A surgery formula for the (smooth) Yamabe invariant

B. Ammann[1] M. Dahl[2] E. Humbert[3]

[1]Universität Regensburg
Germany

[2]Kungliga Tekniska Högskolan, Stockholm
Sweden

[3]Université Henri Poincaré, Nancy
France

Conference on scalar curvature, Münster 2010

(Smooth) Yamabe invariant

Let M be a compact n-dimensional manifold, $n \geq 3$.
Einstein-Hilbert functional

$$\mathcal{E} : \mathcal{M}_1 \to \mathbb{R}, \qquad g \mapsto \int_M \mathrm{scal}^g \, dv^g$$

$\mathcal{M}_1 := \{$metrics on M of volume 1$\}$.
$[g] := \{u^{4/(n-2)} g \mid \mathrm{vol}(u^{4/(n-2)} g) = 1\}$.
Stationary points of $\mathcal{E} : [g] \to \mathbb{R}$ = metrics with constant scalar curvature
Stationary points of $\mathcal{E} : \mathcal{M}_1 \to \mathbb{R}$ = Einstein metrics

Conformal and smooth invariants

Inside a conformal class

$$Y(M,[g]) := \inf_{\tilde{g}\in[g]} \mathcal{E}(\tilde{g}) > -\infty.$$

The conformal Yamabe constant.

$$Y(M,[g]) \leq Y(\mathbb{S}^n)$$

where $\mathbb{S}^n$ is the sphere with the standard structure.

Solution of the Yamabe problem (Trudinger, Aubin, Schoen-Yau)
$\mathcal{E} : [g] \to \mathbb{R}$ attains its infimum.

Remark $Y(M,[g]) > 0$ if and only if $[g]$ contains a metric of positive scalar curvature.

Conformal and smooth invariants

Inside a conformal class

$$Y(M,[g]) := \inf_{\tilde{g}\in[g]} \mathcal{E}(\tilde{g}) > -\infty.$$

The conformal Yamabe constant.

$$Y(M,[g]) \leq Y(\mathbb{S}^n)$$

where $\mathbb{S}^n$ is the sphere with the standard structure.

Solution of the Yamabe problem (Trudinger, Aubin, Schoen-Yau)
$\mathcal{E} : [g] \to \mathbb{R}$ attains its infimum.

Remark $Y(M,[g]) > 0$ if and only if $[g]$ contains a metric of positive scalar curvature.

Conformal and smooth invariants

Inside a conformal class

$$Y(M,[g]) := \inf_{\tilde{g}\in[g]} \mathcal{E}(\tilde{g}) > -\infty.$$

The conformal Yamabe constant.

$$Y(M,[g]) \leq Y(\mathbb{S}^n)$$

where $\mathbb{S}^n$ is the sphere with the standard structure.

Solution of the Yamabe problem (Trudinger, Aubin, Schoen-Yau) $\mathcal{E} : [g] \to \mathbb{R}$ attains its infimum.

Remark $Y(M,[g]) > 0$ if and only if $[g]$ contains a metric of positive scalar curvature.

On the set of conformal classes

$$\sigma(M) := \sup_{[g]\subset\mathcal{M}_1} Y(M,[g]) \in (-\infty, Y(\mathbb{S}^n)]$$

The smooth Yamabe invariant, also called Schoen's σ-constant.

Remark $\sigma(M) > 0$ if and only if M caries a metric of positive scalar curvature.

Supremum attained?

Depends on M.

On the set of conformal classes

$$\sigma(M) := \sup_{[g]\subset\mathcal{M}_1} Y(M,[g]) \in (-\infty, Y(\mathbb{S}^n)]$$

The smooth Yamabe invariant, also called Schoen's σ-constant.

Remark $\sigma(M) > 0$ if and only if M caries a metric of positive scalar curvature.

Supremum attained?

Depends on M.

On the set of conformal classes

$$\sigma(M) := \sup_{[g] \subset \mathcal{M}_1} Y(M,[g]) \in (-\infty, Y(\mathbb{S}^n)]$$

The smooth Yamabe invariant, also called Schoen's σ-constant.

Remark $\sigma(M) > 0$ if and only if M caries a metric of positive scalar curvature.

Supremum attained?

Depends on M.

Example $\mathbb{C}P^2$

The Fubini-Study g_{FS} metric is Einstein and

$$53,31... = \mathcal{E}(g_{\mathrm{FS}}) = Y(\mathbb{C}P^2, [g_{\mathrm{FS}}]) = \sigma(\mathbb{C}P^2).$$

Supremum attained in the Fubini-Study metric.
Claude LeBrun '97

Similar examples

- $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- Gromov & Lawson, Schoen & Yau ≈' 83: Tori $\mathbb{R}^n/\mathbb{Z}^n$. $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$.
- LeBrun '99: All Kähler-Einstein surfaces with non-positive scalar curvature.
- Bray & Neves '04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$.
- Perelman, M. Anderson '06: compact quotients of 3-dimensional hyperbolic space

Example $\mathbb{C}P^2$

The Fubini-Study g_{FS} metric is Einstein and

$$53,31... = \mathcal{E}(g_{\mathrm{FS}}) = Y(\mathbb{C}P^2, [g_{\mathrm{FS}}]) = \sigma(\mathbb{C}P^2).$$

Supremum attained in the Fubini-Study metric.
Claude LeBrun '97

Similar examples

- $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- Gromov & Lawson, Schoen & Yau ≈' 83: Tori $\mathbb{R}^n/\mathbb{Z}^n$. $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$.
- LeBrun '99: All Kähler-Einstein surfaces with non-positive scalar curvature.
- Bray & Neves '04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$.
- Perelman, M. Anderson '06: compact quotients of 3-dimensional hyperbolic space

Example $\mathbb{C}P^2$

The Fubini-Study g_{FS} metric is Einstein and

$$53,31\ldots = \mathcal{E}(g_{\mathrm{FS}}) = Y(\mathbb{C}P^2, [g_{\mathrm{FS}}]) = \sigma(\mathbb{C}P^2).$$

Supremum attained in the Fubini-Study metric.
Claude LeBrun '97

Similar examples

- $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- Gromov & Lawson, Schoen & Yau $\approx'$ 83: Tori $\mathbb{R}^n/\mathbb{Z}^n$. $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$.
- LeBrun '99: All Kähler-Einstein surfaces with non-positive scalar curvature.
- Bray & Neves '04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$.
- Perelman, M. Anderson '06: compact quotients of 3-dimensional hyperbolic space

Example $\mathbb{C}P^2$

The Fubini-Study g_{FS} metric is Einstein and

$$53,31... = \mathcal{E}(g_{\mathrm{FS}}) = Y(\mathbb{C}P^2, [g_{\mathrm{FS}}]) = \sigma(\mathbb{C}P^2).$$

Supremum attained in the Fubini-Study metric.
Claude LeBrun '97

Similar examples

- $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- Gromov & Lawson, Schoen & Yau $\approx' 83$: Tori $\mathbb{R}^n/\mathbb{Z}^n$.
  $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$.
- LeBrun ′99: All Kähler-Einstein surfaces with non-positive scalar curvature.
- Bray & Neves '04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$.
- Perelman, M. Anderson '06: compact quotients of 3-dimensional hyperbolic space

Example where supremum is not attained

Schoen: $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$. The supremum is not attained.

Connected sum of several $S^{n-1} \times S^1$.

Attention

In the case $\sigma(M) \leq 0$ the minimizers are unique. Hence, if a maximizing conformal class exists, then **the unique** minimizing metric in that class is Einstein.

However, in the case $\sigma(M) > 0$, **a** minimizing metric in a maximizing class may be non-Einstein!

Example where supremum is not attained

Schoen: $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$. The supremum is not attained.

Connected sum of several $S^{n-1} \times S^1$.

Attention

In the case $\sigma(M) \leq 0$ the minimizers are unique. Hence, if a maximizing conformal class exists, then **the unique** minimizing metric in that class is Einstein.
However, in the case $\sigma(M) > 0$, **a** minimizing metric in a maximizing class may be non-Einstein!

Some known values of σ

- All examples above.
- Akutagawa & Neves '07: Some non-prime 3-manifolds, e.g.

$$\sigma(\mathbb{R}P^3 \# (S^2 \times S^1)) = \sigma(\mathbb{R}P^3).$$

- Compact quotients of nilpotent Lie groups: $\sigma(M) = 0$.

Unknown cases

- Nontrivial quotients of spheres, except $\mathbb{R}P^3$.
- $S^k \times S^m$, with $k, m \geq 2$.
- No example of dimension ≥ 5 known with $\sigma(M) \neq 0$ and $\sigma(M) \neq \sigma(S^n)$.

Some known values of σ

- All examples above.
- Akutagawa & Neves '07: Some non-prime 3-manifolds, e.g.

$$\sigma(\mathbb{R}P^3\#(S^2 \times S^1)) = \sigma(\mathbb{R}P^3).$$

- Compact quotients of nilpotent Lie groups: $\sigma(M) = 0$.

Unknown cases

- Nontrivial quotients of spheres, except $\mathbb{R}P^3$.
- $S^k \times S^m$, with $k, m \geq 2$.
- No example of dimension ≥ 5 known with $\sigma(M) \neq 0$ and $\sigma(M) \neq \sigma(S^n)$.

Relations to similar invariants...

... to the spectrum of the conformal Laplacian

$$\inf_{\tilde g \in [g]} \lambda_1 \left(4\frac{n-1}{n-2}\Delta^{\tilde g} + \mathrm{scal}^{\tilde g} \right) = \begin{cases} Y(M,[g]) & \text{if } Y(M,[g]) \geq 0 \\ -\infty & \text{if } Y(M,[g]) < 0 \end{cases}$$

... to the Perelman-invariant (Akutagawa, Ishida, LeBrun '06)

$$\begin{aligned} \bar\lambda(M) &:= \sup_{g\in\mathcal{M}_1} \lambda_1(4\Delta^g + \mathrm{scal}^g) \\ &= \begin{cases} \sigma(M) & \text{if } \sigma(M) \leq 0 \\ +\infty & \text{if } \sigma(M) > 0 \end{cases} \end{aligned}$$

... to the $L^{n/2}$-norm of scal

$$\inf_{\tilde g \in \mathcal{M}_1} \|\mathrm{scal}^{\tilde g}\|_{L^{n/2}(\tilde g)} = \begin{cases} |\sigma(M)| & \text{if } \sigma(M) \leq 0 \\ 0 & \text{if } \sigma(M) > 0 \end{cases}$$

$\Longrightarrow$ Hence $\min\{\sigma(M), 0\}$ is determined by an infimum.

Positive scalar curvature

$\sigma(M) > 0$ is a "bordism invariant".
Bordism classes admitting positive scalar curvature metrics form an ideal in the bordism ring.

Questions for given $\epsilon > 0$

1. Is $\sigma(M) > \epsilon$ a "bordism invariant"?
2. Do $\sigma(M) > \epsilon$-classes form a subgroup?
3. .... a subring, an ideal?

Positive scalar curvature

$\sigma(M) > 0$ is a "bordism invariant".
Bordism classes admitting positive scalar curvature metrics form an ideal in the bordism ring.

Questions for given $\epsilon > 0$

1. Is $\sigma(M) > \epsilon$ a "bordism invariant"?
2. Do $\sigma(M) > \epsilon$-classes form a subgroup?
3. .... a subring, an ideal?

Positive scalar curvature

$\sigma(M) > 0$ is a “bordism invariant”.
Bordism classes admitting positive scalar curvature metrics form an ideal in the bordism ring.

Questions for given $\epsilon > 0$

1. Is $\sigma(M) > \epsilon$ a “bordism invariant”?
   Yes for small $\epsilon > 0$, ADH
2. Do $\sigma(M) > \epsilon$-classes form a subgroup?
   Yes for small $\epsilon > 0$, ADH
3. .... a subring, an ideal?

Positive scalar curvature

$\sigma(M) > 0$ is a “bordism invariant”.
Bordism classes admitting positive scalar curvature metrics form an ideal in the bordism ring.

Questions for given $\epsilon > 0$

1. Is $\sigma(M) > \epsilon$ a “bordism invariant”?
   Yes for small $\epsilon > 0$, ADH
2. Do $\sigma(M) > \epsilon$-classes form a subgroup?
   Yes for small $\epsilon > 0$, ADH
3. .... a subring, an ideal?
   Unknown

Positive scalar curvature

$\sigma(M) > 0$ is a “bordism invariant”.
Bordism classes admitting positive scalar curvature metrics form an ideal in the bordism ring.

Questions for given $\epsilon > 0$

1. Is $\sigma(M) > \epsilon$ a “bordism invariant”?
   Yes for small $\epsilon > 0$, ADH
2. Do $\sigma(M) > \epsilon$-classes form a subgroup?
   Yes for small $\epsilon > 0$, ADH
3. .... a subring, an ideal?
   Unknown
4. $\sigma(M) \geq \epsilon > 0 \Longrightarrow \sigma(M \times N) \geq f(\sigma(M), \sigma(N), \dim M, \dim N)$
   $f > 0$
   Unknown, partial results by Akutagawa and Petean

Positive scalar curvature

$\sigma(M) > 0$ is a "bordism invariant".
Bordism classes admitting positive scalar curvature metrics form an ideal in the bordism ring.

Questions for given $\epsilon > 0$

1. Is $\sigma(M) > \epsilon$ a "bordism invariant"?
   Yes for small $\epsilon > 0$, ADH
2. Do $\sigma(M) > \epsilon$-classes form a subgroup?
   Yes for small $\epsilon > 0$, ADH
3. .... a subring, an ideal?
   Unknown
4. $\sigma(M) \geq \epsilon > 0 \Longrightarrow \sigma(M \times N) \geq f(\sigma(M), \sigma(N), \dim M, \dim N)$
   $f > 0$
   Unknown, partial results by Akutagawa and Petean

Main tool for 1. and 2.: Surgery formula for $\sigma(M)$

Surgery

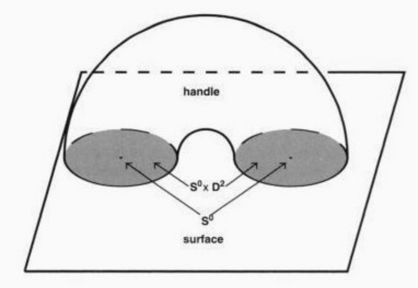
Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M^n$ be an embedding.
We define

$$M_k^\Phi := M \setminus \Phi(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1})/\sim$$

where $/\sim$ means gluing the boundaries via

$$M \ni \Phi(x,y) \sim (x,y) \in S^k \times S^{n-k-1}.$$

We say that M_k^Φ is obtained from M by surgery of dimension k.



Example: 0-dimensional surgery on a surface.

Known surgery formulas

Theorem (Gromov & Lawson '80, Schoen & Yau '79)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M) > 0 \quad \Longrightarrow \quad \sigma(M_k^{\Phi}) > 0.$$

Theorem (Kobayashi '87)

If $k = 0$, *then*

$$\sigma(M_0^{\Phi}) \geq \sigma(M).$$

Theorem (Petean, Petean & Yun '99)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M_k^{\Phi}) \geq \min\{\sigma(M), 0\}.$$

The proof uses the characterization of $\min\{\sigma(M), 0\}$ as an infimum.

Known surgery formulas

Theorem (Gromov & Lawson '80, Schoen & Yau '79)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M) > 0 \quad \Longrightarrow \quad \sigma(M_k^\Phi) > 0.$$

Theorem (Kobayashi '87)

If $k = 0$, *then*

$$\sigma(M_0^\Phi) \geq \sigma(M).$$

Theorem (Petean, Petean & Yun '99)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), 0\}.$$

The proof uses the characterization of $\min\{\sigma(M), 0\}$ as an infimum.

Known surgery formulas

Theorem (Gromov & Lawson '80, Schoen & Yau '79)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M) > 0 \quad \Longrightarrow \quad \sigma(M_k^\Phi) > 0.$$

Theorem (Kobayashi '87)

If $k = 0$, *then*

$$\sigma(M_0^\Phi) \geq \sigma(M).$$

Theorem (Petean, Petean & Yun '99)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), 0\}.$$

The proof uses the characterization of $\min\{\sigma(M), 0\}$ as an infimum.

Main results

Theorem (ADH, # 1)

Let $0 \leq k \leq n-3$. There is a positive constant $L(n,k)$ depending only on n and k such that

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), L(n,k)\}$$

if M_k^Φ is obtained from M by k-dimensional surgery. Furthermore $L(n,0) = Y(\mathbb{S}^n)$.

This theorem implies all three previously known surgery formulas.

Thm # 1 follows directly from Thm # 2.

Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g]).$$

Main results

Theorem (ADH, # 1)

Let $0 \leq k \leq n-3$*. There is a positive constant* $L(n,k)$ *depending only on n and k such that*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), L(n,k)\}$$

if M_k^Φ *is obtained from* M *by* k*-dimensional surgery. Furthermore* $L(n,0) = Y(\mathbb{S}^n)$*.*

This theorem implies all three previously known surgery formulas.

Thm # 1 follows directly from Thm # 2.

Theorem (ADH, #2)

Suppose that M_k^Φ *is obtained from* M *by* k*-dimensional surgery. Then for any metric* g *on* M *there is a sequence of metrics* g_i *on* M_k^Φ *such that*

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g]).$$

Main results

Theorem (ADH, # 1)

Let $0 \leq k \leq n-3$. There is a positive constant $L(n,k)$ depending only on n and k such that

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), L(n,k)\}$$

if M_k^Φ is obtained from M by k-dimensional surgery. Furthermore $L(n,0) = Y(\mathbb{S}^n)$.

This theorem implies all three previously known surgery formulas.

Thm # 1 follows directly from Thm # 2.

Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi, [g_i]) \leq Y(M,[g]).$$

Topological conclusions

$$L(n) := \min\{L(n,1), L(n,2), \ldots, L(n,n-3)\} > 0,$$

$$\bar{\sigma}(M) := \min\{\sigma(M), L(n)\}.$$

Let M_k^Φ be obtained from M by surgery of dimension $k \in \{2, 3, \ldots, n-3\}$, then $\bar{\sigma}(M) = \bar{\sigma}(M_k^\Phi)$.

Goal: Find a bordism invariant!

Topological conclusions

$$L(n) := \min\{L(n,1), L(n,2), \ldots, L(n,n-3)\} > 0,$$

$$\bar{\sigma}(M) := \min\{\sigma(M), L(n)\}.$$

Let M_k^Φ be obtained from M by surgery of dimension $k \in \{2, 3, \ldots, n-3\}$, then $\bar{\sigma}(M) = \bar{\sigma}(M_k^\Phi)$.

Goal: Find a bordism invariant!

Let Γ be a finitely presented group.
Let $\Omega_n^{\mathrm{spin}}(B\Gamma)$ the spin cobordism group over $B\Gamma$.
Any class in $\Omega_n^{\mathrm{spin}}(B\Gamma)$ has a π_1-bijective representative, i.e. it is represented by (M, f), where M is a connected compact spin manifold, and where $f : M \to B\Gamma$ induces an isomorphism $\pi_1(M) \to \pi_1(B\Gamma) = \Gamma$.

Lemma

$n \geq 5$. Let (M_1, f_1) and (M_2, f_2) be spin cobordant over $B\Gamma$ and let (M_2, f_2) be π_1-bijective. Then

$$\bar{\sigma}(M_2) \geq \bar{\sigma}(M_1).$$

We define

$$s_\Gamma([M, f]) := \max \bar{\sigma}(N)$$

where the maximum runs over all (N, h) with

$$[M, f] = [N, h] \in \Omega_n^{\mathrm{spin}}(B\Gamma).$$

Let Γ be a finitely presented group.
Let $\Omega_n^{\mathrm{spin}}(B\Gamma)$ the spin cobordism group over $B\Gamma$.
Any class in $\Omega_n^{\mathrm{spin}}(B\Gamma)$ has a π_1-bijective representative, i.e. it is represented by (M, f), where M is a connected compact spin manifold, and where $f : M \to B\Gamma$ induces an isomorphism $\pi_1(M) \to \pi_1(B\Gamma) = \Gamma$.

Lemma

$n \geq 5$. Let (M_1, f_1) and (M_2, f_2) be spin cobordant over $B\Gamma$ and let (M_2, f_2) be π_1-bijective. Then

$$\bar{\sigma}(M_2) \geq \bar{\sigma}(M_1).$$

We define

$$s_\Gamma([M, f]) := \max \bar{\sigma}(N)$$

where the maximum runs over all (N, h) with

$$[M, f] = [N, h] \in \Omega_n^{\mathrm{spin}}(B\Gamma).$$

$\bar\sigma := \min\{\sigma(M), L(n)\}$
$s_\Gamma([M,f]) = \bar\sigma(M)$ if (M,f) is π_1-bijective
$s_\Gamma : \Omega_n^{\mathrm{spin}}(B\Gamma) \to (-\infty, L(n)]$

Corollary

Let $n \geq 5$ and $\epsilon \in (0, L(n))$. The groups $\{x \in \Omega_n^{\mathrm{spin}}(B\Gamma) \mid s_\Gamma(x) > \epsilon\}$ and $\{x \in \Omega_n^{\mathrm{spin}}(B\Gamma) \mid s_\Gamma(x) \geq \epsilon\}$ are subgroups.

Corollary

Let $n \geq 5$ and M^n simply connected. Then $\sigma(M) \in \{0\} \cup [\epsilon_n, Y(\mathbb{S}^n)]$.

We even get that $\mathrm{image}(\sigma : \Omega_n^{\mathrm{spin}} \to \mathbb{R}) \cap (0, L(n))$ has no accumulation points from above.

$\bar{\sigma} := \min\{\sigma(M), L(n)\}$
$s_\Gamma([M, f]) = \bar{\sigma}(M)$ if (M, f) is π_1-bijective
$s_\Gamma : \Omega_n^{\mathrm{spin}}(B\Gamma) \to (-\infty, L(n)]$

Corollary

Let $n \geq 5$ and $\epsilon \in (0, L(n))$. The groups $\{x \in \Omega_n^{\mathrm{spin}}(B\Gamma) \,|\, s_\Gamma(x) > \epsilon\}$ and $\{x \in \Omega_n^{\mathrm{spin}}(B\Gamma) \,|\, s_\Gamma(x) \geq \epsilon\}$ are subgroups.

Corollary

Let $n \geq 5$ and M^n simply connected. Then $\sigma(M) \in \{0\} \cup [\epsilon_n, Y(\mathbb{S}^n)]$.

We even get that $\mathrm{image}(\sigma : \Omega_n^{\mathrm{spin}} \to \mathbb{R}) \cap (0, L(n))$ has no accumulation points from above.

Application

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.
Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\# \cdots \# M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\# \cdots \# M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$
for $\#\Gamma$ large.

Application

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.
Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\# \cdots \# M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\# \cdots \# M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$
for $\#\Gamma$ large.

Application

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.
Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\#\cdots\#M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\#\cdots\#M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$
for $\#\Gamma$ large.

Application

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.
Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\# \cdots \# M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\# \cdots \# M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$
for $\#\Gamma$ large.

Proof of Theorem # 2

Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g]).$$

Easy part:

$$\liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g])$$

Difficult and important part

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i])$$

For this, we construct a sequence explicitly and study the behavior of the minimizers in $[g_i]$ for $i \to \infty$.

Proof of Theorem # 2

Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M,[g]),L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g]).$$

Easy part:

$$\liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g])$$

Difficult and important part

$$\min\{Y(M,[g]),L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i])$$

For this, we construct a sequence explicitly and study the behavior of the minimizers in $[g_i]$ for $i \to \infty$.

Proof of Theorem # 2

Theorem (ADH, #2)

Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a sequence of metrics g_i on M_k^Φ such that

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi, [g_i]) \leq Y(M,[g]).$$

Easy part:

$$\liminf_{i\to\infty} Y(M_k^\Phi, [g_i]) \leq Y(M,[g])$$

Difficult and important part

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi, [g_i])$$

For this, we construct a sequence explicitly and study the behavior of the minimizers in $[g_i]$ for $i \to \infty$.

Construction of the metrics

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We write close to $S := \Phi(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{round}^{n-k-1}$$

where g_{round}^{n-k-1} is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{round}^{n-k-1}$$

We define a metric

$$g_i = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{round}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

that extends to a metric on M_k^Φ.

Construction of the metrics

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We write close to $S := \Phi(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{round}^{n-k-1}$$

where g_{round}^{n-k-1} is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{round}^{n-k-1}$$

We define a metric

$$g_i = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{round}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

that extends to a metric on M_k^Φ.

Construction of the metrics

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We write close to $S := \Phi(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{round}^{n-k-1}$$

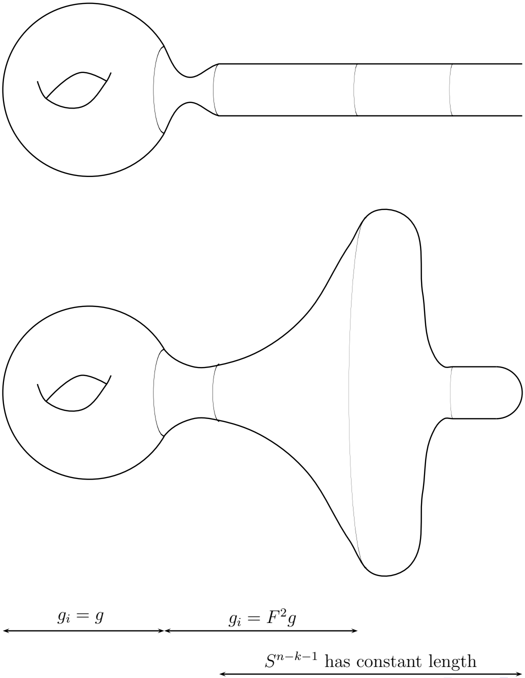
where g_{round}^{n-k-1} is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{round}^{n-k-1}$$

We define a metric

$$g_i = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{round}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

that extends to a metric on M_k^Φ.


$g_i = g$
$g_i = F^2 g$
S^{n-k-1} has constant length

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.

We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{\,g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Then study pointed Gromov-Hausdorff limits.
  Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

$\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.
We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Then study pointed Gromov-Hausdorff limits.
  Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

$\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.
We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{\,g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Then study pointed Gromov-Hausdorff limits.
  Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

$\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$. We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{\,g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Then study pointed Gromov-Hausdorff limits.
  Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

$\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.
We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{\,g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Then study pointed Gromov-Hausdorff limits.
  Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

$\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$

The numbers $L(n, k)$

Additional conditions for $k + 3 = n \geq 6$

$$L(n, k) := \inf_{c \in [0,1]} Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$$

$$Y(N) := \inf_{u \in C_c^\infty(N)} \frac{\int_N 4\frac{n-1}{n-2}|du|^2 + \mathrm{scal}\, u^2}{(\int_N u^p)^{2/p}}$$

Note: $\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1} \cong \mathbb{S}^n \setminus \mathbb{S}^k$.

$k = 0$: $L(n, 0) = Y(\mathbb{R} \times \mathbb{S}^{n-1}) = Y(\mathbb{S}^n)$

$k = 1, \ldots, n-3$: $L(n, k) > 0$

Conjecture #1: $L(n, k) = Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$

Conjecture #2: The infimum in the definition of $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ is attained by an $O(k+1) \times O(n-k)$ invariant function if $-1 < c < 1$.

$O(n-k)$-invariance is difficult,

$O(k+1)$-invariance follows from standard reflection methods

The numbers $L(n,k)$

Additional conditions for $k+3=n\geq 6$

$$L(n,k) := \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$$

$$Y(N) := \inf_{u\in C_c^\infty(N)} \frac{\int_N 4\frac{n-1}{n-2}|du|^2 + \mathrm{scal}\, u^2}{(\int_N u^p)^{2/p}}$$

Note: $\mathbb{H}_1^{k+1}\times\mathbb{S}^{n-k-1}\cong\mathbb{S}^n\setminus\mathbb{S}^k$.

$k=0$: $L(n,0)=Y(\mathbb{R}\times\mathbb{S}^{n-1})=Y(\mathbb{S}^n)$

$k=1,\dots,n-3$: $L(n,k)>0$

Conjecture #1: $L(n,k)=Y(\mathbb{R}^{k+1}\times\mathbb{S}^{n-k-1})$

Conjecture #2: The infimum in the definition of $Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$ is attained by an $O(k+1)\times O(n-k)$ invariant function if $-1<c<1$.

$O(n-k)$-invariance is difficult,
$O(k+1)$-invariance follows from standard reflection methods

The numbers $L(n,k)$

Additional conditions for $k+3=n\geq 6$

$$L(n,k) := \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$$

$$Y(N) := \inf_{u\in C_c^\infty(N)} \frac{\int_N 4\frac{n-1}{n-2}|du|^2+\mathrm{scal}\, u^2}{(\int_N u^p)^{2/p}}$$

Note: $\mathbb{H}_1^{k+1}\times\mathbb{S}^{n-k-1}\cong\mathbb{S}^n\setminus\mathbb{S}^k$.
$k=0$: $L(n,0)=Y(\mathbb{R}\times\mathbb{S}^{n-1})=Y(\mathbb{S}^n)$
$k=1,\ldots,n-3$: $L(n,k)>0$

Conjecture #1: $L(n,k)=Y(\mathbb{R}^{k+1}\times\mathbb{S}^{n-k-1})$

Conjecture #2: The infimum in the definition of $Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$ is attained by an $O(k+1)\times O(n-k)$ invariant function if $-1<c<1$.

$O(n-k)$-invariance is difficult,
$O(k+1)$-invariance follows from standard reflection methods

The numbers $L(n,k)$

Additional conditions for $k+3=n\geq 6$

$$L(n,k) := \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$$

$$Y(N) := \inf_{u\in C_c^\infty(N)} \frac{\int_N 4\frac{n-1}{n-2}|du|^2 + \mathrm{scal}\, u^2}{(\int_N u^p)^{2/p}}$$

Note: $\mathbb{H}_1^{k+1}\times\mathbb{S}^{n-k-1}\cong\mathbb{S}^n\setminus\mathbb{S}^k$.
$k=0$: $L(n,0)=Y(\mathbb{R}\times\mathbb{S}^{n-1})=Y(\mathbb{S}^n)$
$k=1,\ldots,n-3$: $L(n,k)>0$

Conjecture #1: $L(n,k)=Y(\mathbb{R}^{k+1}\times\mathbb{S}^{n-k-1})$

Conjecture #2: The infimum in the definition of $Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$ is attained by an $O(k+1)\times O(n-k)$ invariant function if $-1<c<1$.

$O(n-k)$-invariance is difficult,
$O(k+1)$-invariance follows from standard reflection methods

The numbers $L(n,k)$

Additional conditions for $k+3=n\geq 6$

$$L(n,k) := \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$$

$$Y(N) := \inf_{u\in C_c^\infty(N)} \frac{\int_N 4\frac{n-1}{n-2}|du|^2+\mathrm{scal}\, u^2}{(\int_N u^p)^{2/p}}$$

Note: $\mathbb{H}_1^{k+1}\times\mathbb{S}^{n-k-1}\cong\mathbb{S}^n\setminus\mathbb{S}^k$.
$k=0$: $L(n,0)=Y(\mathbb{R}\times\mathbb{S}^{n-1})=Y(\mathbb{S}^n)$
$k=1,\ldots,n-3$: $L(n,k)>0$

Conjecture #1: $L(n,k)=Y(\mathbb{R}^{k+1}\times\mathbb{S}^{n-k-1})$

Conjecture #2: The infimum in the definition of $Y(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})$ is attained by an $O(k+1)\times O(n-k)$ invariant function if $-1<c<1$.

$O(n-k)$-invariance is difficult,
$O(k+1)$-invariance follows from standard reflection methods

Simplest non-trivial case $n = 4$, $k = 1$.
If Conjecture #2 holds, then Conjecture # 1 as well.

$$L(4,1) = 59,4...$$

$$Y(\mathbb{S}^4) = 61,5...$$

$$Y(\mathbb{S}^2 \times \mathbb{S}^2) = 50,2...$$

One obtains $S^2 \times S^2$ via 1-dimensional surgery from S^4.
Hence

$$\sigma(S^2 \times S^2) \geq 59,4 > 50,2... = Y(\mathbb{S}^2 \times \mathbb{S}^2).$$

Simplest non-trivial case $n = 4$, $k = 1$.
If Conjecture #2 holds, then Conjecture # 1 as well.

$$L(4,1) = 59,4...$$

$$Y(\mathbb{S}^4) = 61,5...$$

$$Y(\mathbb{S}^2 \times \mathbb{S}^2) = 50,2...$$

One obtains $S^2 \times S^2$ via 1-dimensional surgery from S^4.
Hence

$$\sigma(S^2 \times S^2) \geq 59,4 > 50,2... = Y(\mathbb{S}^2 \times \mathbb{S}^2).$$

First ingredient to solve the conjectures:

Theorem (Akutagawa, Große)

If the scalar curvature of $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ *is positive, i.e. if*

$$c^2 k(k+1) < (n-k-1)(n-k-2),$$

then the infimum in the definition of $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ *is attained by a smooth positive* L^p*-function.*