A surgery formula for the (smooth) Yamabe invariant

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(Smooth) Yamabe invariant

Let M be a compact n-dimensional manifold, $n \geq 3$.
The renormalised Einstein-Hilbert functional is

$$\mathcal{E} : \mathcal{M} \to \mathbb{R}, \qquad g \mapsto \frac{\int_M \mathrm{scal}^g \, dv^g}{\mathrm{vol}(M, g)^{(n-2)/n}}$$

$\mathcal{M} := \{\text{metrics on } M\}$.
$[g] := \{u^{4/(n-2)} g \mid u > 0\}$.
Stationary points of $\mathcal{E} : [g] \to \mathbb{R}$ = metrics with constant scalar curvature
Stationary points of $\mathcal{E} : \mathcal{M} \to \mathbb{R}$ = Einstein metrics

Conformal and smooth invariants

Inside a conformal class

$$Y(M,[g]) := \inf_{\tilde{g}\in[g]} \mathcal{E}(\tilde{g}) > -\infty.$$

The conformal Yamabe constant.

$$Y(M,[g]) \leq Y(\mathbb{S}^n)$$

where $\mathbb{S}^n$ is the sphere with the standard structure.

Solution of the Yamabe problem (Trudinger, Aubin, Schoen-Yau) $\mathcal{E} : [g] \to \mathbb{R}$ attains its infimum.

Remark $Y(M,[g]) > 0$ if and only if $[g]$ contains a metric of positive scalar curvature.

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$\mathcal{E} : [g] \to \mathbb{R}$ attains its infimum.

Remark $Y(M,[g]) > 0$ if and only if $[g]$ contains a metric of positive scalar curvature.

On the set of conformal classes

$$\sigma(M) := \sup_{[g]\subset\mathcal{M}} Y(M,[g]) \in (-\infty, Y(\mathbb{S}^n)]$$

The smooth Yamabe invariant, also called Schoen's σ-constant.

Remark $\sigma(M) > 0$ if and only if M caries a metric of positive scalar curvature.

Supremum attained?
Depends on M.

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Depends on M.

Example $\mathbb{C}P^2$

The Fubini-Study g_{FS} metric is Einstein and

$$53,31... = \mathcal{E}(g_{\mathrm{FS}}) = Y(\mathbb{C}P^2, [g_{\mathrm{FS}}]) = \sigma(\mathbb{C}P^2).$$

Supremum attained in the Fubini-Study metric.

LeBrun '97 Seiberg-Witten theory

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Similar examples

- $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- Gromov & Lawson, Schoen & Yau $\approx'$ 83: Tori $\mathbb{R}^n/\mathbb{Z}^n$. $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$. Enlargeable Manifolds
- LeBrun ′99: All Kähler-Einstein surfaces with non-positive scalar curvature.
- Bray & Neves ’04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$.
  Inverse mean curvature flow
- Perelman, M. Anderson ’06: compact quotients of 3-dimensional hyperbolic space
  Ricci flow

Example where supremum is not attained

- Schoen: $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$.
  The supremum is not attained.
- Connected sum of several copies of $S^{n-1} \times S^1$.

Attention

$$\sigma(M) = Y(M, [g]) = \mathcal{E}(g) \leq 0 \implies g \text{ Einstein}$$

$$\sigma(M) = Y(M, [g]) = \mathcal{E}(g) \not\Longrightarrow g \text{ Einstein}$$

In the case $\sigma(M) \leq 0$ the minimizers are unique. Hence, if a maximizing conformal class exists, then **the unique** minimizing metric in that class is Einstein.
However, in the case $\sigma(M) > 0$, **a** minimizing metric in a maximizing class may be non-Einstein!

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Some known values of σ

- All examples above.
- Akutagawa & Neves '07: Some non-prime 3-manifolds, e.g.

$$\sigma(\mathbb{R}P^3 \# (S^2 \times S^1)) = \sigma(\mathbb{R}P^3).$$

- Compact quotients of nilpotent Lie groups: $\sigma(M) = 0$.

Unknown cases

- Nontrivial quotients of spheres, except $\mathbb{R}P^3$.
- $S^k \times S^m$, with $k, m \geq 2$.
- No example of dimension ≥ 5 known with $\sigma(M) \neq 0$ and $\sigma(M) \neq \sigma(S^n)$.

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Positive scalar curvature

$\sigma(M) > 0$ is a "bordism invariant".
Bordism classes admitting positive scalar curvature metrics form an ideal in the bordism ring $\Omega_n^{\mathrm{spin}}(B\pi_1)$.
It is the preimage of an ideal of the map

$$D : \Omega_*^{\mathrm{spin}}(B\pi_1) \to \mathrm{ko}_*(B\pi_1)$$

Questions for fixed $\epsilon > 0$ and π_1

1. Is $\sigma(M) > \epsilon$ a "bordism invariant"?

2. Do $\sigma(M) > \epsilon$-classes form a subgroup?

3. .... a subring, an ideal?

4. $\sigma(M) \geq \epsilon > 0 \Longrightarrow \sigma(M \times N) \geq f(\sigma(M), \sigma(N), \dim M, \dim N)$
   $f > 0$
   Unknown, partial results by Akutagawa and Petean
5. Is the above subgroup a preimage of

$$\Omega_n^{\mathrm{spin}}(B\pi_1) \to \mathrm{ko}_n(B\pi_1)?$$

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Main tool for 1. and 2.: Surgery formula for $\sigma(M)$

Surgery

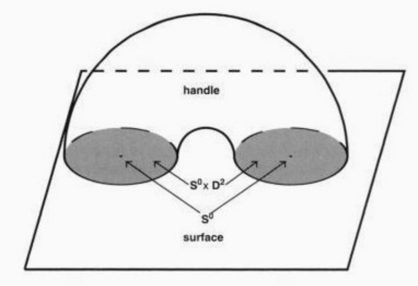
Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M^n$ be an embedding.
We define

$$M_k^\Phi := M \setminus \Phi(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1})/\sim$$

where $/\sim$ means gluing the boundaries via

$$M \ni \Phi(x,y) \sim (x,y) \in S^k \times S^{n-k-1}.$$

We say that M_k^Φ is obtained from M by surgery of dimension k.



Example: 0-dimensional surgery on a surface.

Known surgery formulas

Theorem (Gromov & Lawson '80, Schoen & Yau '79)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M) > 0 \implies \sigma(M_k^{\Phi}) > 0.$$

Theorem (Kobayashi '87)

If $k = 0$, *then*

$$\sigma(M_0^{\Phi}) \geq \sigma(M).$$

Theorem (Petean, Petean & Yun '99)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M_k^{\Phi}) \geq \min\{\sigma(M), 0\}.$$

The proof uses the characterization of $\min\{\sigma(M), 0\}$ as an infimum.

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Main results

Theorem (ADH, # 1)

Let $0 \le k \le n-3$. There is a positive constant $L(n,k)$ depending only on n and k such that

$$\sigma(M_k^\Phi) \ge \min\{\sigma(M), L(n,k)\}$$

if M_k^Φ is obtained from M by k-dimensional surgery. Furthermore $L(n,0) = Y(\mathbb{S}^n)$.

This theorem implies all three previously known surgery formulas.

Thm # 1 follows directly from Thm # 2.

Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M,[g]), L(n,k)\} \le \liminf_{i\to\infty} Y(M_k^\Phi, [g_i]).$$

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Compare different results

Gromov-Lawson-construction

not compatible with conformal structure

Kobayashi-construction

“special case” of our construction
no techniques for dealing with $k > 0$

Petean-construction

Uses the formula

$$\inf_{\tilde{g}\in\mathcal{M}} \|\mathrm{scal}^{\tilde{g}}\|_{L^{n/2}(\tilde{g})} = \begin{cases} |\sigma(M)| & \text{if } \sigma(M) \leq 0 \\ 0 & \text{if } \sigma(M) > 0 \end{cases}$$

Hence $\min\{\sigma(M), 0\}$ is determined by an infimum.

ADH: We had to work with the sup-inf-characterization of σ.

Our knowledge before:
Similar surgery formula holds for a spinorial analogue.
The proof of the spinorial analogue directly lead to a proof in the case $k < (n/2) - 1$. Our proof needed a new “engine” for higher k.

Topological conclusions

$$L(n) := \min\{L(n,1), L(n,2), \ldots, L(n,n-3)\} > 0,$$

$$\bar{\sigma}(M) := \min\{\sigma(M), L(n)\}.$$

Let M_k^Φ be obtained from M by surgery of dimension $k \in \{2,3,\ldots,n-3\}$, then $\bar{\sigma}(M) = \bar{\sigma}(M_k^\Phi)$.

Goal: Find a bordism invariant!
Problems with $k \in \{0,1,n-2,n-1\}$.
Consider $[M,f]$ where $f : M \to B\pi_1 = B\pi_1(M)$ is the classifying map.
Bordisms with maps to $B\pi_1$.

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Bordisms with maps to $B\pi_1$.

Let $\Omega_n^{\text{spin}}(B\pi_1)$ the spin cobordism group over $B\pi_1$.

Lemma

$n \geq 5$. Let (M_1, f_1) and (M_2, f_2) be spin cobordant over $B\pi_1$ and let $f_2 : M_2 \to B\pi_1(M_2)$ be classifying. Then

$$\bar{\sigma}(M_2) \geq \bar{\sigma}(M_1).$$

$\bar{\sigma} := \min\{\sigma(M), L(n)\}$
$s_{\pi_1}([M, f]) := \max\{\bar{\sigma}(M), 0\}$ if f is classifying.
$s_{\pi_1} : \Omega_n^{\text{spin}}(B\pi_1) \to (-\infty, L(n)]$

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Corollary

Let $n \geq 5$ and $\epsilon \in (0, L(n))$. The groups $\{x \in \Omega_n^{\text{spin}}(B\Gamma) \mid s_\Gamma(x) > \epsilon\}$ and $\{x \in \Omega_n^{\text{spin}}(B\Gamma) \mid s_\Gamma(x) \geq \epsilon\}$ are subgroups.

Corollary

Let $n \geq 5$ and M^n simply connected. Then $\sigma(M) \in \{0\} \cup [\epsilon_n, Y(\mathbb{S}^n)]$.

We even get that $\{\sigma(M) \mid : \dim M = n, \quad \pi_1(M) = 1\} \cap (0, L(n))$ has no accumulation points from above.

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Application

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.
Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\# \cdots \# M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\# \cdots \# M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$
for $\#\Gamma$ large.

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We construct a sequence explicitly and study the behavior of the minimizers in $[g_i]$ for $i \to \infty$.

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Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i]).$$

We construct a sequence explicitly and study the behavior of the minimizers in $[g_i]$ for $i \to \infty$.

Construction of the metrics

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We write close to $S := \Phi(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{round}^{n-k-1}$$

where g_{round}^{n-k-1} is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{round}^{n-k-1}$$

We define a metric

$$g_i = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{round}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

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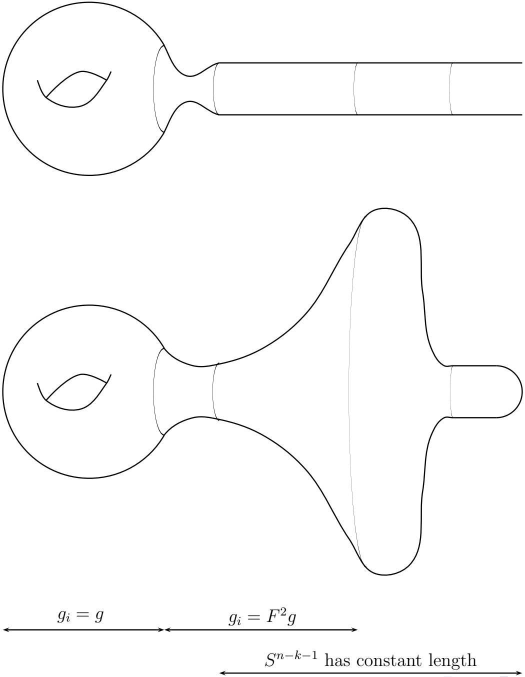

$g_i = g$
$g_i = F^2 g$
S^{n-k-1} has constant length

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.

We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Then study pointed Gromov-Hausdorff limits.
  Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

$\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$

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Major problem

Note that $\mathrm{vol}(M_k^\varphi, g_i) \to \infty$.
And $\mathrm{scal} < 0$ in large regions for $k > (n/2) - 1$.
A priori it might happen that

- $\|u_i\|_{L^p} = 1$
- $\|u_i\|_{L^\infty} \to 0$

Thus all weak limits and blow-up limits would vanish.
Define

$$w(t_0) := \sqrt{\int_{t=t_0} u_i^2 \mathrm{dvol}^{n-1}}$$

$$w''(t) \geq cw(t).$$

Subtle estimates on warped products.
Help from Arthur Besse's book on Einstein manifolds!

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The numbers $L(n,k)$

(Disclaimer: Additional conditions for $k+3=n\geq 6$)

$$L(n,k) := \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1}\times \mathbb{S}^{n-k-1})$$

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Note: $\mathbb{H}_1^{k+1}\times \mathbb{S}^{n-k-1}\cong \mathbb{S}^n\setminus \mathbb{S}^k$.
$k=0$: $L(n,0)=Y(\mathbb{R}\times\mathbb{S}^{n-1})=Y(\mathbb{S}^n)$
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Conjecture #1: $L(n,k)=Y(\mathbb{R}^{k+1}\times\mathbb{S}^{n-k-1})$

Conjecture #2: The infimum in the definition of $Y(\mathbb{H}_c^{k+1}\times \mathbb{S}^{n-k-1})$ is attained by an $O(k+1)\times O(n-k)$ invariant function if $-1<c<1$.

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Simplest non-trivial case $n = 4$, $k = 1$.
If Conjecture #2 holds, then Conjecture #1 will probably follow.

$$L(4,1) = 59,4...$$

$$Y(\mathbb{S}^4) = 61,5...$$

$$Y(\mathbb{S}^2 \times \mathbb{S}^2) = 50,2...$$

One obtains $S^2 \times S^2$ via 1-dimensional surgery from S^4.
Hence

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Theorem (Akutagawa, Große)

If the scalar curvature of $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ *is positive, i.e. if*

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then the infimum in the definition of $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ *is attained by a smooth positive* L^p*-function.*

Other progress

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