UNSTABLE OPERATIONS IN HOMOTOPY K-THEORY

by

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1. Homotopy K-theory via cdh-descent

1.1. — Let k be a field of characteristic zero. We denote by Sm/k and Sch/k the categories of smooth k-schemes and of k-schemes of finite type, respectively. We will write $\mathscr{E}(k)$ for the topos of Nisnevich sheaves on Sm/k, and $\underline{\mathscr{E}}(k)$ for the topos of Nisnevich sheaves on Sch/k. Finally, $\mathscr{E}_{\mathrm{cdh}}(k)$ stands for the topos of sheaves on Sch/k with respect to the cdh-topology. Note that, by Hironaka's resolution of singularity theorem, the cdh-topology is also well defined on Sm/k, and $\mathscr{E}_{\mathrm{cdh}}(k)$ is canonically equivalent to the category of cdh-sheaves on Sm/k.

1.2. — Given a topos $\mathscr{E}$, we denote by $\mathscr{H}_s(\mathscr{E})$ the homotopy category of the Joyal-Jardine model category of simplicial sheaves on $\mathscr{E}$. If I is an interval of $\mathscr{E}$, then we denote by $\mathscr{H}(\mathscr{E})$ the homotopy category of the Morel-Voevodsky model category structure on simplicial sheaves on $\mathscr{E}$ with respect to the interval I (the notation leaves out I because, in practice, at least in what follows, it will always be the same, namely $\mathrm{I} = \mathbf{A}^1$).

1.3. — Let K be the object of $\mathscr{H}_s(\underline{\mathscr{E}}(k))$ which represents algebraic K-theory of k-schemes of finite type: for a k-scheme X, we have a natural isomorphism in the homotopy category of simplicial sets

$$\mathbf{R}Map(\mathrm{X},\mathrm{K}) = \mathbf{R}\Omega(w\mathrm{S}.(\mathrm{Perf}(\mathrm{X}))\,,$$

where the right hand side is Thomason-Trobaugh's version of algebraic K-theory. The fact that K is indeed an object of $\mathscr{H}_s(\underline{\mathscr{E}}(k))$ is a reformulation of the fact that algebraic K-theory satisfies Nisnevich descent.

Note that the cdh-sheafification functor $\underline{\mathscr{E}}(k) \to \mathscr{E}_{\mathrm{cdh}}(k)$ induces a left Quillen functor which preserves weak equivalences, and thus defines a functor

$$\mathscr{H}_s(\underline{\mathscr{E}}(k)) \to \mathscr{H}_s(\mathscr{E}_{\mathrm{cdh}}(k)) \quad , \qquad \mathrm{F} \mapsto \mathrm{F}_{\mathrm{cdh}}\,.$$

In particular, cdh-sheafifying algebraic K-theory defines an object K_{cdh} in $\mathscr{H}_s(\mathscr{E}_{cdh}(k))$. On the other hand, by virtue of a theorem of Haesemeyer, homotopy K-theory satifies cdh-descent over *Sch/k*, which means that the homotopy K-theory simplicial presheaf $\Omega^\infty(KH)$ may be viewed as an object of $\mathscr{H}_s(\mathscr{E}_{cdh}(k))$ (where $KH(X)$ denotes the homotopy K-theory spectrum of X and Ω^∞ stands for the infinite loop space functor; in some sense, $\Omega^\infty(KH)$ is thus a truncated version of usual homotopy K-theory).

Theorem 1.4. — *There is a canonical isomorphism* $K_{cdh} \simeq \Omega^\infty(KH)$ *in* $\mathscr{H}_s(\mathscr{E}_{cdh}(k))$.

Proof. — By construction of KH, there is a canonical morphism of simplicial presheaves $K \to \Omega^\infty(KH)$ over *Sch/k*. As KH satisfies cdh-descent, this map defines by adjunction a morphism $K_{cdh} \to \Omega^\infty(KH)$ in $\mathscr{H}_s(\mathscr{E}_{cdh}(k))$. To prove that the latter is an isomorphism, as smooth k-schemes are generators of $\mathscr{H}_s(\mathscr{E}_{cdh}(k))$, it is sufficient to prove that, for any smooth k-scheme X, the induced map

$$\mathbf{R}Map(X, K_{cdh}) \to \mathbf{R}Map(X, \Omega^\infty(KH)) \simeq \Omega^\infty(KH(X))$$

is an isomorphism in the homotopy category of spaces. But we know that K and $\Omega^\infty(KH)$ agree on smooth k-schemes, so that, as KH satisfies cdh-descent, the restriction of K on *Sm/k* satisies cdh-descent as well. In other words, for any smooth k-scheme X, the natural map

$$K(X) \simeq \mathbf{R}Map(X, K) \to \mathbf{R}Map(X, K_{cdh})$$

is an isomorphism. As the map $K(X) \to \Omega^\infty(KH(X))$ is an isomorphism as well, this achieves the proof. □

2. Unstable operations on K-theories

Here, we will interpret Riou's classification of unstable operations for algebraic K-theory of smooth k-schemes in several directions.

2.1. — Let $\imath : Sm/k \to Sch/k$ be the inclusion functor. We write $\imath^* : \underline{\mathscr{E}}(k) \to \mathscr{E}(k)$ for the corresponding restriction functor, and $\imath_\sharp$ for its left adjoint (so that $\imath_\sharp(X) = X$ for any smooth k-scheme X). The pair $(\imath_\sharp, \imath^*)$ induces a derived adjunction

$$\mathbf{L}\imath_\sharp : \mathscr{H}_s(\mathscr{E}(k)) \rightleftarrows \mathscr{H}_s(\underline{\mathscr{E}}(k)) : \imath^*$$

as well as an $\mathbf{A}^1$-version

$$\mathbf{L}\imath_\sharp : \mathscr{H}(\mathscr{E}(k)) \rightleftarrows \mathscr{H}(\underline{\mathscr{E}}(k)) : \imath^*$$

(note that $\imath^*$ does not need to be derived because it preserves weak equivalences on the nose). In either case, the functor $\mathbf{L}\imath_\sharp$ is fully faithful, because we have the formula

$$1 \simeq \imath^* \mathbf{L}\imath_\sharp .$$

Moreover the functor $\mathbf{L}\imath_\sharp$ preserves finite products. This implies that it is compatible with the plus construction. Moreover, $\imath^*(K)$ represents algebraic K-theory of smooth k-schemes. In other words, we have

$$\mathbf{L}\imath_\sharp \imath^*(K) \simeq \mathbf{L}\imath_\sharp((\mathbf{Z} \times BGL)^+) \simeq (\mathbf{Z} \times \mathbf{L}\imath_\sharp(BGL))^+ \simeq (\mathbf{Z} \times BGL)^+ \simeq K$$

This implies right away the following little computation.

Proposition 2.2. — *For any natural number $n \geq 0$, there is a natural isomorphism*

$$\mathbf{R}Map(\mathrm{K}^n, \mathrm{K}) \simeq \mathbf{R}Map(\imath^*(\mathrm{K})^n, \imath^*(\mathrm{K}))$$

in the homotopy category of spaces.

This means that unstable cohomological operations on algebraic K-theory of k-schemes of finite type are completely characterized by cohomological operations for smooth k-schemes. Therefore, we can use Riou's results as follows[1].

Theorem 2.3. — *For any integer $n \geq 0$ and any natural map*

$$\alpha : \mathrm{K}_0(\mathrm{X})^n \to \mathrm{K}_0(\mathrm{X})$$

defined for smooth k-schemes X, *there is a unique morphism*

$$\bar{\alpha} : \mathrm{K}^n \to \mathrm{K}$$

in $\mathscr{H}_s(\underline{\mathscr{E}}(k))$ which lifts α.

Proof. — By virtue of the preceding proposition, it is sufficient to promote α as a morphism $\imath^*(\mathrm{K})^n \to \imath^*(\mathrm{K})$ in $\mathscr{H}_s(\mathscr{E}(k))$. But, as $\imath^*(\mathrm{K})$ is homotopy invariant, it is sufficient to understand the morphisms from $\imath^*(\mathrm{K})^n$ to $\imath^*(\mathrm{K})$ in the homotopy category of schemes $\mathscr{H}(\mathscr{E}(k))$. By virtue of Riou's theorem, these correspond precisely to the natural transformations $\mathrm{K}_0^n \to \mathrm{K}_0$ defined on smooth k-schemes. □

Proposition 2.4. — *For any integer $n \geq 0$, there is a natural isomorphism*

$$\mathbf{R}Map(\Omega^\infty(\mathrm{KH})^n, \Omega^\infty(\mathrm{KH})) \simeq \mathbf{R}Map(\imath^*(\mathrm{K})^m, \imath^*(\mathrm{K}))$$

in the homotopy category of spaces.

Proof. — By virtue of Theorem 1.4, we have a natural isomorphism

$$\mathbf{R}Map(\Omega^\infty(\mathrm{KH})^n, \Omega^\infty(\mathrm{KH})) \simeq \mathbf{R}Map(\mathrm{K}^n_{\mathrm{cdh}}, \Omega^\infty(\mathrm{KH})),$$

and, as KH satisfies cohomological cdh-descent, we also have a natural isomorphism

$$\mathbf{R}Map(\mathrm{K}^n_{\mathrm{cdh}}, \Omega^\infty(\mathrm{KH})) \simeq \mathbf{R}Map(\mathrm{K}^n, \Omega^\infty(\mathrm{KH})).$$

Therefore, by virtue of Proposition 2.2, it is sufficient to prove that the canonical map

$$\mathbf{R}Map(\mathrm{K}^n, \mathrm{K}) \to \mathbf{R}Map(\mathrm{K}^n, \Omega^\infty(\mathrm{KH}))$$

is an isomorphism. Using the identification $\mathbf{L}_\sharp \imath^*(\mathrm{K}) \simeq \mathrm{K}$, we thus have to prove that the map

$$\mathbf{R}Map(\imath^*\mathrm{K}^n, \imath^*\mathrm{K}) \to \mathbf{R}Map(\imath^*\mathrm{K}^n, \imath^*\Omega^\infty(\mathrm{KH}))$$

is invertible. But this comes from the well known fact that $\imath^*\mathrm{K} \to \imath^*\Omega^\infty(\mathrm{KH})$ is an isomorphism in $\mathscr{H}_s(\mathscr{E}(k))$. □

[1] For this part, we don't need k to be a field of characteristic zero; being a regular noetherian ring is sufficient, for instance.

Corollary 2.5. — *Let $n \geq 0$ be an integer and*

$$\alpha : K_0(X)^n \to K_0(X)$$

be a natural morphism defined for any smooth k-scheme X. Then there is a unique morphism $\bar{\alpha} : K^n \to K$ and a unique morphism $\tilde{\alpha} : \Omega^\infty(KH)^n \to \Omega^\infty(KH)$ in $\mathscr{H}_s(\underline{\mathscr{E}}(k))$ which both lift α. Furthermore, the canonical morphism $c : K \to \Omega^\infty(KH)$ then gives a commutative square

$$\begin{array}{ccc} K^n & \xrightarrow{c^n} & \Omega^\infty(KH)^n \\ {\scriptstyle \bar{\alpha}}\downarrow & & \downarrow{\scriptstyle \tilde{\alpha}} \\ K & \xrightarrow{c} & \Omega^\infty(KH) \end{array}$$

in the homotopy category $\mathscr{H}_s(\underline{\mathscr{E}}(k))$.

Proof. — The existence of $\bar{\alpha}$ comes from Theorem 2.3, and another use of the same theorem together with the previous proposition gives $\tilde{\alpha}$. To check that the square above commutes, it is sufficient to notice that

$$\mathbf{R}Map(K^n, \Omega^\infty(KH)) \simeq \mathbf{R}Map(K^n_{cdh}, \Omega^\infty(KH)) \simeq \mathbf{R}Map(\Omega^\infty(KH)^n, \Omega^\infty(KH))$$

and to use Theorem 2.3 as well as Proposition 2.4 once again. □

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