# The moduli space of Ricci-flat metrics

B. Ammann[1]

[1]Universität Regensburg, Germany

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# History of the subject

Ricci-flat manifolds and parallel spinors have a rich history, e.g.

- the work by Lichnerowicz, Koiso and the Besse group
  ▸ Details
- Bochner, Calabi and Yau $\rightsquigarrow$ Calabi-Yau manifolds
- stability results by McK Wang extended by X. Dai, X. Wang and G. Wei
- recent progress by D. Joyce, M. Haskins, R. Goto, J. Nordström, D. Crowley, S. Goette, K. Kröncke

and many many others
(sorry for not mentioning them!)

# My own relation to the subject

- 2009-2012: H. Weiß and F. Witt.
  Studied a geometric weakly parabolic flow on 7-dimensional manifolds
  Fixed points = $G_2$ metrics
- 2012-2014: B. Ammann, H. Weiß and F. Witt.
  Generalisation to arbitrary dimensions: "spinor flow"
  Fixed points = metrics with parallel spinors
- 2015-2017: Further work on the spinor flow by J. Wittmann (Regensburg) and L. Schiemanowski (Kiel)
- 2015: B. Ammann, K. Kröncke, H. Weiß and F. Witt
  Moduli space of Ricci-flat metrics with parallel spinors
- 2016: D. Wraith. Space of metrics with $\mathrm{scal} \geq 0$
- June 2017: B. Ammann, K. Kröncke, Th. Leistner, A. Lischewski, O. Müller
  Application to a constraint equation in Lorentzian manifolds
- (in progress): B. Ammann, K. Kröncke, H. Weiß:
  the structure bundle

# Introduction

## Goal:

Study the space of all Ricci-flat metrics on a given compact manifold $M$.

## Notation:

$\mathcal{M}(M) := \{\text{Riemannian metrics } g \text{ on } M \text{ with volume } 1\}$
$\mathcal{M}_0(M) := \{g \in \mathcal{M}(M) \mid \mathrm{Ric}^g = 0\}$
$\mathrm{Diff}(M) := \{\text{Diffeomorphisms } M \to M\}$
$\mathrm{Diff}_{\mathrm{Id}}(M)$ is the identity component of $\mathrm{Diff}(M)$.

$\mathrm{Diff}(M)$ acts on $\mathcal{M}(M)$ and on $\mathcal{M}_0(M)$ via pullback

## Main interest:

Moduli space $\mathcal{M}_0(M)/\,\mathrm{Diff}(M)$,
Premoduli space $\mathcal{M}_0(M)/\,\mathrm{Diff}_{\mathrm{Id}}(M)$

## Example 1: Bieberbach manifolds

Flat manifolds $M = \mathbb{R}^n/\Gamma$.
$\Gamma \subset \mathbb{R}^n \rtimes O(n)$, discrete, cocompact, acts without fixed points.
Every Ricci-flat metric on $M$ is flat.

The connected components of $\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ are smooth manifolds of dimension $\geq 1$.

## Example 2: Calabi-Yau manifolds

Calabi-Yau manifolds = Ricci-flat Kähler manifolds.
Their moduli spaces were studied by Tian and Todorov.
E.g. "the" compact 4-manifold $\mathrm{K3}$:

$$\mathrm{K3} := \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{C}P^3.$$

carries a CY metric.

## Main subject of the talk

We will distinuguish two classes of metrics with $\mathrm{Ric} = 0$.

## Definition

A metric $g$ on a compact manifold $M$ is called *structured* if

- the universal covering $\widetilde{M}$ is spin and
- if $\widetilde{M}$ carries a non-zero parallel spinor.

$\mathcal{M}_{\|}(M) := \{g \in \mathcal{M}(M) \mid g \text{ structured}\} \subset \mathcal{M}_0(M)$

$\mathcal{M}_{\|}(M)$ is open and closed in $\mathcal{M}_0(M)$.

## Open Problem

*Is every Ricci-flat metric structured?*

$\mathcal{M}_{\|}(M) \subsetneq \mathcal{M}_0(M)$?

## Theorem A (A.-Kröncke-Weiß-Witt)

*Let $M$ be a closed manifold whose universal covering is spin.*

(1) *The premoduli space is* $\mathcal{M}_{\|}(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ *is a finite-dimensional smooth manifold.*

(2) *On* $\mathcal{M}_{\|}(M)$*, the map* $g \mapsto \operatorname{Hol}(M,g)$ *is locally constant up to conjugation.*

(3) *On* $\mathcal{M}_{\|}(M)$*, the map* $g \mapsto \dim \Gamma_{\|}(\Sigma_g M)$ *is locally constant.*

If $M$ is simply connected and if $g_0$ is a metric of irreducible holonomy, then the theorem was already known for metrics close to $g_0$.

**Sketch of proof:** After passing to a finite covering $\widehat{M} \to M$, we can decompose

$$\widehat{M} = M_1 \times \cdots \times M_r \times T^k$$

into irreducible factors and a torus. Show that deformations "preserve" this structure. Use the known results for the irreducible factors.

# Cheeger-Gromoll splitting theorem

The Cheeger-Gromoll splitting theorem implies:

**Theorem**

*If* $(M, g)$ *is a compact Riemannian manifold with* $\mathrm{Ric}^M \geq 0$*.*
*Then the universal covering* $\widetilde{M}$ *is isometric to*

$$N \times \mathbb{R}^k$$

*where* $N$ *is a simply-connected compact manifold with* $\mathrm{Ric}^N \geq 0$*.*

Note that $k$ is a topological invariant, namely the growth rate of $\pi_1(M)$.

# Killing vector fields

Isometries create singularities in the moduli space.
But luckily we have few isometries:

## Lemma

*Let $K$ be a Killing vector field on a Riemannian manifold, then*

$$\nabla^* \nabla K = \mathrm{Ric}(K)$$

▸ Proof

## Corollary

*If $(M, g)$ is a compact Riemannian manifold with $\mathrm{Ric} \leq 0$, then every Killing vector field is parallel.*

## Proof.

$$\int_M \|\nabla K\|^2 = \int_M \langle \mathrm{Ric}(K), K \rangle \leq 0.$$

### Corollary

*If* $(M, g)$ *is a compact Riemannian manifold with* $\mathrm{Ric} = 0$*.*
*Then* $\widetilde{M}$ *is isometric to*

$$N \times \mathbb{R}^k$$

*where N is a simply-connected compact manifold* $\mathrm{Ric}^N = 0$*,* **and** $\mathrm{Isom}(N)$ **is finite***.*

## Consequences

- It will imply that the singularities of the moduli space $\mathcal{M}_0(M)/\operatorname{Diff}(M)$ are not worse than orbifold singularities. The orbifold singularity at $(M, g_0)$ is modeled by $\mathbb{R}^n/\Gamma$ where $\Gamma := \operatorname{Isom}(M, g_0)/\operatorname{Isom}_{\mathrm{Id}}(M, g_0)$ is finite.
- To get rid of orbifold singularities, we pass to the premoduli space $\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$. This will turn out to be a smooth manifold.
- Every (closed) structured Ricci-flat manifold is finitely covered by a manifold with a parallel spinor.

# Holonomy

## Definition

The *holonomy group*

$\mathrm{Hol}(M, g) := \{P_\gamma \mid \gamma$ path from $p$ to $p$ in $M\} \subset \mathrm{GL}(T_pM)$.

The *restricted holonomy group*

$\mathrm{Hol}_0(M, g) := \{P_\gamma \mid \gamma$ contract. path from $p$ to $p\} = \mathrm{Hol}(\widetilde{M}, g)$.

$(M, g)$ is *irreducible* if $\mathrm{Hol}_0(M, g) \to \mathrm{GL}(T_pM)$ is an irred. repr.

## Theorem (de Rham splitting theorem)

*Let $(\widetilde{M}, \tilde{g})$ be a complete simply-connected Riemannian manifold. Then as a Riemannian product*

$$\widetilde{M} = M_1 \times \cdots \times M_r \times \mathbb{R}^k$$

*where each $M_i$ is a complete non-flat irreducible Riemannian manifold.*

For $\mathrm{Ric} \geq 0$ on a univ. covering $\widetilde{M}$ of a closed $M$ this refines the Cheeger-Gromoll splitting.

## Berger's holonomy list

### Theorem

*If $(M, g)$ is an irreducible Riemannian manifold, $n = \dim M$, then $(M, g)$ is locally symmetric or $\mathrm{Hol}_0(M, g)$ is one of the following:*

(1) $\mathrm{Hol}_0(M, g) = \mathrm{SO}(n)$ *(generic)*

(2) $\mathrm{Hol}_0(M, g) = \mathrm{U}(n/2)$ *(Kähler)*

(3) $\mathrm{Hol}_0(M, g) = \mathrm{SU}(n/2)$ *(Ricci-flat Kähler)*

(4) $\mathrm{Hol}_0(M, g) = \mathrm{Sp}(n/4)$ *(hyper-Kähler)*

(5) $\mathrm{Hol}_0(M, g) = \mathrm{Sp}(n/4) \cdot \mathrm{Sp}(1)$ *(quaternionic-Kähler)*

(6) $\mathrm{Hol}_0(M, g) = \mathrm{G}_2$ *and* $n = 7$

(7) $\mathrm{Hol}_0(M, g) = \mathrm{Spin}(7)$ *and* $n = 8$

There are compact examples in each case.
In cases (3), (4), (6) and (7) we have $\mathrm{Ric} = 0$.
If $M$ is compact with $\mathrm{Ric} = 0$, then we are in case (1), (3), (4), (6) or (7).

# Parallel spinors and holonomy

- If $(M, g)$ carries a parallel spinor, then $(\widetilde{M}, \tilde{g})$ carries a parallel spinor (pullback).
- If $(\widetilde{M}, \tilde{g})$ carries a parallel spinor, then $g$ is a **infinitesimally stable** Ricci-flat metric.
- $(\widetilde{M}, \tilde{g})$ carries a parallel spinor if and only if $\mathrm{Hol}_0(M, g)$ is a product of
  (1) $\{1\} = SO(1)$
  (3) $SU(k)$ (Ricci-flat Kähler)
  (4) $Sp(k)$ (hyper-Kähler)
  (6) $G_2$
  (7) Spin(7)

## Deformations of Ricci-flat metrics

Let $(M, g)$ be a compact Ricci-flat Riemannian manifold.

$$T_g(\mathcal{M}(M)) = \{\text{symmetric } (0,2)\text{-tensors}, \int \mathrm{tr}_g h = 0\}$$

Einstein-Hilbert functional $\mathcal{E}(g) = \int_M \mathrm{scal}^{\,g}\, dv^g$.

$$\mathrm{Hess}_g\, \mathcal{E} : T_g(\mathcal{M}(M)) \times T_g(\mathcal{M}(M)) \to \mathbb{R}$$

The *formal tangent space* of $\mathcal{M}_0(M)$ is

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)) := \ker(\mathrm{Hess}_g\, \mathcal{E}).$$

### Corollary (.../AKWW 2015)

*If $g$ is a structured Ricci-flat metric, then $\mathcal{M}_0(M)$ is a manifold in a neighborhood of $g$ and*

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)) = T_g(\mathcal{M}_0(M)).$$

# Tangent space of the premoduli space

$$h \in T_g(\mathcal{M}(M)) = \{\text{symmetric } (0,2)\text{-tensors}, \int \mathrm{tr}_g h = 0\}$$

$h$ is orthogonal to the conformal class $[g] \Leftrightarrow \mathrm{tr}_g h = 0$
$h$ is orthogonal to the $\mathrm{Diff}(M)$-orbit of $g \Leftrightarrow \mathrm{div}_g h = 0$

$\mathrm{TT} := \{h$ is a symm. $(0,2)$-tensor with $\mathrm{tr}_g h = 0$ and $\mathrm{div}_g h = 0\}$.

The *formal tangent space of the premoduli space* is

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)) := \ker(\operatorname{Hess}_g \mathcal{E}) \cap \mathrm{TT}.$$

## Corollary (.../AKWW 2015)

*If $g$ is a structured Ricci-flat metric, then $\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ is a manifold in a neighborhood of $g\operatorname{Diff}_{\mathrm{Id}}(M)$ and*

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)) = T_g(\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)).$$

## Infinitesimal stability of Ricci-flat metrics

$\mathrm{TT} := \{h$ is a symm. $(0,2)$-tensor with $\mathrm{tr}_g h = 0$ and $\mathrm{div}_g h = 0\}$.

For $h, k \in \mathrm{TT}$ we have

$$(\mathrm{Hess}_g\, \mathcal{E})(h,k) = -\frac{1}{2}\int_M \langle h, \Delta_E k\rangle \,\mathrm{dvol}^g$$

where $\Delta_E = \nabla^*\nabla - 2\mathring{R}$ is the Einstein operator.
A Ricci-flat metric $g$ is called *infinitesimally stable*

$$:\Leftrightarrow (\mathrm{Hess}_g\, \mathcal{E})|_{\mathrm{TT}\times\mathrm{TT}} \leq 0$$
$$\Leftrightarrow \Delta_E \text{ is positive semi-definite}$$

### Theorem (McKenzie Wang, Ind. Univ. Math. J. 1991)

*Structured Ricci-flat metrics are infinitesimally stable.*

More precisely: McK Wang assumes a parallel spinor on $M$. Using what we explained before, the theorem then directly follows as claimed.

### Theorem (X. Dai, X. Wang, G. Wei, Inv. Math. 2005)

*Structured Ricci-flat metrics $g_0$ with irreducible holonomy on a simply connected closed manifold are locally stable, i.e. there is no metric of positive scalar curvature in a neighborhood of $g_0$.*

▸ Historical details

# Kröncke's product formula

## Theorem (Kröncke)

*If* $(M^m, g)$ *and* $(N^n, h)$ *are two infinitesimally stable Ricci-flat manifolds, then* $(M \times N, g + h)$ *is also infinitesimally stable. Furthermore, on* $\mathrm{TT}$ *we have*

$$\ker(\Delta_E^{M\times N}) = \mathbb{R}(n \cdot g - m \cdot h) \oplus \big(\Gamma_{\|}(TM) \odot \Gamma_{\|}(TN)\big) \oplus \ker(\Delta_E^M) \oplus \ker(\Delta_E^N).$$

## Corollary (Kröncke/AKWW)

*The Cheeger-Gromoll/de Rham splitting is preserved under deformations.*

Other Ingredient: The Montgomery-Zippin theorem allows to pass from simply-connected manifolds to arbitrary manifolds.

### Corollary (Kröncke/AKWW)

*Local stability is true for arbitrary closed manifolds.*

### Corollary

*$M$ is a closed manifold such that $\mathcal{M}_{\|}(M)$ has at least $k$ connected components, then $\mathcal{M}_{\|}(M \times M)$ has at least $k^2$ connected components.*

# Application: Metrics of non-negative scalar curvature

Let $M$ be a closed spin manifold.
$\mathrm{Scal}_{>0}(M) := \{g \in \mathcal{M}(M) \mid \mathrm{scal}^g > 0\}$
$\mathrm{Scal}_{\geq 0}(M) := \{g \in \mathcal{M}(M) \mid \mathrm{scal}^g \geq 0\}$

## Theorem (D. Wraith, July 2016 (arxiv))

*If $g_t$ is a path in $\mathrm{Scal}_{\geq 0}(M)$ with $g_0 \in \mathrm{Scal}_{>0}(M)$, then* $\ker D^V_{g(t)} = 0$ *for all $t$ and all Dirac operators twisted with flat bundles $V$.*

Proof: Assume $\ker D^V_{g_1} \ni \varphi \neq 0$. Then $\nabla^{g_1}\varphi = 0$ and $\mathrm{scal}^{g_1} = 0$. Apply Ricci flow to the path $g_t$. Let $\tau \in [0,1]$ be minimal such that $\mathrm{scal}^{g_t} = 0$ for $t \in [\tau, 1]$. Then $\mathrm{Ric}^{g_t} = 0$ for $t \in [\tau, 1]$ and $\mathrm{scal}^{g_{\tau-\epsilon}} > 0$. Thus $g_\tau$ has a parallel spinor. Thus cannot be deformed to $\mathrm{scal} > 0$. $\lightning$

## Consequences

- $\mathrm{Scal}_{\geq 0}(M) = \overline{\mathrm{Scal}_{>0}(M)}^{+} \sqcup \mathcal{M}_{\|}(M) \sqcup \{\text{Unstruct. Ricci flat}\}$
- On $\overline{\mathrm{Scal}_{>0}(M)}^{+} \subset \mathrm{Scal}_{\geq 0}(M)$ the kernel of the Dirac operator vanishes.

Two comments added after the talk:

- $\{\text{Unstruct. Ricci flat}\}$ means the set of some unstructured Ricci-flat metrics not all of them, and it is potentially the empty set.
- $\overline{\mathrm{Scal}_{>0}(M)}^{+}$ is the union of all (path) connected components in $\mathrm{Scal}_{\geq 0}(M)$ which contain an element of $\mathrm{Scal}_{>0}(M)$.

$\mathrm{Scal}_{\geq 0}(M)$ is locally path connected, $\overline{\mathrm{Scal}_{>0}(M)}^{+}$ contains the closure of $\mathrm{Scal}_{>0}(M)$, but it is unknown whether they coincide. I think I said something unprecise in the talk at this point.

## “Corollary”

*Every* **known** *non-trivial element in* $\pi_k(\mathrm{Scal}_{>0}(M))$ *is also non-trivial in* $\pi_k(\mathrm{Scal}_{\geq 0}(M))$.

## “Proof”

*All known proofs showing that elements in* $\pi_k(\mathrm{Scal}_{>0}(M))$ *are non-trivial use index theory. As* $\ker D$ *vanishes not only on* $\mathrm{Scal}_{>0}(M)$ *but on* $\overline{\mathrm{Scal}_{>0}(M)}^{+}$ *the non-triviality extends to* $\overline{\mathrm{Scal}_{>0}(M)}^{+}$.

## Remark

- *I did* **not** *claim that* $\pi_k(\iota) : \pi_k(\mathrm{Scal}_{>0}(M)) \to \pi_k(\mathrm{Scal}_{\geq 0}(M))$ *is injective. The kernel of* $\pi_k(\iota)$ *might be large, but I do not know methods to detect non-trivial elements in* $\ker \pi_k(\iota)$.

# Application: Lorentzian manifolds with parallel spinors

## BBGM parallel transport for spinors

BBGM = Bär-Bourguignon-Gauduchon-Moroianu

Let $g_t$, $t \in [0, T]$ be a path of Riemannian metrics on $M$.

Get metric $G := g_t + dt^2$ on $M \times [0, T]$

## Definition

For any $\varphi \in \Gamma(\Sigma^{g_0} M)$ we define the *BBGM parallel transport* $\mathcal{P}_{0,T}(\varphi)$ as follows:

$$\Sigma^{g_0} M \hookrightarrow \Sigma^G(M \times [0, T])|_{M\times\{0\}}$$

Extend $\varphi$ to $\Phi \in \Gamma(\Sigma^G(M \times [0, T])$ with $\frac{\nabla}{dt}\Phi = 0$.

$$\mathcal{P}_{0,T}(\varphi) := \Phi_{M\times\{T\}} \in \Gamma(\Sigma^{g_T} M).$$

## Theorem B (A., Kröncke, Müller 2017)

*Assume $g_t$ is a family of Ricci-flat metrics on a closed manifold $M$, and $\mathrm{div}^{g_t} \frac{d}{dt} g_t = 0$. Let $\varphi \in \Gamma(\Sigma^{g_0} M)$ be parallel. Then $\mathcal{P}_{0,t}(\varphi) \in \Gamma(\Sigma^{g_t} M)$ is parallel for all $t$.*

Using "good" identifications $\Phi \in \Gamma(\Sigma N)$, $N := M \times [0, T]$.

$$\frac{\partial}{\partial t} \cdot \Phi = i\Phi$$

$$\nabla^N_{(\partial/\partial s)} \Phi = 0$$

$$\nabla^N_X \Phi = \frac{1}{2} W(X) \cdot \frac{\partial}{\partial s} \cdot \Phi = \frac{i}{2} W(X) \cdot \Phi$$

These are solutions for the constraint equations for Lorentzian manifolds with a parallel spinor.

Consequence (& Lischewski und Leistner):
There is a time- and space-oriented Lorentzian manifold $Q$ of dimension $\dim M + 2$ with

- $(N, g_t + dt^2)$ is a Cauchy hypersurface for $Q$,
- $Q$ carries a parallel spinor whose associated vector field is lightlike (and parallel).

Expected picture:

$$\left\{\begin{array}{l}\text{Curves in}\\ \ \mathcal{M}_{\|}(M)/\operatorname{Diff}_{\mathrm{Id}}(M)\\ +\text{ scaling functions}\end{array}\right\} \overset{\text{1-to-1}}{\longleftrightarrow} \left\{\begin{array}{l}\text{Cauchy data for}\\ \text{Lorentzian manifolds with}\\ \text{a lightlike parallel spinor}\end{array}\right\}$$

$$\overset{\text{1-to-1}}{\longleftrightarrow} \left\{\begin{array}{l}\text{Lorentzian manifolds with}\\ \text{a lightlike parallel spinor}\\ +\text{ choice of a Cauchy surface}\end{array}\right\}$$

# Some related literature


- B. Ammann, K. Kröncke, H. Weiss, and F. Witt, *Holonomy rigidity for Ricci-flat metrics*, `http://arxiv.org/abs/1512.07390`
- A. Besse, *Einstein manifolds*, Springer, Berlin, 1987.
- R. Goto, *Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler, $\mathrm{G}_2$ and $\mathrm{Spin}(7)$ structures*, Internat. J. Math. **15** (2004), no. 3, 211–257.
- K. Kröncke *On infinitesimal Einstein deformations*, Differ. Geom. Appl. **38** (2015), 41–57. `http://arxiv.org/abs/1508.00721`
- J. Nordström, *Ricci-flat deformations of metrics with exceptional holonomy*, Bull. Lond. Math. Soc. **45** (2013), 1004–1018. `http://arxiv.org/abs/1008.0663`.

# Literature continued


- M. WANG, *Preserving parallel spinors under metric deformations*, Ind. Math. J. **40** (1991), 815–844.
- X. DAI, X. WANG, G. WEI, *On the Stability of Riemannian Manifold with Parallel Spinors*, Invent. Math. **161** (2005), 151–176.
- D. WRAITH, *Non-negative versus positive scalar curvature*, `http://arxiv.org/abs/1607.00657`

# Besse group

The authors of the book "Einstein manifolds" by "Arthur Besse" are:

Averous, Bérard-Bergery, Berger, Bourguignon, Derdzinski, DeTurck, Gauduchon, Hitchin, Houillot, Karcher, Kazdan, Koiso, Lafontaine, Pansu, Polombo, Thorpe, Valère

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# Proof of the Lemma for Killing fields

Let $K$ be a Killing field on a Riemannian manifold. Then $Y \mapsto \nabla_Y K$ is skew-symmetric, and thus $Y \mapsto \nabla^2_{X,Y} K$ is also skew-symmetric for fixed $X$.

$$\begin{aligned}
0 &= g(R(X,Y)K,W) + g(R(Y,W)K,X) + g(R(W,X)K,Y) \\
&= g(\nabla^2_{X,Y}K,W) - g(\nabla^2_{Y,X}K,W) + g(R(Y,W)K,X) \\
&+ g(\nabla^2_{W,X}K,Y) - g(\nabla^2_{X,W}K,Y) \\
&= 2g(\nabla^2_{X,Y}K,W) + 2g(R(Y,W)K,X)
\end{aligned}$$

Thus

$$\nabla^2_{X,Y}K + R(K,X)Y = 0.$$

Taking the trace we get

$$\nabla^*\nabla K = \mathrm{Ric}(K).$$

# Historical comments about stability

- Infinitesimal stability: This was proven by McK Wang for manifolds with parallel spinors. The general case of structured Ricci-flat manifolds then follows using the fact that any closed structured Ricci-flat manifold is finitely covered by one with parallel spinor. Dai-Wang-Wei reprove this. Their argument for infinite fundamanetal group can be simplified using this finite covering.
- For local stability Dai-Wang-Wei require simple-connectedness (an assumption which is not hard to remove). Their proof however also uses irreducibility. The irreducibility is used at the step to argue that every formal deformation integrates to a deformation. In reducibe metrics Dai-Wang-Wei's arguments only yield proofs for formal deformations of product type.

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