The Yamabe invariant

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Einstein-Hilbert functional

Let M be a compact n-dimensional manifold, $n \geq 3$.
The renormalised Einstein-Hilbert functional is

$$\mathcal{E} : \mathcal{M} \to \mathbb{R}, \qquad \mathcal{E}(g) := \frac{\int_M \mathrm{scal}^g \, dv^g}{\mathrm{vol}(M, g)^{(n-2)/n}}$$

$\mathcal{M} := \{\text{metrics on } M\}$.
$[g] := \{u^{4/(n-2)} g \mid u > 0\}$.

Stationary points of $\mathcal{E} : [g] \to \mathbb{R}$ = metrics with constant scalar curvature

Stationary points of $\mathcal{E} : \mathcal{M} \to \mathbb{R}$ = Einstein metrics

Conformal Yamabe constant

Inside a conformal class

$$Y(M,[g]) := \inf_{\tilde{g}\in[g]} \mathcal{E}(\tilde{g}) > -\infty.$$

This is the *conformal Yamabe constant*.

$$Y(M,[g]) \leq Y(\mathbb{S}^n)$$

where $\mathbb{S}^n$ is the sphere with the standard structure.

Solution of the Yamabe problem (Trudinger, Aubin, Schoen-Yau)
$\mathcal{E} : [g] \to \mathbb{R}$ attains its infimum.

Remark $Y(M,[g]) > 0$ if and only if $[g]$ contains a metric of positive scalar curvature.

Obata's theorem

Theorem (Obata)

Assume:

- *M is connected and compact*
- *g_0 is an Einstein metric on M*
- *$g = u^{4/(n-2)} g_0$ with scal^g constant*
- *(M, g_0) not conformal to $\mathbb{S}^n$*

Then u is constant.

Conclusion If g_0 is Einstein, then $\mathcal{E}(g_0) = Y(M, [g_0])$.
This conclusion also holds if g_0 is a non-Einstein metric with $\mathrm{scal} = const \leq 0$ (Maximum principle).

So in these two cases, we have determined $Y(M, [g_0])$.
However in general it is difficult to get explicit "good" lower bounds for $Y(M, [g_0])$.

On the set of conformal classes

$$\sigma(M) := \sup_{[g]\subset\mathcal{M}} Y(M,[g]) \in (-\infty, Y(\mathbb{S}^n)]$$

The smooth Yamabe invariant.
Introduced by O. Kobayashi and R. Schoen.

Remark $\sigma(M) > 0$ if and only if M caries a metric of positive scalar curvature.

Supremum attained?
Depends on M.

Example $\mathbb{C}P^2$

The Fubini-Study g_{FS} metric is Einstein and

$$53.31... = \mathcal{E}(g_{\mathrm{FS}}) = Y(\mathbb{C}P^2, [g_{\mathrm{FS}}]) = \sigma(\mathbb{C}P^2).$$

Supremum attained in the Fubini-Study metric.

LeBrun '97 Seiberg-Witten theory
LeBrun & Gursky '98 Twisted Dirac operators

Similar examples

- $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- Gromov & Lawson, Schoen & Yau $\approx'$ 83: Tori $\mathbb{R}^n/\mathbb{Z}^n$.
  $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$. Enlargeable Manifolds
- LeBrun $'$99: All Kähler-Einstein surfaces with non-positive scalar curvature. Seiberg-Witten theory
- Bray & Neves '04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$.
  Inverse mean curvature flow
- Perelman, M. Anderson '06 (sketch), Kleiner-Lott '08 compact quotients of 3-dimensional hyperbolic space
  Ricci flow

Example where supremum is not attained

Schoen: $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$.
The supremum is not attained.

Some known values of σ

- All examples above.
- Akutagawa & Neves ’07: Some non-prime 3-manifolds, e.g.

$$\sigma(\mathbb{R}P^3 \# (S^2 \times S^1)) = \sigma(\mathbb{R}P^3).$$

- Compact quotients of nilpotent Lie groups: $\sigma(M) = 0$.

Unknown cases

- Nontrivial quotients of spheres, except $\mathbb{R}P^3$.
- $S^k \times S^m$, with $k, m \geq 2$.
- No example of dimension ≥ 5 known with $\sigma(M) \neq 0$ and $\sigma(M) \neq \sigma(S^n)$.

Positive scalar curvature $\Leftrightarrow$ psc $\Leftrightarrow \sigma(M) > 0$

Suppose $n \geq 5$.

1. $\sigma(M) > 0$ is a “bordism invariant”.
2. Bordism classes admitting psc metrics form a subgroup in the bordism group $\Omega_n^{\text{spin}}(B\pi_1)$.
3. If $P^p \xrightarrow{\pi} B^b$ is a fiber bundle, equipped with a family of vertical metrics $(g_p)_{p \in B}$ with $\mathrm{scal}^{g_p} > 0 \ \forall p \in B$, then $\sigma(P) > 0$.

Guiding questions of our work, $\epsilon > 0$

1. Is $\sigma(M) > \epsilon$ a “bordism invariant”?
   Yes for $0 < \epsilon < \Lambda'_n$, $\Lambda'_5 = 45.1$, $\Lambda'_6 = 49.9$, ADH
2. Do $\sigma(M) > \epsilon$-classes form a subgroup?
   Yes for $0 < \epsilon < \Lambda'_n$, ADH
3. If $P^p \stackrel{\pi}{\to} B^b$ is a fiber bundle, equipped with a family of vertical metrics $(g_p)_{p\in B}$ with $Y(\pi^{-1}(p), [g_p]) > 0$, $f = p - b \geq 3$, $b = \dim B \geq 3$, then

$$\sigma(P)^p \geq c_{b,f} \left(\min_{p\in B} Y(\pi^{-1}(p), [g_p]) \right)^f .$$

ADH + M. Streil

Explicit values for Λ_n

Theorem (ADH)

Let M be a compact simply connected manifold, $n = \dim M$. Then

$$n = 5: \quad 45.1 = \Lambda_5' \leq \sigma(M) \leq \sigma(S^5) = 78.9\ldots.$$

$$n = 6: \quad 49.9 = \Lambda_6' \leq \sigma(M) \leq \sigma(S^6) = 96.2\ldots.$$

Gap theorems

Theorem (ADH)

Let M is a 2-connected compact manifold of dimension $n \geq 5$.
If $\alpha(M) \neq 0$, then $\sigma(M) = 0$.
If $\alpha(M) = 0$, then

$n =$	5	6	7	8	9	10	11
$\sigma(M) \geq$	78.9	87.6	74.5	92.2	109.2	97.3	135.9
$\sigma(S^n) =$	78.9	96.2	113.5	130.7	147.8	165.0	182.1

Theorem (ADH)

Let Γ be group whose homology is finitely generated in each degree. In the case $n \geq 5$, we know that

$$\{\sigma(M) \mid \pi_1(M) = \Gamma,\ \dim M = n\} \cap [0, \Lambda_n]$$

is a well-ordered set (with respect to the standard order $\leq$). In other words: there is no sequence of n-dimensional manifolds M_i with $\pi_1(M_i) = \Gamma$ such that $\sigma(M_i) \in [0, \Lambda_n]$ and such that $\sigma(M_i)$ is strictly decreasing.

On the other hand it is conjectured that

$$\sigma(S^n/\Gamma) \to 0 \quad \text{for} \quad \#\Gamma \to \infty$$

Techniques

Key ingredients

(1) A monotonicity formula for surgery, ADH

(2) A lower bound for products, ADH

Other techniques

(3a) Rearranging functions on $\mathbb{H}^r_c \times \mathbb{S}^s$ to test functions on $\mathbb{R}^r \times \mathbb{S}^s$, ADH

(3b) Conformal Yamabe constants of $Y(\mathbb{R}^2 \times \mathbb{S}^{n-2})$, Petean-Ruiz

(4) Are L^p-solutions of the Yamabe equation on complete manifolds already L^2? Results by ADH

(5) Obata's theorem about constant scalar metrics conformal to Einstein manifolds

(6) Standard bordism techniques: Smale, ..., Gromov-Lawson, Stolz

(1) A Monotonicity formula for surgery

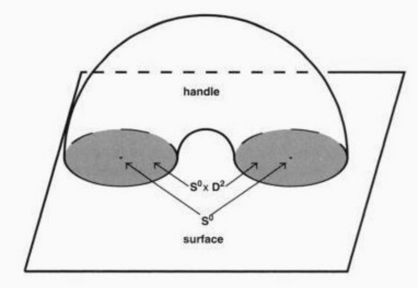
Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M^n$ be an embedding.
We define

$$M_k^\Phi := M \setminus \Phi(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1}) / \sim$$

where $/ \sim$ means gluing the boundaries via

$$M \ni \Phi(x, y) \sim (x, y) \in S^k \times S^{n-k-1}.$$

We say that M_k^Φ is obtained from M by surgery of dimension k.



Example: 0-dimensional surgery on a surface.

Let M_k^Φ be obtained from M by k-dimensional surgery, $0 \leq k \leq n-3$.

Theorem (ADH, # 1)

There is $\Lambda_{n,k} > 0$ *with*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), \Lambda_{n,k}\}$$

Furthermore $\Lambda_{n,0} = Y(\mathbb{S}^n)$.

Special cases were already proved by Gromov-Lawson, Schoen-Yau, Kobayashi, Petean.
Thm # 1 follows directly from Thm # 2.

Theorem (ADH, #2)

For any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\lim_{i\to\infty} Y(M_k^\Phi, [g_i]) = \min\left\{Y(M, [g]), \Lambda_{n,k}\right\}.$$

Construction of the metrics

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We write close to $S := \Phi(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g^{n-k-1}_{round}$$

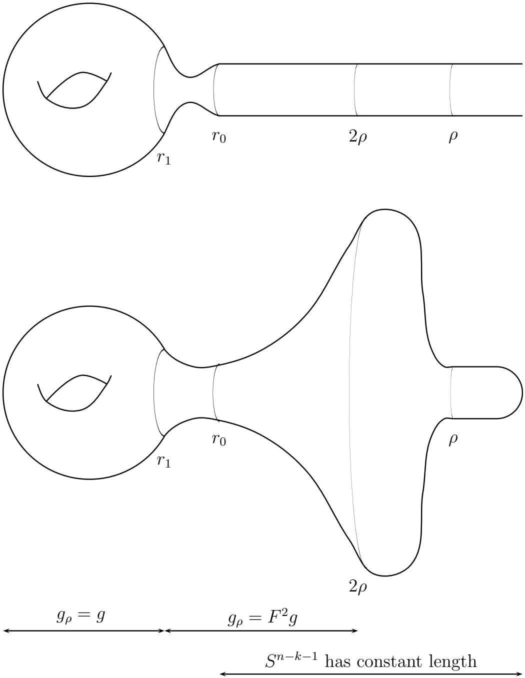
where g^{n-k-1}_{round} is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g^{n-k-1}_{round}$$

We define a metric

$$g_i = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (\rho, r_0) \\ f^2(t) g|_S + dt^2 + g^{n-k-1}_{round} & \text{for } r < \rho \end{cases}$$

that extends to a metric on M^Φ_k.



r_0
2ρ
ρ
r_1
r_0
ρ
r_1
2ρ
$g_\rho = g$
$g_\rho = F^2 g$
S^{n-k-1} has constant length

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.
We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Gromov-Hausdorff convergence of pointed spaces.
  Limit spaces:

$$\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}, \quad c \in [0,1]$$

  $\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$
  Then $\liminf \lambda_i \geq Y(\mathbb{M}_c)$.

The numbers $\Lambda_{n,k}$

(Disclaimer: Additional conditions for $k+3=n\geq 7$
See Ammann–Große 2016 for some related questions)

$$\Lambda_{n,k} := \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$$

$$Y(N) := \inf_{u\in C_c^\infty(N)} \frac{\int_N 4\frac{n-1}{n-2}|du|^2+\mathrm{scal}\, u^2}{(\int_N u^p)^{2/p}}$$

Note: $\mathbb{H}_1^{k+1}\times\mathbb{S}^{n-k-1}\cong\mathbb{S}^n\setminus\mathbb{S}^k$.
$k=0$: $\Lambda_{n,k}=Y(\mathbb{R}\times\mathbb{S}^{n-1})=Y(\mathbb{S}^n)$
$k=1,\ldots,n-3$: $\Lambda_{n,k}>0$

$\Lambda_n := \min\{\Lambda_{n,2},\ldots,\Lambda_{n,n-3}\}$

Notation: Λ'_n is a positive explicit lower bound for Λ_n.

Conjecture #1: $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$

Conjecture #2: The infimum in the definition of $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ is attained by an $O(k+1) \times O(n-k)$ invariant function if $0 \leq c < 1$.

$O(n-k)$-invariance is difficult,
$O(k+1)$-invariance follows from standard reflection methods

Comments If we assume Conjecture #2, then Conjecture #1 reduces to an ODE and $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ can be calculated numerically. Assuming Conjecture #2, a maple calculation confirmed Conjecture #1 for all tested n, k and c.
The conjecture **would** imply:

$$\sigma(S^2 \times S^2) \geq \Lambda_{4,1} = 59.4...$$

Compare this to

$$Y(\mathbb{S}^4) = 61.5... \qquad Y(\mathbb{S}^2 \times \mathbb{S}^2) = 50.2... \qquad \sigma(\mathbb{C}P^2) = 53.31...$$

Values for $\Lambda_{n,k}$

More values of $\Lambda_{n,k}$

n	k	$\Lambda_{n,k} \geq$ known	$\Lambda_{n,k} =$ conjectured	$Y(\mathbb{S}^n)$
3	0	43.8	43.8	43.8
4	0	61.5	61.5	61.5
4	1	38.9	59.4	61.5
5	0	78.9	78.9	78.9
5	1	56.6	78.1	78.9
5	2	45.1	75.3	78.9
6	0	96.2	96.2	96.2
6	1	> 0	95.8	96.2
6	2	54.7	94.7	96.2
6	3	49.9	91.6	96.2
7	0	113.5	113.5	113.5
7	1	> 0	113.2	113.5
7	2	74.5	112.6	113.5
7	3	74.5	111.2	113.5
7	4	> 0	108.1	113.5

(2) A lower bound for products

$a_n := 4(n-1)/(n-2)$

Theorem (ADH)

Let (V, g) *and* (W, h) *be Riemannian manifolds of dimensions* $v, w \geq 3$*. Assume that* $Y(V, [g]) \geq 0$, $Y(W, [h]) \geq 0$ *and that*

$$\frac{\mathrm{Scal}^g + \mathrm{Scal}^h}{a_{v+w}} \geq \frac{\mathrm{Scal}^g}{a_v} + \frac{\mathrm{Scal}^h}{a_w}. \tag{1}$$

Then,

$$\frac{Y(V \times W, [g+h])}{(v+w)a_{v+w}} \geq \left(\frac{Y(V, [g])}{v a_v}\right)^{\frac{v}{m}} \left(\frac{Y(W, [h])}{w a_w}\right)^{\frac{w}{m}}.$$

Main technique: Iterated Hölder inequality.

How good is this bound?

$$b_{v,w} \leq \frac{Y(V \times W, [g + \ h])}{(v+w)\left(\frac{Y(V,[g])}{v}\right)^{\frac{v}{v+w}}\left(\frac{Y(W,[h])}{w}\right)^{\frac{w}{v+w}}} \leq 1,$$

$$b_{v,w} := \frac{a_{v+w}}{{a_v}^{v/(v+w)}{a_w}^{w/(v+w)}} < 1.$$

$b_{v,w}$	w=3	w=4	w=5	w=6	w=7
v= 3	0.625	0.7072..	0.7515..	0.7817..	0.8042..
4	0.7072..	0.7777..	0.8007..	0.8367..	0.8537..
5	0.7515..	0.8007..	0.8427..	0.8631..	0.8772..
6	0.7817..	0.8367..	0.8631..	0.88	0.8921..
7	0.8042..	0.8537..	0.8772..	0.8921..	0.9027..

Application to $\Lambda_{n,k}$

$\mathbb{H}_c^{k+1}$ conformal to a subset of $\mathbb{S}^{k+1}$

$\Rightarrow \quad Y(\mathbb{H}_c^{k+1}) = Y(\mathbb{S}^{k+1})$

Thus for $2 \leq k \leq n-k-4$:

$$\begin{aligned} \Lambda_{n,k} &= \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \\ &\geq n\, b_{k+1,n-k-1} \left(\frac{Y(\mathbb{S}^{k+1})}{k+1}\right)^{(k+1)/n} \left(\frac{Y(\mathbb{S}^{n-k-1})}{n-k-1}\right)^{(n-k-1)/n} \end{aligned}$$

Applying the product formula to fiber bundles

Assume $F^f \to P^n \to B^b$ is a fiber bundle, with a psc-metric g_F on F, structure group in $Isom(F)$, $f = \dim F \geq 3$.

Shrink the psc metric g_F on F.

We see: $\sigma(P) \geq Y((F, g_F) \times \mathbb{R}^b)$ (M. Streil, PhD thesis).
For $b \geq 3$:

$$Y((F, g_F) \times \mathbb{R}^b) \geq n\, b_{f,b} \left(\frac{Y(F, [g_F])}{f} \right)^{f/n} \left(\frac{Y(\mathbb{S}^b)}{b} \right)^{b/n}$$

If g_F carries an Einstein metric, then Petean-Ruiz can provide lower bounds for $Y(F \times \mathbb{R}, [g_F + dt^2])$ and $Y(F \times \mathbb{R}^2, [g_F + dt^2 + ds^2])$.

Important building blocks

For the following manifolds we have lower bounds on the smooth Yamabe invariant and the conformal Yamabe constant.

- Smooth Yamabe invariant of total spaces of bundles with fiber $\mathbb{C}P^2$. These total spaces generate the oriented bordism classes.
- Smooth Yamabe invariant of total spaces of bundles with fiber $\mathbb{H}P^2$. These total spaces generate the kernel of $\alpha : \Omega_n^{\mathrm{spin}} \to KO_n$
- Conformal Yamabe constant of Einstein manifolds: $\mathrm{SU}(3)/\mathrm{SO}(3)$, $\mathbb{C}P^2$, $\mathbb{H}P^2$
- $\mathbb{H}P^2 \times \mathbb{R}$, $\mathbb{H}P^2 \times \mathbb{R}^2$, $\mathbb{C}P^2 \times \mathbb{R}$, $\mathbb{C}P^2 \times \mathbb{R}$ Petean-Ruiz
- Conformal Yamabe constant of $\mathbb{R}^2 \times \mathbb{S}^{n-2}$. Particularly important for $n = 4, 5, 9, 10$. Petean-Ruiz
- Conformal Yamabe constant of $\mathbb{R}^3 \times \mathbb{S}^2$. Petean-Ruiz

Thanks for your attention!

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Possible application to $\mathbb{C}P^3$

Lemma

Assume that the surgery monotoncity formula holds for the conjectured values

$$\Lambda_{6.2} = 94.7\ldots \qquad \Lambda_{6.3} = 91.6\ldots$$

Then $\sigma(\mathbb{C}P^3) \geq \min\{\Lambda_{6.2}, \Lambda_{6.3}\} \geq 91.6\ldots.$

Compare to the Fubini-Study metric g_{FS}

$\mu(\mathbb{C}P^3, [g_{FS}]) = 82.9864\ldots.$

Proof.

$\mathbb{C}P^3$ is spin-bordant to S^6. Find such a bordism W such that that W is 2-connected. Then one can obtain $\mathbb{C}P^3$ by surgeries of dimension 2 and 3 out of S^6. □

Application to connected sums

Assume that M is compact, connected of dimension at least 5 with $0 < \sigma(M) < \min\{\Lambda_{n,1}, \dots \Lambda_{n,n-3}\} =: \widehat{\Lambda}_n$. Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\# \cdots \# M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\# \cdots \# M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, \widehat{\Lambda}_n)$ for $\#\Gamma$ large.

Application to connected sums $M\#N$

Assume that M and N are compact, connected of dimension at least 5 with

$$0 < \sigma(N) > \sigma(M) < \widehat{\Lambda}_n.$$

Then

$$\sigma(M) = \sigma(M\#N).$$

More values of $\Lambda_{n,k}$

Back

n	k	$\Lambda_{n,k} \geq$ known	$\Lambda_{n,k} =$ conjectured	$Y(\mathbb{S}^n)$
8	0	130.7	130.7	130.7
8	1	> 0	130.5	130.7
8	2	92.2	130.1	130.7
8	3	95.7	129.3	130.7
8	4	92.2	127.9	130.7
8	5	> 0	124.7	130.7
9	0	147.8	147.8	147.8
9	1	109.2	147.7	147.8
9	2	109.4	147.4	147.8
9	3	114.3	146.9	147.8
9	4	114.3	146.1	147.8
9	5	109.4	144.6	147.8
9	6	> 0	141.4	147.8

Back

n	k	$\Lambda_{n,k} \geq$ known	$\Lambda_{n,k} =$ conjectured	$Y(\mathbb{S}^n)$
10	0	165.0		165.02
10	1	102.6		165.02
10	2	126.4		165.02
10	3	132.0		165.02
10	4	133.3		165.02
10	5	132.0		165.02
10	6	126.4		165.02
10	7	> 0		165.02
11	0	182.1		182.1
11	1	> 0		182.1
11	2	143.3		182.1
11	3	149.4		182.1
11	4	151.3		182.1
11	5	151.3		182.1
11	6	149.4		182.1
11	7	143.3		182.1
11	8	> 0		182.1