1}^{n} -\frac{\partial^2}{\partial {x_i}^2}.$$

For a short introduction into the the main facts about generalized Laplacians needed in the talk, we refer to [1] and the literature cited therein, as e.g. the lecture books [5] and [2].

A generalized Laplacian has an extension to an operator from the Sobolev space $\Gamma_{H^2}(E)$ to $\Gamma_{L^2}(E)$, also denoted by Δ. The space $\Gamma_{H^2}(E)$ is a dense linear subspace of the Hilbert space $\Gamma_{L^2}(E)$, and $\Delta : \Gamma_{H^2}(E) \subset \Gamma_{L^2}(E) \to \Gamma_{L^2}(E)$ is then an unbounded self-adjoint operator in $\Gamma_{L^2}(E)$.

Theorem 1.3. *There exists an orthonormal basis* $\varphi_1, \varphi_2, \dots$ *of* $\Gamma_{L^2}(E)$ *and real numbers* $\lambda_1, \lambda_2, \dots$ *such that*

$$\Delta\varphi_k = \lambda_k \cdot \varphi_k,$$

$\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \nearrow +\infty$. *Each* λ_k *is repeated only finitely many times. All* φ_k *are smooth.*

Theorem 1.4 (Weyl)**.** *Let* $\Delta : \Gamma(E) \to \Gamma(E)$ *be a generalized Laplacian over an* n*-dimensional compact Riemannian manifold. For each* $\lambda \in \mathbb{R}$ *let* $N(\lambda)$ *be the number of eigenvalues of* Δ *less than* λ*. Then*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} = \frac{\mathrm{rank} E \cdot \mathrm{vol}(M)}{(4\pi)^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}+1\right)}.$$

In particular

$$\lambda_k \sim C k^{2/n}.$$

2 Motivation

For any generalized Laplacian Δ we want to define its determinant

$$\text{“}\det\text{''}\Delta = \lambda_1\lambda_2\lambda_3\cdots.$$

Besides the inner-mathematical interest, it is important for quantum field theory to define such determinants, as these determinants are parts of natural action functional in quantum field theory, see e.g. [3] for details.

On the other hand it is obvious that the infinite product $\lambda_1\lambda_2\lambda_3\cdots$ does not converge, so the above definition does not make sense. We have to find a new way to interpret this formula, to *regularize* this divergent product.

Consider a positive definit self-adjoint operator P on a finite-dimensional Hilbert space. Let $\lambda_1, \dots, \lambda_n$ its eigenvalues and $\varphi_1, \dots, \varphi_n$ its eigenvectors. The functional calculus provides for $z \in \mathbb{C}$ an operator P^z with

$$P^z(\varphi_k) := (\lambda_k)^z \varphi_k = e^{z \log \lambda_k} \varphi_k.$$

We define

$$\zeta_P(z) := \mathrm{tr}\, P^{-z} = \sum_{k=1}^{n} (\lambda_k)^{-z} = \sum_{k=1}^{n} e^{-z \log \lambda_k}.$$

Then

$$\zeta_P'(0) = \frac{d}{dz}\Big|_{z=0} \zeta_P(z) = \sum_{k=1}^{n} (-\log \lambda_k) e^{-0 \log \lambda_k} = -\log \left(\prod_{k=1}^{n} \lambda_k \right)$$

Thus

$$\det P = e^{-\zeta_P'(0)}.$$

Strategy: Define the spectral zeta function ζ_Δ for any generalized Laplacian Δ. Then define

$$\det \Delta = e^{-\zeta_\Delta'(0)}$$

3 Definition of spectral zeta functions

In this section we mainly follow [2, Section 9.6].

Let Δ be a generalized Laplacian with notation as above.

We fix a number $\Lambda \geq 0$. Let m be the smallest integer with $\lambda_m > \Lambda$. Let $\mathcal{H}$ be the closed subspace of $\Gamma_{L^2}(E)$ generated by $(\varphi_k)_{k \geq m}$. For the operator

$$P := \Delta|_{\mathcal{H}} : \mathcal{H} \cap \Gamma_{H^2}(E) \subset \mathcal{H} \to \mathcal{H}.$$

we can define complex powers, i.e. for $z \in \mathbb{C}$, there is a unique closed, densely defined operator P^z in $\mathcal{H}$ such that $P^z(\varphi_k) = (\lambda_k)^z \varphi_k$ for $k \geq m$. The operator is bounded if $\Re z \leq 0$, compact for $\Re z < 0$, and because of Weyl's theorem about the asymptotic of eigenvalues, it is trace-class for $\Re z < -n/2$. The trace of such operators is usually written as Tr in order to distinguish it from the pointwise trace tr of endomorphism valued functions as for example $M \to \mathrm{End}(E)$.

Definition 3.1. *The spectral zeta function is*

$$\zeta_{\Delta,\Lambda}(z) := \mathrm{Tr}\, P^{-z}.$$

Example 3.2. *Assume that $M = S^1 = \mathbb{R}/(2\pi\mathbb{Z}) \ni [t]$ is the circle of perimeter 2π, and let Δ be the Laplace-Beltrami operator. Then $\lambda_1 = 0$, and $\lambda_{2k} = \lambda_{2k+1} = k^2$ and $\varphi_1 \equiv 1/(2\pi)$, $\varphi_{2k} = \frac{1}{\pi} \cos kt$ $\varphi_{2k+1} = \frac{1}{\pi} \sin kt$. Then for $\Lambda = 0$ we have $m = 2$, and*

$$\zeta_{\Delta,0}(z) = \sum_{j=2}^{\infty} \lambda_j^{-z} = 2 \sum_{k=1}^{\infty} \frac{1}{k^{2z}} = 2\zeta(2z)$$

where ζ is the Riemann zeta function.

A priori it is only defined for $z \in \mathbb{C}$ with $\Re z > n/2$, but we can extend it meromorphically.

Theorem 3.3. *The spectral zeta function has a meromorphic extension $\zeta_{\Delta,\Lambda} : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. This function is holomorphic in 0 and all poles are elements of $\{n/2 - \ell \,|\, \ell \in \mathbb{N}_0\}$.*

The proof uses heat kernel methods and the Mellin transform.

We then define

$$\det{}' \Delta := e^{-\zeta_{\Delta,0}'(0)}.$$

Here we write $\det'$ instead of det in order to indicate that it is not the determinant on the full space, but only on $\mathcal{H}$.

It is also interesting to analyse what happens when we change the cut-off value from Λ to $\tilde{\Lambda}$. Let

$$\Lambda < \lambda_m \le \lambda_{m+1} \cdots \le \lambda_{\tilde{m}-1} \le \tilde{\Lambda} < \lambda_{\tilde{m}}.$$

Then $\operatorname{Tr} P^{-z} = \operatorname{Tr} \tilde{P}^{-z} + \sum_{j=m}^{\tilde{m}-1} \lambda_j^{-z}$, and this implies

$$e^{-\zeta'_{\Delta,\Lambda}(0)} = \lambda_m \cdot \lambda_{m+1} \cdots \lambda_{\tilde{m}-1} \cdot e^{-\zeta'_{\Delta,\tilde{\Lambda}}(0)}.$$

Thus the following definition for generalized Laplacians Δ does not depend on the choice of $\Lambda \ge 0$:

$$\det_{\text{full}} \Delta = \lambda_1 \cdot \lambda_2 \cdots \lambda_{m-1} \cdot e^{-\zeta'_{\Delta,\Lambda}(0)}.$$

However, obviously if 0 is a priori known to be an eigenvalue, as e.g. for the Laplace-Beltrami operator, then $\det'$ carries non-trivial information in contrast to $\det_{\text{full}}$.

Remark 3.4. *One can define zeta functions and determinants in much larger generality, e.g. for arbitrary positive self-adjoint elliptic (pseudo-)differential operators. However in general*

$$\det(P_1 \circ P_2) \neq \det P_1 \det P_2,$$

which is known as the multiplicative anomaly, see e.g. [4] and [7] for more information.

4 Heat kernel

We briefly summarize the heat kernel methods on compact manifolds, see [1] for details. For simplicity we assume $0 = \Lambda < \lambda_1$.

Definition 4.1. *For $\psi \in E_y$, $x, y \in M$, $t > 0$, we define*

$$k_t(x,y)(\psi) := \sum_{j=1}^{\infty} e^{-t\lambda_j} \langle \varphi_j(y), \psi \rangle \, \varphi_j(x).$$

Thus $k_t(x,y) \in \operatorname{Hom}(E_y, E_x)$. It is called the heat kernel *of Δ on M.*

For $t_0 > 0$, the heat kernel and all its t-derivatives converge uniformly in $t \ge t_0$ in all H^k-norms and all C^k-norms. In particular, $k_t(x,y)$ is smooth in t, x, and y, and we can differentiate term by term.

For every $t > 0$ the section $k_t(x,y)$ is the integral kernel of the operator $e^{-t\Delta}$. If we define for $\Psi \in \Gamma(E)$,

$$\psi_t(x) := \int_M k_t(x,y)(\Psi(y)) = \left(e^{-t\Delta}\Psi\right)(x),$$

then on $M \times [0,\infty)$ we have a solution of the equation

$$\left(\frac{d}{dt} + \Delta\right)\psi_t = 0$$

with initial condition $\psi_0 = \Psi$.

The operator $\frac{d}{dt} + \Delta$ is called the *heat operator* as it describes the propagation of heat, in case that E is the trivial real line bundle with trivial connection.

For $t \to \infty$ the heat kernel converges exponentially to zero. The asymptotics for $t \to 0$ is well-understood, see e.g. [1], and in particular it is an important tool for proving the Atiyah-Singer index theorem, and it is an important ingredient in Connes' approach to describe the Standard Model in terms of noncommutative geometry.

For our presentation we are only interested in the trace of $e^{-t\Delta}$. We use $\operatorname{Tr} e^{-t\Delta} = \int_M k_t(x,x)\,dx$. We obtain the following asymptotic expansion for $t \to 0$:

$$\sum_{j=1}^{\infty} e^{-t\lambda_j} = \operatorname{Tr} e^{-t\Delta} \sim \sum_{\ell=0}^{\infty} a_\ell t^{-(n/2)+\ell}$$

where $a_\ell \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, are real numbers obtained by integrating geometric data.

For example if E is a flat bundle, then

$$a_0 := \frac{\operatorname{rank} E}{(4\pi)^{\frac{n}{2}}} \operatorname{vol}(M), \qquad a_1 := \frac{\operatorname{rank} E}{6(4\pi)^{\frac{n}{2}}} \int\limits_M \operatorname{scal}(x).$$

5 Mellin transform

Again we cite [2, Section 9.6] as reference.

For a function $f \in C^\infty((0,\infty))$ with suitable behavior close to 0 and ∞ we define the *Mellin transform* as

$$M[f](z) = \frac{1}{\Gamma(z)} \int_0^\infty f(t) t^{z-1}\,dt.$$

Recall

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt.$$

Substituting $\tau = \lambda t$, for $\lambda > 0$ one easily sees

$$\int_0^\infty e^{-t\lambda} t^{z-1}\,dt = \Gamma(z)\lambda^{-z}.$$

This implies — provided that all integrals converge —

$$\begin{aligned} M[t \mapsto \operatorname{Tr} e^{-t\Delta}](z) &= \frac{1}{\Gamma(z)} \int_0^\infty \operatorname{Tr}(e^{-t\Delta}) t^{z-1}\,dt \\ &= \sum_{j=1}^{\infty} \frac{1}{\Gamma(z)} \int_0^\infty (e^{-t\lambda_j}) t^{z-1}\,dt \\ &= \sum_{j=1}^{\infty} (\lambda_j)^{-z} = \operatorname{Tr} \Delta^{-z} = \zeta_{\Delta,0}(z) \end{aligned}$$

Proposition 5.1. *Assume $f \in C^\infty((0,\infty))$ has for $t \to 0$ the asymptotic expansion*

$$f(t) \sim \sum_{\ell=0}^{\infty} a_\ell t^{-(n/2)+\ell}.$$

Further we assume

$$|f(t)| \leq Ce^{-t\lambda_0}$$

for some constants $C > 0$, $\lambda_0 > 0$ *and all* $t \geq 1$. *Then*

- *The Mellin transform* $M[f](z)$ *is well-defined thorugh the integral above for* $\Re z > n/2$,
- $M[f](z)$ *has a meromorphic extension to* $\mathbb{C}$ *with poles contained in*

$$\left\{ \frac{n}{2} - \ell \;\middle|\; \ell \in \mathbb{N}_0 \right\},$$

- $M[f](z)$ *is holomorphic in* 0 *with*

$$M[f](0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ a_{n/2} & \text{if } n \text{ is even} \end{cases}$$

- *If* $(n/2) - \ell \notin \mathbb{Z}_{\leq 0}$, *then the residue of* $M[f](z)$ *at* $(n/2) - \ell$ *is*

$$\frac{a_\ell}{\Gamma(\frac{n}{2} - \ell)}.$$

For a proof see the 1st Lemma in Section 9.6 of [2].

As the function $t \mapsto \mathrm{Tr}(e^{-t\Delta})$ satisfies the assumptions of the proposition, the spectral zeta function the theorem is proven, and the residues of the poles reflect the coefficients of the asymptotic expansion of the trace of the heat kernel.

References

[1] Ammann, B., and Bär, C. The Einstein-Hilbert action as a spectral action. In *Noncommutative Geometry and the Stadard Model of Elementary particle Physics, F. Scheck, H. Upmeier, W. Werner (Ed.).* Springer Verlag, 2002. `http://www.mathematik.uni-regensburg.de/ammann/preprints/hessel99/`.

[2] Berline, N., Getzler, E., and Vergne, M. *Heat kernels and Dirac operators.* Springer-Verlag, 1991.

[3] Dunne, G. Functional determinants in quantum field theory. Preprint ArXiv, `http://arxiv.org/abs/0711.1178`, 2007.

[4] Kontsevich, M., and Vishik, S. Determinants of elliptic pseudo-differential operators. Preprint ArXiv, `http://arxiv.org/abs/hep-th/9404046`, 1994.

[5] Lawson, H. B., and Michelsohn, M.-L. *Spin Geometry.* Princeton University Press, Princeton, 1989.

[6] Minakshisundaram, S., and Pleijel, Å. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canadian J. Math. 1* (1949), 242–256.

[7] Okikiolu, K. The multiplicative anomaly for determinants of elliptic operators. *Duke Math. J. 79* (1995), 723–750.

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