A surgery formula for the smooth Yamabe invariant

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Einstein-Hilbert functional

History

Surgery

Main results

Topological conclusions

Another Application

Comments on the proofs

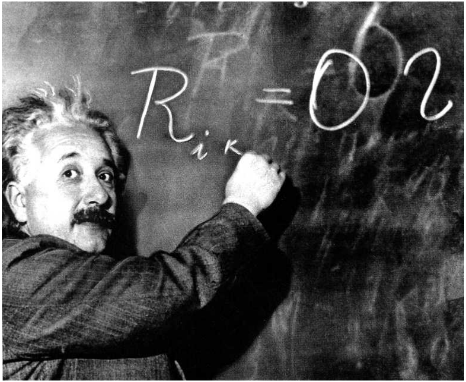
The numbers $L(n, k)$

Related result for the Dirac operator

Einstein's equation

Einstein's General Relativity.

What is the equation that describes evolution of spacetime (without matter)?



Answer: Einstein metrics

$$\mathrm{Ric} = \kappa g,$$

where κ is constant.

Lagrangian formulation

What is the associated Lagrange functional?

David Hilbert

Let $\mathcal{M}$ be the set of all semi-Riemannian metrics on M^n. The Einstein equations are stationary points of the Einstein-Hilbert-functional

$$\begin{aligned} \mathcal{E} : \mathcal{M} &\to \mathbb{R}, \\ g &\mapsto \frac{\int_M \mathrm{scal}^g \, dv^g}{\left(\int_M dv^g\right)^{(n-2)/n}} \end{aligned}$$

among compactly supported perturbations.

The Setting

Now: Let M be n-dimensional, $n \geq 3$.
Search for Einstein metrics on M.
These are the stationary points of

$$\mathcal{E} : \mathcal{M}_1 \to \mathbb{R}, \qquad g \mapsto \int_M \mathrm{scal}^g \, dv^g$$

$\mathcal{M}_1 := \{\text{metrics on } M \text{ of volume } 1\}$.
$[g] := \{u^{4/(n-2)} g \mid \mathrm{vol}(u^{4/(n-2)} g) = 1\}$.

Definitions

Inside a conformal class:

$$Y(M,[g]) := \inf_{\tilde{g}\in[g]} \mathcal{E}(\tilde{g}) > -\infty$$

is the conformal Yamabe invariant. We have

$$Y(M,[g]) \leq Y(\mathbb{S}^n)$$

where $\mathbb{S}^n$ is the sphere with the standard structure.

On the set of conformal classes:

$$\sigma(M) := \sup_{[g]\subset\mathcal{M}_1} Y(M,[g]) \in (-\infty, Y(\mathbb{S}^n)]$$

The smooth Yamabe invariant, also called Schoen's σ-constant.

Remark $\sigma(M) > 0$ if and only if M caries a metric of positive scalar curvature.

Yamabe's idea

Find stationary points of $\mathcal{E}$ with a minimax principle:

- Find a minimizer in each conformal class, i.e. $\tilde{g} \in [g]_1$ with $\mathcal{E}(\tilde{g}) = Y(M, [g])$. Such metrics always exist, have constant scalar curvature, and are called Yamabe metrics. They are unique if $Y(M, [g_0]) < 0$.
- Find a maximizing conformal class $[g_{\mathrm{max}}]$, i.e. $Y(M, [g_{\mathrm{max}}]) = \sigma(M)$.
  2nd step failed: Many manifolds do not carry Einstein metrics, e.g. $S^2 \times S^1$.

But:

- $\sigma(M)$ is linked to many geometric quantities
- Very challenging to calculate $\sigma(M)$,
  many interresting techniques used
- Our result: $\max\{\min\{\sigma(M), L(n)\}, 0\}$ is a bordism invariant
  Key step: a surgery formula

Relations to similar invariants...

... to the spectrum of the conformal Laplacian

$$\inf_{\tilde{g}\in[g]} \lambda_1\left(4\frac{n-1}{n-2}\Delta^{\tilde{g}} + \mathrm{scal}^{\tilde{g}}\right) = \begin{cases} Y(M,[g]) & \text{if } Y(M,[g]) \geq 0 \\ -\infty & \text{if } Y(M,[g]) < 0 \end{cases}$$

... to the Perelman-invariant (Akutagawa, Ishida, LeBrun '06)

$$\begin{aligned} \bar{\lambda}(M) &:= \sup_{g\in\mathcal{M}_1} \lambda_1(4\Delta^g + \mathrm{scal}^g) \\ &= \begin{cases} \sigma(M) & \text{if } \sigma(M) \leq 0 \\ +\infty & \text{if } \sigma(M) > 0 \end{cases} \end{aligned}$$

... to the $L^{n/2}$-norm of scal

$$\inf_{\tilde{g}\in\mathcal{M}_1} \|\mathrm{scal}^{\tilde{g}}\|_{L^{n/2}(\tilde{g})} = \begin{cases} |\sigma(M)| & \text{if } \sigma(M) \leq 0 \\ 0 & \text{if } \sigma(M) > 0 \end{cases}$$

$\Longrightarrow$ Hence $\min\{\sigma(M), 0\}$ is determined by an infimum.

Example $\mathbb{C}P^2$

The Fubini-Study $g_{\rm FS}$ metric is Einstein and

$$53,31... = \mathcal{E}(g_{\rm FS}) = Y(\mathbb{C}P^2, [g_{\rm FS}]) = \sigma(\mathbb{C}P^2).$$

Claude LeBrun ’97, *Seiberg-Witten theory*

Similar examples

- $\sigma(S^n) = n(n-1)\omega_n^{2/n}$.
- Gromov & Lawson, Schoen & Yau $\approx' 83$: Tori $\mathbb{R}^n/\mathbb{Z}^n$. $\sigma(\mathbb{R}^n/\mathbb{Z}^n) = 0$. *Enlargeable manifolds*
- LeBrun ′99: All Kähler-Einstein surfaces with non-positive scalar curvature.
- Bray & Neves ’04: $\mathbb{R}P^3$. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$. *Penrose inequality, Huisken-Illmanen techniques*
- Perelman, M. Anderson ’06: compact quotients of 3-dimensional hyperbolic space, *Ricci flow*

Other cases when $\sigma(M)$ is known

- Akutagawa & Neves '07: Some non-prime 3-manifolds, e.g.

$$\sigma(\mathbb{R}P^3 \# (S^2 \times S^1)) = \sigma(\mathbb{R}P^3).$$

- Compact quotients of nilpotent Lie groups: $\sigma(M) = 0$.

Unknown cases

- Nontrivial quotients of spheres, except $\mathbb{R}P^3$.
- $S^k \times S^m$, with $k, m \geq 2$.
- No example of dimension ≥ 5 known with $\sigma(M) \neq 0$ and $\sigma(M) \neq \sigma(S^n)$.

Surgery

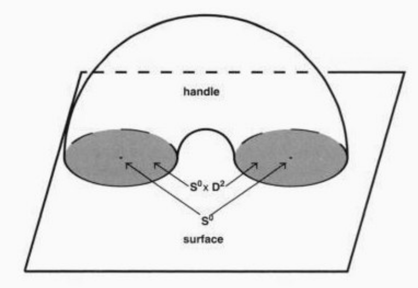
Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We define

$$M_k^\Phi := M \setminus \Phi(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1})/\sim$$

where $/\sim$ means gluing the boundaries via

$$M \ni \Phi(x, y) \sim (x, y) \in S^k \times S^{n-k-1}.$$

We say that M_k^Φ is obtained from M by surgery of dimension k.



Example: 0-dimensional surgery on a surface.

Known surgery formulas

Theorem (Gromov & Lawson '80, Schoen & Yau '79)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M) > 0 \quad \Longrightarrow \quad \sigma(M_k^\Phi) > 0.$$

Theorem (Kobayashi '87)

If $k = 0$, *then*

$$\sigma(M_0^\Phi) \geq \sigma(M).$$

Theorem (Petean & Yun '99)

If $0 \leq k \leq n-3$, *then*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), 0\}.$$

The proof uses the characterization of $\min\{\sigma(M), 0\}$ as an infimum.

Main results

Theorem (ADH, # 1)

Let $0 \leq k \leq n-3$. *There is a positive constant* $L(n,k)$ *depending only on n and k such that*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), L(n,k)\}.$$

Furthermore $L(n,0) = Y(\mathbb{S}^n)$.

This theorem implies all three previously known surgery formulas.
Thm # 1 follows directly from Thm # 2.

Theorem (ADH, #2)

Let $0 \leq k \leq n-3$. *Then for any metric g on M there is a sequence of metrics* g_i *on* M_k^Φ *such that*

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g]).$$

Topological conclusions

From now on $n \geq 5$.

$$L(n) := \min\{L(n,1), L(n,2), \ldots, L(n,n-3)\}.$$

If $k \in \{2,3,\ldots,n-3\}$, then

$$\min\{\sigma(M), L(n)\} = \min\{\sigma(M_k^\Phi), L(n)\}.$$

There is no sequence of simply connected manifolds $(M_i | i \in \mathbb{N})$ of fixed dimension n such that

$$L(n) > \sigma(M_1) > \sigma(M_2) > \sigma(M_3) > \ldots \geq 0.$$

We obtain bordism invariants, e.g.

$$s_\Gamma : \Omega_n^{spin}(B\Gamma) \to \mathbb{R}.$$

The group $\{x \in \Omega_n^{\mathrm{spin}}(B\Gamma) \,|\, s_\Gamma(x) > \epsilon\}$ is a subgroup.

Another Application

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.
Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\# \cdots \# M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\# \cdots \# M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$ for $\#\Gamma$ large.

Proof of Theorem # 2

Theorem (ADH, #2)

Let $0 \leq k \leq n-3$*. Then for any metric g on M there is a sequence of metrics* g_i *on* M_k^Φ *such that*

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g]).$$

Easy part:

$$\liminf_{i\to\infty} Y(M_k^\Phi,[g_i]) \leq Y(M,[g])$$

Difficult and important part

$$\min\{Y(M,[g]), L(n,k)\} \leq \liminf_{i\to\infty} Y(M_k^\Phi,[g_i])$$

For this, we construct a sequence $[g_i]$ explicitly and study the behavior of the minimizers in $[g_i]$ for $i \to \infty$.

Construction of the metrics

Let $\Phi : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We write close to $S := \Phi(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{round}^{n-k-1}$$

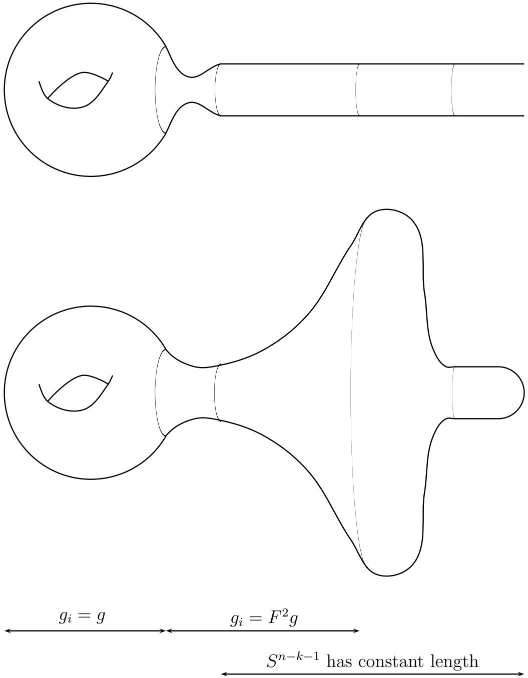
where g_{round}^{n-k-1} is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{round}^{n-k-1}$$

We define a metric

$$g_i = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{round}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

that extends to a metric on M_k^Φ.



$g_i = g$
$g_i = F^2 g$
S^{n-k-1} has constant length

Proof of Theorem #2, continued

Any class $[g_i]$ contains a minimizing metric written as $u_i^{4/(n-2)} g_i$.
We obtain a PDE:

$$4\frac{n-1}{n-2}\Delta^{g_i} u_i + \mathrm{scal}^{g_i} u_i = \lambda_i u_i^{\frac{n+2}{n-2}}$$

$$u_i > 0, \qquad \int u_i^{2n/(n-2)} dv^{g_i} = 1, \qquad \lambda_i = Y([g_i])$$

This sequence might:

- Concentrate in at least one point. Then $\liminf \lambda_i \geq Y(\mathbb{S}^n)$.
- Concentrate on the old part $M \setminus S$. Then $\liminf \lambda_i \geq Y([g])$.
- Concentrate on the new part.
  Then study pointed Gromov-Hausdorff limits.
  Limit spaces:

$$\mathbb{M}_c := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$$

$\mathbb{H}_c^{k+1}$: simply connected, complete, $K = -c^2$

The numbers $L(n,k)$

$$L(n,k) := \inf_{c\in[0,1]} Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$$

Note: $\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1} \cong \mathbb{S}^n \setminus \mathbb{S}^k$.
$k = 0$: $L(n,0) = Y(\mathbb{R} \times \mathbb{S}^{n-1}) = Y(\mathbb{S}^n)$
$k = 1, \ldots, n-3$: $L(n,k) > 0$

Conjecture #1: $L(n,k) = Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$

Conjecture #2: For determining $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ it suffices to consider $O(n-k)$ invariant functions.

Suppose Conjecture #2 is true.
Then one can even restrict to $O(k+1) \times O(n-k)$-invariant functions.
The determination of $L(n,k)$ then reduces to solving ODEs.

$n = 4$, $k = 1$. $\mathbb{H}^2_c \times \mathbb{S}^2$.
Conj. #2 implies Conj. # 1.
It follows $L(4,1) = 59,4....$
Compare to $\sigma(S^4) = Y(\mathbb{S}^4) = 61,5....$

One obtains $S^2 \times S^2$ via 1-dimensional surgery from S^4.
Hence

$$\sigma(S^2 \times S^2) \geq 59,4 > 50,2... = Y(\mathbb{S}^2 \times \mathbb{S}^2).$$

Related result for the Dirac operator

Theorem (ADH, #2)

$0 \leq k \leq n-2$. *Suppose that M_k^Φ is obtained from M by k-dimensional surgery, which is compatible with orientation and spin structure. Then for any metric g on M there is a sequence of metrics $\bar{g}$ on M_k^Φ such that*

$$\dim\ker D^{M_k^\Phi,\bar{g}} \leq \dim\ker D^{M,g}$$

The Atiyah-Singer index theorem implies for any closed spin manifold M

$$\dim\ker D^{M,\bar{g}} \geq |\alpha(M)| \qquad (*)$$

with $\alpha(M) = \hat{A}(M)$ if $4|n$.

Corollary

Equality in (*) holds for generic metrics, if M is connected.

If $n \not\equiv 1,2 \mod 8$, g generic, then

- $D|_{\Sigma^+}$ injective and $D|_{\Sigma^-}$ surjective, or
- $D|_{\Sigma^-}$ injective and $D|_{\Sigma^+}$ surjective

End of the talk.
Thank you!

Some Details about topological conclusions

$$L(n) := \min\{L(n,1), L(n,2), \dots, L(n,n-3)\},$$

$$\bar{\sigma}(M) := \min\{\sigma(M), L(n)\}.$$

Let M_k^Φ be obtained from M by surgery of dimension $k \in \{2, 3, \dots, n-3\}$, then $\bar{\sigma}(M) = \bar{\sigma}(M_k^\Phi)$.

Goal: Find a bordism invariant!

Let Γ be a finitely presented group.
Let $\Omega_n^{\text{spin}}(B\Gamma)$ the spin cobordism group over $B\Gamma$.
Any class in $\Omega_n^{\text{spin}}(B\Gamma)$ has a π_1-bijective representative, i.e. it is represented by (M, f), where M is a connected compact spin manifold, and where $f : M \to B\Gamma$ induces an isomorphism $\pi_1(M) \to \pi_1(B\Gamma) = \Gamma$.

Lemma

$n \geq 5$. Let (M_1, f_1) and (M_2, f_2) be spin cobordant over $B\Gamma$ and let (M_2, f_2) be π_1-bijective. Then

$$\bar{\sigma}(M_2) \geq \bar{\sigma}(M_1).$$

We define

$$s_\Gamma([M, f]) := \max \bar{\sigma}(N)$$

where the maximum runs over all (N, h) with

$$[M, f] = [N, h] \in \Omega_n^{\text{spin}}(B\Gamma).$$

$\bar{\sigma} := \min\{\sigma(M), L(n)\}$
$s_\Gamma([M, f]) := \bar{\sigma}(M)$ if (M, f) is π_1-bijective
$s_\Gamma : \Omega_n^{\text{spin}}(B\Gamma) \to (-\infty, L(n)]$

Corollary

Let $n \geq 5$ and $\epsilon \in (0, L(n))$. The groups $\{x \in \Omega_n^{\text{spin}}(B\Gamma) \mid s_\Gamma(x) > \epsilon\}$ and $\{x \in \Omega_n^{\text{spin}}(B\Gamma) \mid s_\Gamma(x) \geq \epsilon\}$ are subgroups.

Applications

Let $n \geq 5$. Take M^n with $\sigma(M) \in (0, L(n))$.
Let $p, q \in \mathbb{N}$ be relatively prime. Then

$$\sigma(\underbrace{M\# \cdots \# M}_{p \text{ times}}) = \sigma(M)$$

or

$$\sigma(\underbrace{M\# \cdots \# M}_{q \text{ times}}) = \sigma(M).$$

Are there such manifolds M?
Schoen conjectured: $\sigma(S^n/\Gamma) = \sigma(S^n)/(\#\Gamma)^{2/n} \in (0, L(n))$ for $\#\Gamma$ large.

Dimension $n = 3$

- Einstein 3-manifolds with $\kappa = 0$ are flat.
- Einstein 3-manifolds with $\kappa > 0$ are quotients of spheres.
- Einstein 3-manifolds with $\kappa < 0$ are quotients of hyperbolic spaces.

More about: Examples where Yamabe's idea fails

There is no Einstein metric on
non-prime 3-manifolds and
$S^2 \times S^1$.

Schoen: $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$. The supremum is not attained.

Other problem In the case $\sigma(M) < 0$ the minimizers are unique. Hence, if a maximizing conformal class exists, then **the unique** minimizing metric in that class is Einstein.
However, in the case $\sigma(M) > 0$, **a** minimizing metric in a maximizing class may be non-Einstein!