The moduli space of Ricci-flat metrics

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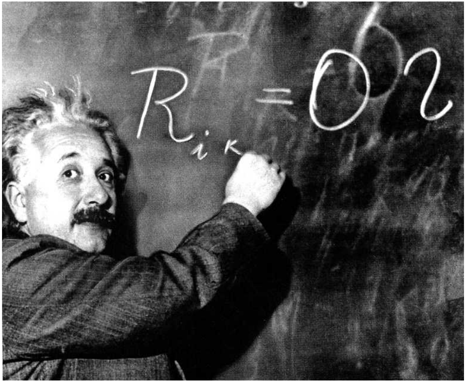
Einstein's equation

1915: Einstein's General Relativity.
What is the equation that describes evolution of spacetime (without matter and without cosmological constant)?



Answer: Ricci-flat metrics

$$\mathrm{Ric} = 0.$$

Lagrangian formulation

What is the associated Lagrange functional?

David Hilbert, 1915

Let $\mathcal{M}$ be the set of all semi-Riemannian metrics on M^n. A metric g_0 is a stationary point of the Einstein-Hilbert-functional

$$\begin{aligned} \mathcal{E} : \mathcal{M} &\to \mathbb{R}, \\ g &\mapsto \int_M \mathrm{scal}^g \, dv^g \end{aligned}$$

among compactly supported perturbations, if and only if g_0 is Ricci-flat.

Introduction

Goal:

Study the space of all Ricci-flat metrics on a given compact manifold M.

Notation:

$\mathcal{M}(M) := \{\text{Riemannian metrics } g \text{ on } M \text{ with volume } 1\}$
$\mathcal{M}_0(M) := \{g \in \mathcal{M}(M) \mid \mathrm{Ric}^g = 0\}$
$\mathrm{Diff}(M) := \{\text{Diffeomorphisms } M \to M\}$
$\mathrm{Diff}_{\mathrm{Id}}(M)$ is the identity component of $\mathrm{Diff}(M)$.

$\mathrm{Diff}(M)$ acts on $\mathcal{M}(M)$ and on $\mathcal{M}_0(M)$ via pullback

Main interest:

Moduli space $\mathcal{M}_0(M)/\mathrm{Diff}(M)$,
Premoduli space $\mathcal{M}_0(M)/\mathrm{Diff}_{\mathrm{Id}}(M)$

Simplest Examples: Bieberbach manifolds

Flat manifolds $M = \mathbb{R}^n/\Gamma$.

$\Gamma \subset \mathbb{R}^n \rtimes O(n)$, discrete, cocompact, acts without fixed points.

Ricci-flat deformations of flat manifolds are also flat.

The "flat connected components" of $\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ are smooth manifolds of dimension ≥ 1.

Are there non-flat Ricci-flat compact manifolds?

Samuel Bochner thought they do not exists.
Eugenio Calabi was his PhD student (1950).
Calabi sketched how to get counterexamples (1954-57).
Proof by Shing-Tung Yau (1978, Fields Medal 1982).

Example The compact 4-manifold $\mathrm{K3}$:

$$\mathrm{K3} := \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{C}P^3.$$

carries a Ricci-flat Kähler metric.
Manifolds with Ricci-flat Kähler metrics are called *Calabi-Yau manifolds*.

Main subject of the talk

We will distinuguish two classes of metrics with $\mathrm{Ric} = 0$.

Definition

A metric g on a compact manifold M is called *good* if

- the universal covering $\widetilde{M}$ is spin and
- if $\widetilde{M}$ carries a non-zero parallel spinor.

$\mathcal{M}_{\|}(M) := \{g \in \mathcal{M}(M) \mid g \text{ good.}\} \subset \mathcal{M}_0(M)$

$\mathcal{M}_{\|}(M)$ is open and closed in $\mathcal{M}_0(M)$.

Open Problem

Is every Ricci-flat metrics good?

$\mathcal{M}_{\|}(M) \subsetneq \mathcal{M}_0(M)$?

Main results

(1) The premoduli space is $\mathcal{M}_{||}(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ is a finite-dimensional smooth manifold.

(2) On $\mathcal{M}_{||}(M)$, the map $g \mapsto \operatorname{Hol}(M, g)$ is locally constant up to conjugation.

(3) On $\mathcal{M}_{||}(M)$, the map $g \mapsto \dim \Gamma_{||}(\Sigma_g M)$ is locally constant.

- Many old results by Lichnerowicz, Koiso and the Besse group ▸ Details
- Much progress was obtained recently by D. Joyce, M. Haskins, R. Goto, J. Nordström, D. Crowley, S. Goette, K. Kröncke and many others
- We just proved a small missing piece.

Cheeger-Gromoll splitting theorem

Let $(\widetilde{M}, \tilde{g})$ be a complete Riemannian manifold.
A *line* is a geodesic $c : \mathbb{R} \to \widetilde{M}$ such that

$$d(c(t), c(s)) = |t - s| \qquad \forall t, s \in \mathbb{R}.$$

Theorem (Cheeger-Gromoll)
If $(\widetilde{M}, \tilde{g})$ is a complete Riemannian manifold which contains a line and if $\mathrm{Ric}^{\widetilde{M}} \geq 0$, then $\widetilde{M} = N \times \mathbb{R}$ as a Riemannian product and N is complete with $\mathrm{Ric}^{N} \geq 0$.

Remark If M is a compact Riemannian manifold. Then $\widetilde{M}$ contains a line if and only if $\pi_1(M)$ is infinite.

By induction we get:

Corollary

If (M, g) *is a compact Riemannian manifold with* $\mathrm{Ric}^M \geq 0$*.*
Then the universal covering $\widetilde{M}$ *is isometric to*

$$N \times \mathbb{R}^k$$

where N *is a simply-connected compact manifold with* $\mathrm{Ric}^N \geq 0$*.*

Note that k is a topological invariant, namely the growth rate of $\pi_1(M)$.

Killing vector fields

Goal: Understand the moduli spaces
$\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M) \supset \mathcal{M}_{\|}(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$

Problematic:

Compare to the (pre)-moduli of all metrics

- $\mathcal{M}(M)/\operatorname{Diff}(M)$ is infinite dimensional.
  No problem for $\mathcal{M}_0(M)$, as $\mathrm{Ric} = 0$ is weakly elliptic.
  Weak ellipticity yields $\dim(\mathcal{M}_0(M)/\operatorname{Diff}(M)) < \infty$.
- $\mathcal{M}(M)/\operatorname{Diff}(M)$ is highly singular
  Example: $\mathcal{M}(M)/\operatorname{Diff}(S^n)$ and $\mathcal{M}(M)/\operatorname{Diff}_{\mathrm{Id}}(S^n)$ have a bad singularity at g_{round}.

Reason: Isometries create singularities in the moduli space.

Back to $\mathcal{M}_0(M)$

Thus: Isometries create singularities in the moduli space.
But luckily we have few isometries:

Lemma

Let K be a Killing vector field on a Riemannian manifold, then

$$\nabla^* \nabla K = \mathrm{Ric}(K)$$

▸ Proof

Corollary

If (M, g) is a compact Riemannian manifold with $\mathrm{Ric} \leq 0$, then every Killing vector field is parallel.

Proof.

$$\int_M \|\nabla K\|^2 = \int_M \langle \mathrm{Ric}(K), K \rangle \leq 0.$$

Corollary

If (M, g) *is a compact Riemannian manifold with* $\mathrm{Ric} \leq 0$ *without parallel vector fields, then* $\mathrm{Isom}(M, g)$ *is finite.*

Consequences

- This will simplify the arguments a lot.
- However if $\mathrm{Isom}(M, g_0)$ is not connected, $\Gamma := \mathrm{Isom}(M, g_0)/\,\mathrm{Isom}_{\mathrm{Id}}(M, g_0)$, then we expect the moduli $\mathcal{M}_0(M)/\,\mathrm{Diff}(M)$ to have an orbifold singularity in $[g_0]$, i.e. locally it is modeled by $\mathbb{R}^k/\Gamma$.
- Instead we discuss $\mathcal{M}_0(M)/\,\mathrm{Diff}_{\mathrm{Id}}(M)$. This has good chances to be a manifold.

Holonomy

Definition

For a Riemannian manifold (M, g), $p \in M$ we define the *holonomy group*

$\mathrm{Hol}(M, g) := \{P_\gamma \mid \gamma$ a path from p to p in $M\} \subset \mathrm{GL}(T_pM)$

and the *restricted holonomy group*

$\mathrm{Hol}_0(M, g) := \{P_\gamma \mid \gamma$ a contractible path from p to p in $M\}$.

$\mathrm{GL}(T_pM) \cong \mathrm{GL}(n, \mathbb{R})$; $\mathrm{Hol}_0(M, g) = \mathrm{Hol}(\widetilde{M}, g)$

Examples

- (M, g) carries a Kähler structure iff $\mathrm{Hol}(M, g) \subset \mathrm{U}(n/2)$.
- (M, g) carries a Kähler structure and is Ricci-flat iff $\mathrm{Hol}_0(M, g) \subset \mathrm{SU}(n/2)$ and $\mathrm{Hol}(M, g) \subset \mathrm{U}(n/2)$.

De Rham Splitting Theorem

Definition

A Riemannian manifold (M, g) is *irreducible* if $\mathrm{Hol}(M, g) \to \mathrm{GL}(T_p M)$ is an irreducible representation.

Theorem

Let $(\widetilde{M}, \tilde{g})$ be a complete simply-connected Riemannian manifold. Then as a Riemannian product

$$\widetilde{M} = M_1 \times \cdots \times M_r \times \mathbb{R}^k$$

where each M_i is a complete non-flat irreducible Riemannian manifold.

M_i and $\mathbb{R}^k$ are called the *de Rahm factors* of $\widetilde{M}$.

If $\mathrm{Ric} \geq 0$ and $\widetilde{M}$ is the covering of a compact M, then all M_i are compact with $\mathrm{Ric} \geq 0$
(de Rham splitting = Cheeger-Gromoll splitting)

Berger's holonomy list

Theorem

If (M, g) *is an irreducible Riemannian manifold,* $n = \dim M$*, then* (M, g) *is locally symmetric or* $\mathrm{Hol}_0(M, g)$ *is one of the following:*

(1) $\mathrm{Hol}_0(M, g) = \mathrm{SO}(n)$ *(generic)*

(2) $\mathrm{Hol}_0(M, g) = \mathrm{U}(n/2)$ *(Kähler)*

(3) $\mathrm{Hol}_0(M, g) = \mathrm{SU}(n/2)$ *(Ricci-flat Kähler)*

(4) $\mathrm{Hol}_0(M, g) = \mathrm{Sp}(n/4)$ *(hyper-Kähler)*

(5) $\mathrm{Hol}_0(M, g) = \mathrm{Sp}(n/4) \cdot \mathrm{Sp}(1)$ *(quaternionic-Kähler)*

(6) $\mathrm{Hol}_0(M, g) = \mathrm{G}_2$ *and* $n = 7$

(7) $\mathrm{Hol}_0(M, g) = \mathrm{Spin}(7)$ *and* $n = 8$

There are compact examples in each case.
In cases (3), (4), (6) and (7) we have $\mathrm{Ric} = 0$.
If M is compact with $\mathrm{Ric} = 0$, then we are in case (1), (3), (4), (6) or (7).

Parallel spinors

If M carries a Riemannian metric and a spin structure, then one can define the spinor bundle $\Sigma M \to M$.
If dim M is even, in the sense of bundles with metric and connection

$$\Sigma M \otimes \Sigma M = \Lambda^{\bullet} T^* M \otimes_{\mathbb{R}} \mathbb{C}$$

If dim M is odd:

$$\Sigma M \otimes \Sigma M = \Lambda^{\text{even}} T^* M \otimes_{\mathbb{R}} \mathbb{C}$$

Parallel spinor = (non-zero) parallel section of ΣM.
Strong restrictions!

If (M, g) carries a parallel spinor, then

- $\mathrm{Ric} = 0$ and
- $\mathrm{Hol}_0(M, g)$ is a product of
  (1) $\{1\} = \mathrm{SO}(1)$
  (3) $\mathrm{SU}(k)$ (Ricci-flat Kähler)
  (4) $\mathrm{Sp}(k)$ (hyper-Kähler)
  (6) G_2
  (7) $\mathrm{Spin}(7)$
- g is a **stable** Ricci-flat metric

For these conclusions it is sufficient that $(\widetilde{M}, \tilde{g})$ carries a parallel spinor.

Deformations of Ricci-flat metrics

Let (M, g) be a compact Ricci-flat Riemannian manifold.

$$T_g(\mathcal{M}(M)) = \{\text{symmetric } (0,2)\text{-tensors}, \int \mathrm{tr}_g h = 0\}$$

Recall $\mathcal{E}(g) = \int_M \mathrm{scal}^{\,g}\, dv^g$.

$$\mathrm{Hess}_g\,\mathcal{E} : T_g(\mathcal{M}(M)) \times T_g(\mathcal{M}(M)) \to \mathbb{R}$$

The *formal tangent space* is

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)) := \ker(\mathrm{Hess}_g\,\mathcal{E}).$$

Corollary (.../AKWW 2015)

If g is a good Ricci-flat metric, then $\mathcal{M}_0(M)$ is a manifold in a neighborhood of g and

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)) = T_g(\mathcal{M}_0(M)).$$

Tangent space of the premoduli space

$$h \in T_g(\mathcal{M}(M)) = \{\text{symmetric } (0,2)\text{-tensors}, \int \mathrm{tr}_g h = 0\}$$

h is orthogonal to the conformal class $[g] \Leftrightarrow \mathrm{tr}_g h = 0$
h is orthogonal to the $\mathrm{Diff}(M)$-orbit of $g \Leftrightarrow \mathrm{div}_g h = 0$

$\mathrm{TT} := \{h \text{ is a symm. } (0,2)\text{-tensor with } \mathrm{tr}_g h = 0 \text{ and } \mathrm{div}_g h = 0\}.$

The *formal tangent space of the premoduli space* is

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)) := \ker(\operatorname{Hess}_g \mathcal{E}) \cap \mathrm{TT}.$$

Corollary (.../AKWW 2015)

If g is a good Ricci-flat metric, then $\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ is a manifold in a neighborhood of $g \operatorname{Diff}_{\mathrm{Id}}(M)$ and

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)) = T_g(\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)).$$

Stability of Ricci-flat metrics

$\mathrm{TT} := \{h$ is a symm. $(0,2)$-tensor with $\mathrm{tr}_g h = 0$ and $\mathrm{div}_g h = 0\}$.

For $h, k \in \mathrm{TT}$ we have

$$(\mathrm{Hess}_g\, \mathcal{E})(h,k) = -\frac{1}{2}\int_M \langle h, \Delta_E k\rangle \,\mathrm{dvol}^g$$

where $\Delta_E = \nabla^*\nabla - 2\mathring{R}$ is the Einstein operator.
A Ricci-flat metric g is called *stable*

$$:\Leftrightarrow (\mathrm{Hess}_g\, \mathcal{E})|_{\mathrm{TT}\times\mathrm{TT}} \leq 0$$
$$\Leftrightarrow \Delta_E \text{ is positive semi-definite}$$

Theorem (McKenzie Wang; *Ind. Univ. Math. J.* 1991)

Good Ricci-flat metrics are stable.

Reproved by X. Dai, X. Wang, G. Wei; *Invent. Math.* 2005

Results

Let M be a compact manfold.

(1) Then $\mathcal{M}_{||}(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ is a smooth manifold.

(2) The function

$$\begin{array}{rcl} \mathcal{M}_{||}(M) & \to & \{\mathrm{Subgrp}(\mathrm{GL}(n,\mathbb{R}))\}/\sim \\ g & \mapsto & [\mathrm{Hol}(M,g)], \end{array}$$

where $\sim$ is conjugation, is locally constant.

(3) If M is spin, then the function

$$\begin{array}{rcl} \mathcal{M}_{||}(M) & \to & \mathbb{R} \\ g & \mapsto & \dim \Gamma_{||}(\Sigma_g M) \end{array}$$

is locally constant.

Results, continued

Our original motivation was:

(4) If M is spin, then the spinorial energy functional

$$\begin{array}{rcl} \mathcal{S}(M) & \to & \mathbb{R} \\ (g,\varphi) & \mapsto & \displaystyle\frac{1}{2}\int_M \|\nabla^g\varphi\|^2 \,\mathrm{dvol}^g \end{array}$$

is a Morse-Bott function.

Here $\mathcal{S}(M)$ is the set of the pairs (g,φ) where g is a Riemannian metric on M and where $\varphi \in \Gamma(\Sigma^g M)$ is of constant length equal to 1, i.e. $\|\varphi\| \equiv 1$.

Interpretations of $\mathcal{S}(M)$

$\mathcal{S}(M)$ has various interpretations in different dimensions:

Examples

- For $\dim M = 2$ there is a double $\mathcal{S}(M) \to \mathrm{Imm}^{\mathrm{form}}(M, \mathbb{R}^3)$ where $\mathrm{Imm}^{\mathrm{form}}(M, \mathbb{R}^3)$ is the set of formal immersions, and

$$\mathrm{Imm}(M, \mathbb{R}^3) \hookrightarrow \mathrm{Imm}^{\mathrm{form}}(M, \mathbb{R}^3)$$

  is a homotopy equivalence.
- For $\dim M = 3$ the set $\mathcal{S}(M)$ is a double covering of the space of trivializations of TM.
- For $\dim M = 7$ the set $\mathcal{S}(M)$ is a double covering of the space of G_2-structures (which are compatible with the spin structure).

Who proved these results?

- Most of the hard work was already done before us.
  If M is simply-connected and g irreducible, then the results were known by classical work (e.g. McKenzie Wang et al.) and work of Tian-Todorov, D. Joyce, R. Goto, J. Nordström
  E.g. Goto and Nordström proved the smoothness of the premoduli in this case.
- Reducible manifolds (i.e. products) were not so clear:
  H.-J. Hein could explain to me essential steps in the CY-case using Tian-Todorov theory.
- Our collaboration provided the missing steps.
- Key argument for our proof: Kröncke's product formula.
  Much simpler than the Tian-Todorov approach
- Subtle arguments for the non-simply-connected case.
  E.g. problematic: Full holonomy groups of good Ricci-flat metrics are not classified in the reducible case.

Kröncke's product formula

Theorem (Kröncke)

If (M^m, g) and (N^n, h) are two stable Ricci-flat manifolds, then $(M \times N, g + h)$ is also stable.
Furthermore, on TT we have

$$\ker(\Delta_E^{M\times N}) = \mathbb{R}(n \cdot g - m \cdot h) \oplus \left(\Gamma_{\|}(TM) \odot \Gamma_{\|}(TN)\right) \oplus \ker(\Delta_E^M) \oplus \ker(\Delta_E^N).$$

Corollary

The Cheeger-Gromoll/de Rham splitting is preserved under deformations.

Some related literature

- B. Ammann, K. Kröncke, H. Weiss, and F. Witt, *Holonomy rigidity for Ricci-flat metrics*, http://arxiv.org/abs/1512.07390
- A. Besse, *Einstein manifolds*, Springer, Berlin, 1987.
- R. Goto, *Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler, G_2 and $\mathrm{Spin}(7)$ structures*, Internat. J. Math. **15** (2004), no. 3, 211–257.
- K. Kröncke *On infinitesimal Einstein deformations*, Differ. Geom. Appl. **38** (2015), 41–57. http://arxiv.org/abs/1508.00721
- J. Nordström, *Ricci-flat deformations of metrics with exceptional holonomy*, Bull. Lond. Math. Soc. **45** (2013), 1004–1018. http://arxiv.org/abs/1008.0663.
- M. Wang, *Preserving parallel spinors under metric deformations*, Ind. Math. J. **40** (1991), 815–844.

Besse group

The authors of the book “Einstein manifolds” by “Arthur Besse” are:

Averous, Bérard-Bergery, Berger, Bourguignon, Derdzinski, DeTurck, Gauduchon, Hitchin, Houillot, Karcher, Kazdan, Koiso, Lafontaine, Pansu, Polombo, Thorpe, Valère

Back

Proof of the Lemma for Killing fields

Let K be a Killing field on a Riemannian manifold. Then $Y \mapsto \nabla_Y K$ is skew-symmetric, and thus $Y \mapsto \nabla^2_{X,Y} K$ is also skew-symmetric for fixed X.

$$
\begin{aligned}
0 &= g(R(X,Y)K,W) + g(R(Y,W)K,X) + g(R(W,X)K,Y) \\
&= g(\nabla^2_{X,Y}K,W) - g(\nabla^2_{Y,X}K,W) + g(R(Y,W)K,X) \\
&\quad + g(\nabla^2_{W,X}K,Y) - g(\nabla^2_{X,W}K,Y) \\
&= 2g(\nabla^2_{X,Y}K,W) + 2g(R(Y,W)K,X)
\end{aligned}
$$

Thus

$$\nabla^2_{X,Y}K + R(K,X)Y = 0.$$

Taking the trace we get

$$\nabla^*\nabla K = \mathrm{Ric}(K).$$