CONTROLLED OBJECTS IN LEFT-EXACT ∞-CATEGORIES AND THE NOVIKOV CONJECTURE

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Abstract. We associate to every $G$-bornological coarse space $X$ and every left-exact $\infty$-category with $G$-action a left-exact infinity-category of equivariant $X$-controlled objects. Postcomposing with algebraic K-theory leads to new equivariant coarse homology theories. This allows us to apply the injectivity results for assembly maps by Bunke, Engel, Kasprowski and Winges to the algebraic K-theory of left-exact $\infty$-categories.

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1. Introduction

The main goal of this paper is to provide new examples of functors on the orbit category of a group to which the split injectivity results for assembly maps shown in [BEKWc] apply. To this end, we introduce and study left-exact ∞-categories of controlled objects over G-bornological coarse spaces with the aim of constructing new examples of equivariant coarse homology theories in the sense of [BEKWa].

The split injectivity theorems of [BEKWc] concern functors on the orbit category of a group which satisfy conditions comprised in the notion of a CP-functor. Being a CP-functor requires that the functor extends to an equivariant coarse homology theory in a particular way, and that this equivariant coarse homology theory has various additional properties.

1.1. Functors on the orbit category and split injectivity of assembly maps. Let $G$ be a group and let $G\text{Orb}$ denote the orbit category of $G$, i.e., the category of transitive $G$-sets and equivariant maps. If $\mathcal{F}$ is a family of subgroups of $G$ (Definition 6.44), then let $G\mathcal{F}\text{Orb}$ denote the full subcategory of $G\text{Orb}$ of $G$-sets with stabilisers in $\mathcal{F}$ (Definition 6.45).

Consider two families $\mathcal{F}'$ and $\mathcal{F}$ of subgroups of $G$ such that $\mathcal{F}' \subseteq \mathcal{F}$. Furthermore, consider a functor $M: G\text{Orb} \to M$ with a cocomplete target ∞-category. Then we have a relative assembly map (see Definition 6.50)

$$\text{Ass}_{\mathcal{F}', M}: \colim_{G\mathcal{F}'\text{Orb}} M \to \colim_{G\mathcal{F}\text{Orb}} M.$$

It is a morphism between objects of $M$ and induced by the inclusion of the index categories of the colimits in (1.1).

The natural question about the assembly map is under which conditions on the data $G$, $\mathcal{F}$, $\mathcal{F}'$, and $M$ it is an equivalence. Depending on the functor and the choice of the families, the answer to this question in special cases is predicted by the Baum–Connes conjecture or the Farrell–Jones conjecture. The case of the Farrell–Jones conjecture will be reviewed below in Example 1.2.

One could also ask the weaker question whether the assembly map admits a left inverse. In a series of papers culminating in [BEKWc] we developed an axiomatic approach to such split injectivity theorems. In addition to [BEKWc], we refer to Section 6.5 for the complete statements.
In order to provide a glimpse of how these theorems look like, we formulate such an injectivity statement for the functor $K_G^C: \text{GOorb} \to \text{Sp}$ (see Definition 6.22 and Definition 6.26) associated to a left-exact $\infty$-category with $G$-action $C$.

**Theorem 1.1.** Assume:

1. $G$ admits a finite-dimensional CW-model for the classifying space $E_{\text{Fin}}^G$.
2. $G$ is a finitely generated subgroup of a linear group over a commutative ring with unit or of a virtually connected Lie group.

Then the assembly map

$$\text{Ass}_{\text{Fin},K_G^C}^\text{All}: \colim_{\text{GFinOrb}} K_G^C \to \colim_{\text{GOorb}} K_G^C$$

admits a left inverse.

Using that $\ast$ is the final object of $\text{GOorb}$ we can identify the target of this assembly map with $K_G^C(\ast)$.

The assumptions on the data for the assembly map required in the theorems listed in Section 6.5 can be separated into assumptions on the group $G$ and the families $\mathcal{F}', \mathcal{F}$ on the one hand, and the assumption on the functor $M$ being a CP-functor, see Definition 6.19, on the other hand. The contribution of the present paper is the verification that the functor $K_G^C$ is a hereditary CP-functor (Corollary 6.30). Theorem 1.1 now follows by applying Theorem 6.55 with $K_G^C$ in place of $M$.

By varying the left-exact $\infty$-category $C$, the present paper contributes many new examples of CP-functors, but it does not enlarge the class of groups for which injectivity results are known.

**Example 1.2.** Our main focus is on the equivariant $K$-theory of left-exact categories $K_G^C$, but the motivating and guiding example for our approach is the equivariant algebraic $K$-theory functor

$$K_A^G: \text{GOorb} \to \text{Sp}$$

associated to an object of $\text{Fun}(BG, \text{Add})$, i.e., an additive category $A$ with a strict $G$-action. This functor has first been constructed in [DL98]. We refer to this case as the linear case as opposed to the derived case.

Let $BG$ be the category with one object having the group of automorphisms $G$. The group $G$ with its $G$-action by left translations is an object of $\text{GOorb}$ with group of automorphisms $G$ (acting by right translation). The $G$-set $G$ therefore provides an embedding.

\[ j: BG \to \text{GOorb} \]

(1.2)

Let $A_\infty$ be $A$ considered as an object of $\text{Fun}(BG, \text{Add}_\infty)$, where $\text{Add}_\infty$ is the large $\infty$-category of small additive categories (actually a 2-category) obtained from the category of small additive categories and additive functors by inverting equivalences. Denote the left Kan extension of $A_\infty$ along $j$ by

\[ \text{Ind}_G^G(A_\infty): \text{GOorb} \to \text{Add}_\infty \]

Finally, let

\[ K^{A_\infty}: \text{Add}_\infty \to \text{Sp} \]

be the non-connective $K$-theory functor for additive categories (constructed by Pedersen–Weibel [PW85] and Schlichting [Sch04], see also [BEKWd, Sec. 3.2] for
the factorisation over $\text{Add}_\infty$. We then define the composed functor
\begin{equation}
K_A^G := K^\text{Add} \circ \text{Ind}^G(A_\infty): G\text{Orb} \to \text{Sp}
\end{equation}
(compare with Definition 6.22).

The linear version of the $K$-theoretic Farrell–Jones conjecture asserts that the assembly map
\begin{equation}
\text{Ass}^\text{All}_{V_{\text{v cyc}}} K_A^G : \colim_{G_{V_{\text{v cyc}}}} K_A^G \to K_A^G(\ast)
\end{equation}
is an equivalence, where $V_{\text{v cyc}}$ is the family of virtually cyclic subgroups and we calculated the colimit in the target of (1.1) using that $\ast$ is the final object of $G_{\text{All}}\text{Orb}$. The Farrell–Jones conjecture is known in many cases, see [Lüch, Sec. 7.7] for an overview.

As mentioned above, we derived various split injectivity theorems about assembly maps in [BEKwa, BEKWc]. These theorems rely on the assumption that $M$ is a (hereditary) CP-functor (Definitions 6.19 and 6.24).

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Let $A$ be an additive category with $G$-action.

**Theorem 1.3** ([BEKwa, Ex. 1.10],[BEKWc, Ex. 2.6]). The functor $K_A^G$ is a hereditary CP-functor.

The argument for this result given in [BEKwa] is short but heavily uses results from [BEKWa] and the quite technical paper [BEKWd]. As explained in Example 1.6 below, the results of the present paper also provide an independent argument for Theorem 1.3.

The transition from the linear to the derived case consists of replacing additive categories by left-exact $\infty$-categories.

An $\infty$-category is called left-exact if it contains a zero object, i.e., an object which is initial and final at the same time, and if it admits all finite limits. A morphism between left-exact $\infty$-categories is a left-exact functor, i.e., a functor which preserves finite limits. Let $\text{Cat}^{\text{Lex}_\infty, \ast}$ denote the large subcategory of $\text{Cat}_\infty$ of small left-exact $\infty$-categories, see Example 7.2. Note that $\text{Cat}^{\text{Lex}_\infty, \ast}$ contains the $\infty$-category $\text{Cat}^{\text{Lex}_\infty}$ of small stable $\infty$-categories as a full subcategory. It admits all small limits and colimits (Propositions 7.9 and 7.18).

Our input data is a left-exact $\infty$-category $C$ with an action of $G$, i.e., an object of $\text{Fun}(BG, \text{Cat}^{\text{Lex}_\infty, \ast})$. Define the functor
\begin{equation}
\text{Ind}^G(C): G\text{Orb} \to \text{Cat}^{\text{Lex}_\infty, \ast}
\end{equation}
as the left Kan extension of $C$ along $j$ from (1.2).

**Example 1.4.** Let $R$ in $\text{Alg}(\text{Sp})$ be an associative ring spectrum, and let $\text{Mod}(R)$ be its stable $\infty$-category of right modules. In the following, we use this example in order to name various $\infty$-categories appearing in the present paper, and to provide typical objects of them. The $\infty$-category $\text{Mod}(R)$ is presentable. Its subcategory $\text{Mod}(R)^{\text{perf}}$ of $\text{Mod}(R)$ compact objects belongs to $\text{Cat}^{\text{Lex}_\infty}$. We then consider the subcategory $\text{Mod}_{\omega}(R)$ of $\text{Mod}(R)$ of $\omega$-presentable $R$-modules consisting of those modules which can be written as $\omega$-filtered colimits of objects of $\text{Mod}(R)^{\text{perf}}$. Then $\text{Mod}_{\omega}(R)$ is a pointed, stable and $\omega$-presentable $\infty$-category, and its opposite $\text{Mod}_{\omega}(R)^{\text{op}}$ is an object of the subcategory $\text{Cat}^{\text{Lex}_\infty, \ast}_\infty$ of $\text{CAT}^{\text{Lex}_\infty, \ast}_\infty$ (Example 7.8). The subcategory of cocompact objects (Definition 7.7)
\[ \text{Mod}_\infty(R)^{\text{op},\omega} \simeq \text{Mod}(R)^{\text{perf},\text{op}} \] is then an example of an object of \( \text{Cat}_{\infty,*}^{\text{Lex}} \) (or, since it is idempotent complete, actually of \( \text{Cat}_{\infty,*}^{\text{Lex,perf}} \) (Example 7.4)).

We consider \( \text{Mod}_\infty(R)^{\text{op},\omega} \) as a constant functor in \( \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}}) \), i.e., as a left-exact \( \infty \)-category with trivial \( G \)-action. In this case, one can calculate the value of \( \text{Ind}^G(\text{Mod}_\infty(R)^{\text{op},\omega}) \) on an orbit \( G/H \) in \( GO\text{rb} \) explicitly. Let \( R[H] \) denote the group ring of \( H \) with coefficients in \( R \). Then one can check that

\[
\text{Ind}^G(\text{Mod}_\infty(R)^{\text{op},\omega})(G/H) \simeq \text{Mod}_\infty(R[H])^{\text{op},\omega}.
\]

The algebraic \( K \)-theory functor for left-exact \( \infty \)-categories (Definition 6.26) will be defined as the composition

\[
K : \text{Cat}_{\infty,*}^{\text{Lex}} \xrightarrow{\widetilde{\text{Sp}}} \text{Cat}_\infty^{\text{ex}} \xrightarrow{K^{\text{st}}} \text{Sp},
\]

where \( \widetilde{\text{Sp}} \) is the stabilisation functor (Definition 7.42) and \( K^{\text{st}} \) is the algebraic \( K \)-theory functor for stable \( \infty \)-categories [BGT13, Def. 9.6], see Definition 6.26. The functor \( K \) from (1.8) is an example of a finitary localising invariant (Definition 6.7), which in addition preserves products (Proposition 6.29). Consider the composed functor (Definition 6.22)

\[
KCG := K \circ \text{Ind}^G(C): GO\text{rb} \to \text{Sp}.
\]

The following is our main result about the functor \( KCG \). It is the derived analogue of Theorem 1.3. Let \( C \) be in \( \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}}) \).

**Theorem 1.5 (Corollary 6.30).** The functor \( KCG \) is a hereditary \( CP \)-functor.

**Example 1.6.** Let \( A \) be in \( \text{Fun}(BG, \text{Add}) \). Then we can form the dg-category of bounded chain complexes \( Ch^b(A) \) in \( A \). By [BC, Prop. 2.7] the localisation \( Ch^b(A)_\infty \) of \( Ch^b(A) \) at the homotopy equivalences is a stable \( \infty \)-category. Taking the \( G \)-action into account we obtain an object \( Ch^b(A)_\infty \) of \( \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}}) \) which can serve as input for our theory.

Note that the functor \( Ch^b(-)_\infty : \text{Add} \to \text{Cat}_{\infty,*}^{\text{Lex}} \) preserves equivalences and therefore descends to a functor (denoted by the same symbol)

\[
Ch^b(-)_\infty : \text{Add}_\infty \to \text{Cat}_{\infty,*}^{\text{Lex}}.
\]

By Corollary 7.68, we have an equivalence

\[
\text{Ind}^G(Ch^b(A)_\infty) \simeq Ch^b(\text{Ind}^G(A_\infty)).
\]

The Gillet–Waldhausen theorem [TT90, Theorem 1.11.7] implies the relation

\[
K^{\text{Add}} \simeq K \circ Ch^b(-)_\infty
\]

between the \( K \)-theory functors from (1.3) and (1.8). See [BGT13, Sec. 9] for a comparison of different constructions of algebraic \( K \)-theory functors.

This discussion shows:

**Corollary 1.7.** There is an equivalence

\[
KA_G \simeq KCh^b(A)_\infty,G
\]

of functors \( GO\text{rb} \to \text{Sp} \).

Consequently, Theorem 1.5 strictly generalises Theorem 1.3. ✷
Example 1.8. Consider the presentable ∞-category of pointed spaces $\text{Spc}_*$. The full subcategory $\text{Spc}^{op,\omega}_*$ of cocompact objects in its opposite belongs to $\text{Cat}^{\text{Lex},\infty}_{*}$. This is a non-stable example which can serve as an input for our theory.

Example 1.9. In [BKW], we studied the equivariant Waldhausen $A$-theory functor $A_P : \text{GOrb} \to \text{Sp}$, $S \mapsto A(P \times_G S)$ associated to a $G$-principal bundle with total space $P$.

Theorem 1.10 ([BKW, Thm. 5.17]). $A_P$ is a hereditary CP-functor.

Theorem 1.10 is a special case of Theorem 1.5 as follows: Let $\ell : \text{Top} \to \text{Spc}$ be the canonical localisation functor. Every topological $G$-space $Q$ (i.e., an object of $\text{Fun}(BG, \text{Top})$) gives rise to an object $\ell(Q)$ in $\text{Fun}(BG, \text{Spc})$. Since $\text{Cat}^{\text{Lex},\infty}_{*}$ is cocomplete, it is tensored over spaces, so we can form the functor $K(\ell(Q) \otimes \text{Spc}^{op,\omega})_G : \text{GOrb} \to \text{Sp}$. It is a hereditary CP-functor by Theorem 1.5. In Section 7.7 we will show that the functors $K(\ell(Q) \otimes \text{Spc}^{op,\omega})_G$ and $A_P$ are equivalent for every principal $G$-bundle $P$.

We conclude this introduction with an indication how Theorem 1.5 is shown. The first condition for a CP-functor (Definition 6.19) is that its target ∞-category is stable, complete, cocomplete, and compactly generated. Note that the ∞-category of spectra $\text{Sp}$ has all these properties.

The remaining conditions for $KCG$ being a CP-functor require that it extends (in a particular way, see below) to an equivariant coarse homology theory $E$, and that this equivariant coarse homology theory is continuous, and strongly additive, and admits transfers. It is the realisation of this condition which connects the study of CP-functors with the construction of equivariant coarse homology theories using controlled object functors. In the following, we need the category $G\text{BornCoarse}$ of $G$-bornological coarse spaces (see Section 2.4) and the notion of an equivariant coarse homology theory $E : G\text{BornCoarse} \to \text{Sp}$ (Definition 6.2). These notions were introduced in [BEKWA] in the precise form needed.

The phrase “extends in a particular way” means that for every $S$ in $G\text{Orb}$ there is a natural equivalence

\begin{equation}
KCG(S) \simeq E(G_{\text{can},\min} \otimes S_{\text{min},\max})
\end{equation}

where we refer to Example 2.38 and Example 2.39 for the notation appearing in the argument of $E$.

We show, as a consequence of Proposition 5.2, that the functor $KCG$ is the restriction of the equivariant coarse homology theory

$KCA^G : G\text{BornCoarse} \to \text{Sp}$

defined in Definition 6.33. More precisely, for every $S$ in $G\text{Orb}$ there is a natural equivalence

\begin{equation}
KCG(S) \simeq KCA^G(S_{\text{min},\max})
\end{equation}

But this is not yet the correct “particular way” of extending as indicated in (1.10). The correct equivariant coarse homology theory which has to be taken for $E$ is the coarse algebraic $K$-homology $KCA^G : G\text{BornCoarse} \to \text{Sp}$.
with coefficients in $\mathbb{C}$ which is defined in Definition 6.31. By Corollary 5.4, we get a natural equivalence
\[ K\mathcal{A}_G(S_{\min,max}) \simeq K\mathcal{A}^G(G_{can,min} \otimes S_{\min,max}), \]
which together with (1.11) gives the required natural equivalence
\[ K\mathcal{C}_G(S) \simeq K\mathcal{A}^G(G_{can,min} \otimes S_{\min,max}). \]
We now have to see that the equivariant coarse homology theory $K\mathcal{A}^G$ is strongly additive, continuous, and admits transfers.

Continuity is built into the definition of $K\mathcal{A}^G$, see Section 4.1.

The condition of strong additivity (Definition 3.37) heavily depends on the fact that the $K$-theory functor preserves products (Proposition 6.29). At the moment, we do not know any other non-trivial finitary localising invariant (Definition 6.7) with this property.

Finally, the existence of transfers heavily depends on the construction of $K\mathcal{A}^G$ via categories of controlled objects. We refer to Section 4.4 for the details. This section is the derived analogue of the paper [BEKWd] covering the linear case.

1.2. Controlled objects and coarse homology theories. Coarse geometry was invented by J. Roe [Roe03, Roe88, Roe93]. Partially motivated by the study of assembly maps, controlled topology has been developed e.g. in [CP95, BFJR04, BLR08, BL11] as a parallel branch. Eventually, it has been observed in [HPR96, Mit01, BEa] and other places that one can interpret controlled topology as a part of coarse geometry via the cone construction.

In [BEc] (the non-equivariant case) and [BEKWa] (the equivariant case), we provided a formal framework for coarse geometry and axiomatised the notion of a (equivariant) coarse homology theory (see Definition 6.2). This framework subsumes the proper metric spaces studied in classical coarse geometry and the cones considered in controlled topology, but also allows more general constructions.

More specifically, in [BEKWa] we introduced the category of $G$-bornological coarse spaces $G\operatorname{BornCoarse}$ (Definition 2.35). These are $G$-sets equipped with a compatible $G$-bornology and $G$-coarse structure, see Definitions 2.15, 2.32 and 2.34. The bornology is used to encode local finiteness conditions, while the coarse structure captures the large-scale geometry. While in the classical definition by Roe the bounded sets are determined by the coarse structure, in the case of $G$-bornological spaces there is much more freedom for the choice of the bornology.

Recall that in homotopy theory one studies topological spaces (or simplicial sets) up to weak equivalence. Analogously, the homotopy theory of bornological coarse spaces studies bornological coarse spaces up to coarse equivalence (Remark 3.2 and flasques (Definition 3.4). Homotopical invariants of $G$-bornological coarse spaces which in addition satisfy an appropriate version of excision are given by the evaluation of equivariant coarse homology theories
\[ E : G\operatorname{BornCoarse} \to M, \]
where $M$ is a cocomplete stable $\infty$-category, e.g., the category of spectra. In [BEKWa], we constructed the universal equivariant coarse homology theory
\[ \operatorname{Yo}^S : G\operatorname{BornCoarse} \to G\operatorname{Sp}.\lambda \]
which takes values in the stable $\infty$-category $G\operatorname{Sp}.\lambda$ of equivariant coarse motivic spectra.
For another attempt to axiomatise coarse homology theories we refer to [Mit01]. The examples of coarse homology theories prior to [BEc] (the non-equivariant case) and [BEKWa] (the equivariant case) were constructed under more restrictive assumptions on the spaces, and often only as group-valued functors satisfying a weaker set of axioms. The most relevant properties were coarse invariance and versions of excision. Some versions of the vanishing on flasques property was considered as a particular property of the example. This applies for example to the coarse ordinary homology and coarse topological $K$-homology which were defined as group-valued functors on the category of proper metric spaces and proper controlled maps [Roe93], [Roe96]. The algebraic $K$-theory functors in controlled topology were usually defined on spaces which are cones over topological spaces [Wei02],[BLR08], but sometimes also for general metric space as in [PW85].

It turned out that the construction of all these examples could be modified in order to fit our notion of coarse homology theory. We refer to [BEc], [BEKWa], [BC] for the cases of ordinary coarse homology, topological coarse $K$-homology, and coarse algebraic $K$-homology with coefficients in an additive category.

In the present paper, we construct functors

$$V: G\text{BornCoarse} \to \text{Cat}^{\text{Lex}}_{\infty,*}$$

which associate to $X$ in $G\text{BornCoarse}$ a left-exact $\infty$-category of $X$-controlled objects in a (previously chosen) category $C$ in $\text{Fun}(BG, \text{Cat}^{\text{Lex}}_{\infty,*})$ (Definition 4.19). These constructions are designed such that if

$$Hg: \text{Cat}^{\text{Lex}}_{\infty,*} \to M$$

is a homological functor (Definition 6.7), e.g., the composition of a finitary localising invariant on stable $\infty$-categories with the stabilisation functor, then the composition

$$Hg \circ V: G\text{BornCoarse} \to M$$

is an equivariant coarse homology theory.

The idea to use controlled objects to produce coarse homology theories is natural and has been used in previous examples. The first case is probably the use of controlled Alexander chains in Roe’s construction of ordinary coarse homology. Controlled objects in an additive category were used to construct the controlled or coarse versions of algebraic $K$-theory of additive categories, see e.g. [PW85], [Wei02], [BLR08], [BFJR04], [BEKWa]. In an analogous fashion, coarse topological $K$-homology has been constructed using controlled objects in $C^*$-categories, see [BEc], [Bun19], [BEb]. Non-linear versions of categories of controlled objects, namely $X$-controlled retractive spaces over some auxiliary space, have been used to construct controlled $A$-theory [Wei02], [UW19], and an equivariant coarse homology theory extending equivariant $A$-theory in [BKW].

In all these examples, the coefficient category $C$ is an ordinary category $C$. In the present paper, we start with an object $C$ in $\text{Fun}(BG, \text{Cat}^{\text{Lex}}_{\infty,*})$ (Example 7.8). In the following, we explain how we associate to $X$ in $G\text{BornCoarse}$ a category of $X$-controlled objects in $C$.

Let $X$ be a $G$-set. In the first step, consider the $\infty$-category

$$\text{PSh}_C(X) := \text{Fun}(P_X^{\text{op}}, C)$$

(see (2.1)) of contravariant functors from the poset $P_X$ of subsets of $X$ to $C$. The group $G$ acts on $X$ (and hence on $P_X$) as well as on $C$. With the induced $G$-action,
$\mathbf{PSh}_C(X)$ is again an object of $\mathbf{Fun}(BG, \mathbf{Cat}^{\text{Lex}}_{\infty, \ast})$. We set

$$\mathbf{PSh}_C^G(X) := \lim_{BG} \mathbf{PSh}_C(X).$$

The construction depends functorially on $X$ in $\mathbf{GSet}$ and produces a functor

$$\mathbf{PSh}_C^G : \mathbf{GSet} \to \mathbf{Cat}^{\text{Lex}}_{\infty, \ast}.$$ 

Using the forgetful functor $\mathbf{GCoarse} \to \mathbf{GSet}$ we can view $\mathbf{PSh}_C^G$ as a functor defined on the category $\mathbf{GCoarse}$ of $G$-coarse spaces (Definition 2.18). If $X$ is in $\mathbf{GCoarse}$, we then use the coarse structure $C_X$ of $X$ in order to define a subcategory $\mathbf{Sh}_C^G(X)$ of sheaves in $\mathbf{PSh}_C^G(X)$. For every invariant entourage $U$ in $C_X^G$, consider the Grothendieck topology $\tau_U$ generated by $U$-covering families (Definition 2.2) and let

$$\mathbf{Sh}_C^U(X) \subseteq \mathbf{PSh}_C(X)$$

be the full subcategory of $\tau_U$-sheaves. It is an object of $\mathbf{Fun}(BG, \mathbf{CAT}_{\infty, \ast}^{\text{Lex}})$ (Example 7.6). Define the object

$$\mathbf{Sh}_C(X) := \bigcup_{U \in C_X^G} \mathbf{Sh}_C^U(X)$$

in $\mathbf{Fun}(BG, \mathbf{CAT}_{\infty, \ast}^{\text{Lex}})$ (Example 7.5). By applying $\lim_{BG}$, we get the objects

$$\mathbf{Sh}_C^{U,G}(X) \text{ in } \mathbf{CAT}_{\infty, \ast}^{\text{Lex}} \text{ and } \mathbf{Sh}_C^G(X) \text{ in } \mathbf{CAT}_{\infty, \ast}^{\text{Lex}}.$$ 

The excision property of sheaves is encoded in the Glueing Lemma (Lemma 2.14). The construction of $\mathbf{Sh}_C^G(X)$ depends functorially on the $G$-coarse space $X$ and thus produces a functor

$$\mathbf{Sh}_C^G : \mathbf{GCoarse} \to \mathbf{CAT}_{\infty, \ast}^{\text{Lex}}.$$ 

Morphisms between sheaves are local on $X$. So the functor $\mathbf{Sh}_C^G$ on $\mathbf{GCoarse}$ is far from being coarsely invariant.

We will introduce morphisms which propagate in the $X$-direction by performing a localisation in the realm of left-exact $\infty$-categories (Definition 7.37). If $V$ is an invariant entourage of $X$ containing the diagonal, then we can define a $G$-equivariant functor of posets (see (2.9) for details) and a natural transformation

$$V(-) : \mathcal{P}_X \to \mathcal{P}_X, \quad V(-) \to \text{id}.$$ 

The induced functor $V_*$ on presheaves preserves sheaves and descends to an endofunctor $V_*^G$ on $\mathbf{Sh}_C^G(X)$. We now form the labelled object $(\mathbf{Sh}_C^G(X), W_X)$ of $\mathbf{CAT}_{\infty, \ast}^{\text{Lex}}$, where $W_X$ is generated by the morphisms $M \to V_*^G M$ for all $M$ in $\mathbf{Sh}_C^G(X)$. We then define the object

$$\tilde{V}_C^G(X) := W_X^{-1} \mathbf{Sh}_C^G(X)$$

by localisation in $\mathbf{CAT}_{\infty, \ast}^{\text{Lex}}$ (Definition 2.75). Some effort is needed to show that this construction is covariantly functorial for $X$ in $\mathbf{GCoarse}$ and $C$ in $\mathbf{Fun}(BG, \mathbf{Cat}_{\infty, \ast}^{\text{Lex}})$ and finally leads to the functor $\tilde{V}_C^G$ from (2.86). It is important to observe that the construction also has a contravariant functoriality for a restricted class of morphisms in $\mathbf{GCoarse}$ called coarse coverings (Definition 2.23 and Lemma 2.85). We now have a functor

$$\tilde{V}_C^G : \mathbf{GCoarse} \to \mathbf{CAT}_{\infty, \ast}^{\text{Lex}}.$$
It is excisive (in an appropriate sense) and coarsely invariant, but its values are still too large.

Using the forgetful functor \( \text{GBornCoarse} \to \text{GCoarse} \) we can view \( \hat{\mathcal{V}}^G_C \) as a functor on \( \text{GBornCoarse} \) (Definition 2.35). For \( X \) in \( \text{GBornCoarse} \) we now use the bornology \( \mathcal{B}_X \) on \( X \) in order to define a full subcategory

\[
\mathcal{V}^G_C(X) \subseteq \hat{\mathcal{V}}^G_C(X)
\]

of objects represented by equivariantly small sheaves. In the non-equivariant case, the natural condition on a sheaf to be small is that it sends the bounded subsets of \( X \) (i.e., the elements of the bornology \( \mathcal{B}_X \)) to cocompact objects in \( C \). This assumption is not sufficient in the equivariant setting. For example, we would like \( \mathcal{V}^G_C(G_{\text{can, min}}) \) to be equivalent, at least up to idempotent completion, to the category \( C^{G, \omega} \) via the global sections functor. Unless we explicitly require that the evaluation of a sheaf on a \( G \)-bounded subset, i.e., the \( G \)-orbit of a bounded subset, is cocompact in \( C \), the image of the global sections functor will not even be contained in \( C^{G, \omega} \). Since the condition must also be compatible with the contravariant functoriality for coverings, we are forced to require that an equivariant small sheaf evaluates to cocompact objects in \( C \) on \( H \)-bounded subsets for all subgroups \( H \) of \( G \) (see Proposition 2.67 in particular). This construction finally leads to the functor

\[
\mathcal{V}^G_C : \text{GBornCoarse} \to \text{Cat}^\text{Lex}_\infty, *
\]

We now obtain our first version (Definition 4.19) of a functor of equivariant \( X \)-controlled objects in \( C \)

\[
\mathcal{V}^{G,c}_C : \text{GBornCoarse} \to \text{Cat}^\text{Lex}_\infty, *
\]

by forcing continuity Definition 4.4 on \( \mathcal{V}^G_C \). Essentially, this means that we force the value of the functor on a \( G \)-bornological coarse space \( X \) to be determined by its values on locally finite subsets of \( X \).

In order to get the second version, we first apply the construction above to the trivial group leading to \( \mathcal{V}^c_C \). If we apply the functor to \( X \) in \( \text{GBornCoarse} \), then by functoriality we get an object \( \mathcal{V}^c_C(X) \) in \( \text{Fun}(BG, \text{Cat}^\text{Lex}_\infty, *) \) and set

\[
\mathcal{V}^c_{C,G}(X) := \colim_{BG} \mathcal{V}^c_C(X).
\]

This yields a functor

\[
\mathcal{V}^c_{C,G} : \text{GBornCoarse} \to \text{Cat}^\text{Lex}_\infty, *
\]

The properties of both functors \( \mathcal{V}^{G,c}_C \) and \( \mathcal{V}^c_{C,G} \) are stated in Corollaries 4.20 and 4.21. Upon composition with the algebraic \( K \)-theory functor, they yield equivariant coarse homology theories

\[
K\mathcal{X}^G_C := K \circ \mathcal{V}^{G,c}_C : \text{GBornCoarse} \to \text{Sp},
\]

see Corollary 6.32, and

\[
K\mathcal{X}^G_{C,G} := K \circ \mathcal{V}^c_{C,G} : \text{GBornCoarse} \to \text{Sp},
\]

see Corollary 6.34. Each of these theories features some additional properties which for example enable us to prove Theorem 1.5.
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2. Sheaves on bornological coarse spaces

We adopt the following notational conventions throughout the entire article: $\text{Cat}^{\text{LEX}}_{\infty,*}$ denotes the subcategory of the very large $\infty$-category of large pointed $\infty$-categories whose objects are opposites of pointed, compactly generated presentable $\infty$-categories, and whose morphisms are right adjoint functors whose left adjoints commute with cofiltered limits (so that taking opposite categories induces an equivalence $\text{Pr}^{\text{LEX}}_{\infty,*} \simeq \text{Cat}^{\text{LEX}}_{\infty,*}$). Furthermore, $\text{Cat}^{\text{LEX}}_{\infty,*}$ denotes the large $\infty$-category of small pointed, left-exact $\infty$-categories and left-exact functors. For more details and further explanations, see Section 7.1.

2.1. Presheaves. Let $X$ be a set. By $\mathcal{P}_X$ we denote the poset of subsets of $X$ with the inclusion relation. For $C$ in $\text{Cat}^{\text{LEX}}_{\infty,*}$ we consider the functor category

\[ \text{PSh}_C(X) := \text{Fun}(\mathcal{P}_X^\text{op}, C) \]

called the $\infty$-category of $C$-valued presheaves on $X$. It is again an object of $\text{Cat}^{\text{LEX}}_{\infty,*}$.

A map of sets $f: X \to X'$ gives rise to the inverse image map $f^{-1}(-): \mathcal{P}_{X'} \to \mathcal{P}_X$ of posets. By precomposition it induces a morphism

\[ \hat{f}_*: \text{PSh}_C(X) \to \text{PSh}_C(X') \]

in $\text{Cat}^{\text{LEX}}_{\infty,*}$ between the presheaf categories.

A morphism $\phi: C \to C'$ in $\text{Cat}^{\text{LEX}}_{\infty,*}$ gives rise to a morphism

\[ \hat{\phi}_*: \text{PSh}_C(X) \to \text{PSh}_{C'}(X) \]

in $\text{Cat}^{\text{LEX}}_{\infty,*}$ by postcomposition with $\phi$. These constructions can be turned into a functor

\[ \text{PSh}: \text{Set} \times \text{Cat}^{\text{LEX}}_{\infty,*} \to \text{Cat}^{\text{LEX}}_{\infty,*}. \]

By taking images, the map $f$ also induces the morphism of posets $f(-): \mathcal{P}_X \to \mathcal{P}_{X'}$. The relations $f(f^{-1}(Y')) \subseteq Y'$ for all $Y'$ in $\mathcal{P}_{X'}$, and $Y \subseteq f^{-1}(f(Y))$ for all $Y$ in $\mathcal{P}_X$ provide the counit and the unit of an adjunction

\[ f(-): \mathcal{P}_X \Rightarrow \mathcal{P}_{X'}: f^{-1}(-) \]

between poset morphisms. We get an induced adjunction

\[ \hat{f}^*: \text{PSh}_C(X') \Rightarrow \text{PSh}_C(X): \hat{f}_* \]

of functors between the presheaf categories, where $\hat{f}^*$ is given by precomposition with $f(-)$. It is also a morphism in $\text{Cat}^{\text{LEX}}_{\infty,*}$.

If $\phi: C \to C'$ is a morphism in $\text{Cat}^{\text{LEX}}_{\infty,*}$, then $\hat{\phi}_*$ is a morphism in $\text{Cat}^{\text{LEX}}_{\infty,*}$ and therefore fits into an adjunction

\[ \hat{\phi}^*: \text{PSh}_{C'}(X) \Rightarrow \text{PSh}_C(X): \hat{\phi}_*, \]
where \( \hat{\phi}^* \) preserves cofiltered limits (Example 7.8).

Consider a subset \( V \) of \( X \times X \) (called entourage) such that \( \text{diag}(X) \subseteq V \). For every \( Y \) in \( P_X \) we define the \( V \)-thickening

\[
V[Y] := \{ x \in X \mid (\exists y \in Y \mid (x, y) \in V) \}
\]

and the \( V \)-thinning

\[
V(Y) := \{ x \in X \mid V[\{x\}] \subseteq Y \} .
\]

Thickening and thinning induce morphisms of posets

\[
V[-], V(-): P_X \to P_X .
\]

We define the morphisms

\[
V^*, V_*: \mathbf{PSh}_C(X) \to \mathbf{PSh}_C(X)
\]

through precomposition with \( V[-] \) and \( V(-) \), respectively. The relations \( Y \subseteq V(V[Y]) \) and \( V[V(Y)] \subseteq Y \) for all \( Y \) in \( P_X \) provide the unit and counit of an adjunction

\[
V[-]: P_X \rightleftarrows P_X : V(-)
\]

between endofunctors of \( P_X \). We therefore get an induced adjunction

\[
V^*: \mathbf{PSh}_C(X) \rightleftarrows \mathbf{PSh}_C(X) : V_*
\]

between the presheaf categories.

Let \( G \) be a group and let \( G\text{-Set} := \mathbf{Fun}(BG, \text{Set}) \) be the category of \( G \)-sets. The functor \( \mathbf{PSh} \) from (2.3) induces a functor \( \mathbf{PSh}^G \) defined as the following composition:

\[
\mathbf{PSh}^G: G\text{-Set} \times \mathbf{Fun}(BG, \mathbf{Cat}^\text{LEX}_{\infty, *}) \xrightarrow{\mathbf{PSh}} \mathbf{Fun}(BG \times BG, \mathbf{Cat}^\text{LEX}_{\infty, *}) \xrightarrow{\text{diag}_{BG}} \mathbf{Fun}(BG, \mathbf{Cat}^\text{LEX}_{\infty, *}) \xrightarrow{\text{lim}_{BG}} \mathbf{Cat}^\text{LEX}_{\infty, *} .
\]

The first morphism is given by postcomposition with \( \mathbf{PSh} \), the functor \( \text{diag}_{BG}: BG \to BG \times BG \) is the diagonal embedding, and the limit over \( BG \) exists since \( \mathbf{Cat}^\text{LEX}_{\infty, *} \) is complete (Example 7.8). We write

\[
\mathbf{PSh}_C^G: G\text{-Set} \to \mathbf{Cat}^\text{LEX}_{\infty, *}
\]

for the specialisation of the functor \( \mathbf{PSh}^G \) from (2.12) at \( C \) in \( \mathbf{Fun}(BG, \mathbf{Cat}^\text{LEX}_{\infty, *}) \).

In the following, we repeatedly use that a \( G \)-equivariant adjunction induces an adjunction after passing to the limit over \( BG \).

If \( f: X \to X' \) is a morphism in \( G\text{-Set} \), and if \( C \) is in \( \mathbf{Fun}(BG, \mathbf{Cat}^\text{LEX}_{\infty, *}) \), then by passing to the limit over \( BG \) the adjunction (2.5) induces an adjunction

\[
\tilde{f}^*, \tilde{f}_*: \mathbf{PSh}_{C}(X') \rightleftarrows \mathbf{PSh}_{C}(X) : \tilde{f}^* , \tilde{f}_* .
\]

Similarly, if \( X \) is in \( G\text{-Set} \) and the entourage \( V \) is \( G \)-invariant, then the adjunction (2.11) induces an adjunction

\[
V^*, V_*: \mathbf{PSh}_C^G(X) \rightleftarrows \mathbf{PSh}_C^G(X) : V_* .
\]
Finally, if $\phi : C \to C'$ is a morphism in $\Fun(BG, \Cat_{\infty, s}^{\LEX})$, then the adjunction (2.6) induces an adjunction

$$
\phi^* \phi' : \Sh_{\infty}^{\LEX}(C) \cong \Sh_{\infty}^{\LEX}(C') \phi^* \phi'.
$$

The right adjoints $f^*_s$, $V^*_s$, and $\phi^*_s$ in these adjunctions are morphisms in $\Cat_{\infty, s}^{\LEX}$, and their left adjoints $\phi_*^G$, $V_*^G$, and $\phi_*^G$ preserve cofiltered limits.

2.2. Sheaves. Let $X$ be a set with an entourage $U$.

**Definition 2.1.** A subset $B$ of $X$ is called $U$-bounded if $B \times B \subseteq U$. ♣

Let $Y$ be in $\mathcal{P}_X$, and let $\mathcal{Y}$ be a family of subsets of $Y$.

**Definition 2.2.** The family $\mathcal{Y}$ is a $U$-covering family of $Y$ if for every $U$-bounded subset $B$ of $X$ there exists a member $Y$ of $Y$ such that $B \subseteq Y$. ♣

The collections of $U$-covering families of the elements of $\mathcal{P}_X$ determines a Grothendieck topology $\tau^U$ on $\mathcal{P}_X$. For $C$ in $\Cat_{\infty, s}^{\LEX}$, we let $\Sh_{\infty}^{\LEX}(X)$ denote the full subcategory of $\Sh_{\infty}^{\LEX}(X)$ of $i^U$-sheaves (we will also say $U$-sheaves).

**Remark 2.3.** Let $Y$ be in $\mathcal{P}_X$ and $\mathcal{Y}$ be a $U$-covering family. Then we consider the associated sieve $S_Y$. It is the full subcategory of $(\mathcal{P}_X)/_Y$ consisting of objects $Z \to Y$ such that $Z$ is contained in a member of $\mathcal{Y}$.

An object $M$ in $\Sh_{\infty}^{\LEX}(X)$ is a $U$-sheaf if and only if the canonical morphism

$$
M(Y) \to \lim_{(Z \to Y) \in (S_Y)^{op}} M(Z)
$$

is an equivalence for all $Y$ in $\mathcal{P}_X$ and all $U$-covering families $\mathcal{Y}$ of $Y$. ♣

The covering family of $Y$ consisting of all $U$-bounded subsets refines every other $U$-covering family. As a consequence it is easy to construct the $U$-sheafification functor.

Let $i : \mathcal{P}_{X}^{\bset} \to \mathcal{P}_X$ be the inclusion of the sub-poset of $\mathcal{P}_X$ of $U$-bounded elements. Then we have an adjunction

$$
i^* : \Sh_{\infty}^{\LEX}(X) \to \Fun(\mathcal{P}_{X}^{\bset, op}, C) : i_* ,
$$

where $i^*$ is the restriction, and $i_*$ is the right Kan extension functor along $i$ (which exists since $C$ is complete). We set

$$
L^U := i_* i^* .
$$

**Lemma 2.4.** We have an adjunction

$$
L^U : \Sh_{\infty}^{\LEX}(X) \cong \Sh_{\infty}^{\LEX}(X) : \text{incl} .
$$

Moreover, $L^U$ preserves small limits.

**Proof.** The unit of the adjunction (2.16) provides a morphism $\text{id} \to L^U$. We must show that this transformation presents $L^U$ as the $U$-sheafification functor.

Since $i$ is fully faithful, the counit of the adjunction (2.16) is an equivalence $i^* i_* \simeq \text{id}$. This implies that $L^U L^U \simeq L^U$. It remains to show that $L^U$ takes values in $U$-sheaves, and that for any $U$-sheaf $M$ the morphism $M \to L^U M$ is an equivalence.

Let $M$ be in $\Sh_{\infty}^{\LEX}(X)$. We must show that $L^U M$ is a $U$-sheaf. Since the covering family by $U$-bounded subsets refines every other $U$-covering family it suffices to check the sheaf condition for the sieves associated to this family. Thus let $Y$ be in
\( \mathcal{P}_X \). The sieve associated to the covering family of all \( U \)-bounded subsets is the slice category \( (\mathcal{P}_X^{(U \text{bd})})/Y \) (Remark 2.3). We must show that

\[
(L^U M)(Y) \to \lim_{B \in ((\mathcal{P}_X^{(U \text{bd})})/Y)_{U}} (L^U M)(B)
\]

is an equivalence. Using the pointwise formula for the right Kan extension in order to express \( L^U M \) this morphism is equivalent to the morphism

\[
\lim_{B \in ((\mathcal{P}_X^{(U \text{bd})})/Y)_{U}} M(B) \to \lim_{B \in ((\mathcal{P}_X^{(U \text{bd})})/Y)_{U}} \lim_{B' \in ((\mathcal{P}_X^{(U \text{bd})})/Y)_{U}} M(B') .
\]

We now observe that for \( B \) in \( (\mathcal{P}_X^{(U \text{bd})})/Y \) the final object of \( (\mathcal{P}_X^{(U \text{bd})})/B \) is \( B \). Consequently, we can replace the inner limit in the target of (2.20) by the evaluation at \( B \), and then (2.20) becomes the identity. This shows that the morphism (2.19) is an equivalence.

We now consider a \( U \)-sheaf \( M \) and show that \( M \to L^U M \) is an equivalence. Indeed, the evaluation of this morphism at \( Y \) in \( \mathcal{P}_X \) is equivalent to

\[
M(Y) \to \lim_{B \in ((\mathcal{P}_X^{(U \text{bd})})/Y)_{U}} M(B)
\]

which is an equivalence by the sheaf condition.

Since the functors \( i^* \) and \( i_* \) preserve small limits, so does \( L^U \).

The full subcategory \( \text{Sh}_C(X) \) of \( \text{PSh}_C(X) \) is closed under small limits and will be considered as an object of \( \text{CAT}^{\text{LEX}}_{\infty,*} \) (Example 7.6).

Let \( \phi: C \to C' \) be a morphism in \( \text{CAT}^{\text{LEX}}_{\infty,*} \).

**Lemma 2.5.** The functor \( \hat{\phi}_* \) from (2.6) preserves \( U \)-sheaves, and we have an adjunction

\[
L^U \hat{\phi}^*: \text{Sh}_C^U(X) \rightleftarrows \text{Sh}_{C'}^U(X) : \hat{\phi}_* .
\]

Moreover, \( L^U \hat{\phi}^* \) preserves small cofiltered limits.

**Proof.** Since \( \phi \) preserves small limits it is clear that \( \hat{\phi}_* \) preserves \( U \)-sheaves. It then follows from Lemma 2.4 and the adjunction (2.6) that the left adjoint is given by the claimed formula. The last assertion follows from this formula and the fact that \( L^U \) and \( \hat{\phi}^* \) preserve small cofiltered limits.

Consider a map of sets \( f: X \to X' \). Let \( U' \) be an entourage of \( X' \) such that \( f(U) \subseteq U' \) (where \( f(U) \) abbreviates \( f \times f)(U) \)).

**Lemma 2.6.** The functor \( \hat{f}_* \) from (2.5) sends \( U \)-sheaves to \( U' \)-sheaves, and we have an adjunction

\[
L^U \hat{f}^*: \text{Sh}_C^U(X') \rightleftarrows \text{Sh}_{C'}^U(X) : \hat{f}_* .
\]

Moreover, \( L^U \hat{f}^* \) preserves small limits.

**Proof.** The map \( f^{-1}(-): \mathcal{P}_{X'} \to \mathcal{P}_X \) sends \( U' \)-covering families to \( U \)-covering families. Indeed, let \( Y' \) be a \( U' \)-covering family of \( Y' \) in \( \mathcal{P}_{X'} \), and consider a \( U' \)-bounded subset \( B \) of \( f^{-1}(Y') \). Then \( f(B) \) is a \( U \)-bounded subset of \( Y' \) and hence contained in a member of \( Y' \). In view of (2.4) this implies that \( B \) is contained in a member of \( f^{-1}(Y') \).

---

1In order to shorten the notation we denote the objects of \( (\mathcal{P}_X^{(U \text{bd})})/Y \) by \( B \) instead of \( B \to Y \).
We now conclude that \( \hat{f} \) sends \( U \)-sheaves to \( U' \)-sheaves. To this end, we consider \( M \) in \( \text{Sh}_C^U(X) \), an element \( Y' \) of \( \mathcal{P}_{X'} \), and a \( U' \)-covering family \( \mathcal{V}' \) of \( Y' \). The functor \( f^{-1}(\cdot): S_{Y'} \to S_{f^{-1}(Y')} \) induces the second of the following chain of morphisms:

\[
M(f^{-1}(Y')) \to \lim_{Z \in (S_{f^{-1}(Y')})^{\text{op}}} M(Z) \\
\to \lim_{Z' \in (S_{Y'})^{\text{op}}} M(f^{-1}(Z'))
\]

The first map is an equivalence as a consequence of the \( U \)-sheaf condition on \( M \) since \( f^{-1}(\mathcal{V}') \) is a \( U' \)-covering family. The adjunction (2.4) induces an adjunction \( f^{-1}: S_{Y'}^{\text{op}} \cong S_{f^{-1}(Y')}^{\text{op}} : f(\cdot) \). Since left adjoints are final, this shows that the second map is also an equivalence. Consequently, the composition of both maps is an equivalence. This verifies the sheaf condition for \( \hat{f} \), \( M \) on \( Y' \) and the \( U' \)-covering family \( \mathcal{V}' \).

It is now clear that the left adjoint is given by the claimed formula. The last assertion follows from this formula and the fact that \( L^U \) and \( \hat{f}^* \) preserve small limits. \( \square \)

The inverse of an entourage \( V \) and the composition of entourages \( U, V \) of \( X \) are defined by

\[
(2.23) \quad V^{-1} := \{(y, x) \in X \times X | (x, y) \in V\}
\]

and

\[
(2.24) \quad UV := \{(x'', x) \in X \times X | (\exists x' \in X \mid (x'', x') \in U \land (x', x) \in V)\}.
\]

Let \( V \) be an entourage of \( X \) with \( \text{diag}(X) \subseteq V \) and assume that \( U' \) is an entourage of \( X \) such that \( VUV^{-1} \subseteq U' \).

**Lemma 2.7.** The functor \( V_* \) from (2.11) sends \( U \)-sheaves to \( U' \)-sheaves, and we have an adjunction

\[
(2.25) \quad L^U V^* : \text{Sh}_C^U(X) \rightleftarrows \text{Sh}_C^{U'}(X) : V_*.
\]

Moreover, \( L^U V^* \) preserves small limits.

**Proof.** As in the proof of **Lemma 2.6**, the second assertion is a formal consequence of the first. To prove the first assertion, one first checks that \( V(\cdot) \) sends \( U' \)-covering families to \( U \)-covering families. Let \( Y \) be in \( \mathcal{P}_{X} \), and let \( \mathcal{V}' \) be a \( U' \)-covering family of \( Y \). We consider a \( U \)-bounded subset \( B \) of \( V(Y) \). Then \( U[B] \) is \( VUV^{-1} \)-bounded, so in particular \( U' \)-bounded and therefore contained in a member of \( \mathcal{V}' \). In view of (2.10) this implies that \( B \) is contained in a member of \( U(\mathcal{V}') \). We now argue as in the proof of **Lemma 2.6** with the adjunction (2.4) replaced by (2.10) that \( V_* \) sends \( U \)-sheaves to \( U' \)-sheaves. \( \square \)

Let \( G \) be a group, let \( X \) be in \( G\text{-Set} \), and let \( C \) be in \( \text{Fun}(C, \text{Cat}^{\text{LEX}}_{\infty, *}) \). Assume that \( U \) is a \( G \)-invariant entourage of \( X \). Under this condition the action of \( G \) on \( X \) preserves \( U \)-sheaves by **Lemma 2.6**. We define

\[
(2.26) \quad \text{Sh}_C^{U,G}(X) := \lim_{BG} \text{Sh}_C^U(X),
\]

where the limit is interpreted in \( \text{CAT}_{\infty} \). Recall the definition of \( \text{CAT}^{\text{LEX}}_{\infty, *} \) from **Example 7.6**.

**Lemma 2.8.** We have \( \text{Sh}_C^{U,G}(X) \in \text{CAT}^{\text{LEX}}_{\infty, *} \).
Proof: We must show that \( \mathbf{Sh}^U_C(X) \) admits small limits. Note that \( \mathbf{Sh}_C^U(X) \) is a \( BG \)-indexed diagram of full subcategories of the diagram \( \mathbf{PSh}_C(X) \) in the functor category \( \mathbf{Fun}(BG, \mathbf{Cat}_{\mathbf{LEX}}^\infty) \). By Lemma 7.12 we conclude that \( \mathbf{Sh}_C^U(X) \) is a full subcategory of \( \mathbf{PSh}^G_C(X) \). It consists of all presheaves whose evaluations (i.e., the image under \( e \) and the relation \( \hat{L} \)) Moreover, we conclude the assertion.

As seen in the proof of Lemma 2.8, \( \mathbf{Sh}_C^U(X) \) is a full subcategory of \( \mathbf{PSh}_C^G(X) \).

**Corollary 2.9.** The adjunction (2.18) induces an adjunction

\[
L^U_G : \mathbf{PSh}_C^G(X) \rightleftarrows \mathbf{Sh}_C^U(X) : \text{incl}.
\]

Moreover, \( L^U_G \) preserves small limits.

Let \( \phi : C \to C' \) be a morphism in \( \mathbf{Fun}(BG, \mathbf{Cat}_{\mathbf{LEX}}^\infty, \mathbf{LEX}) \), and let \( U \) be an invariant entourage of \( X \).

**Corollary 2.10.** The adjunction (2.21) induces an adjunction

\[
(2.27) \quad L^U_G \hat{\phi}^*: \mathbf{Sh}_C^U(X) \rightleftarrows \mathbf{Sh}_C^U(X) : \hat{\phi}^*.
\]

Moreover, \( L^U_G \hat{\phi}^* \) preserves small cofiltered limits.

Let \( V \) be an invariant entourage of \( X \) such that \( \text{diag}(X) \subseteq V \). Then \( V \) and \( V^* \) in (2.25) are equivariant. Assume that \( U' \) is an invariant entourage of \( X \) which in addition satisfies \( VUV^{-1} \subseteq U' \).

**Corollary 2.11.** The adjunction (2.25) induces an adjunction

\[
L^U_G V^*: \mathbf{Sh}_C^{U'}(X) \rightleftarrows \mathbf{Sh}_C^U(X) : V^*.
\]

Moreover, \( L^U_G V^* \) preserves small limits.

Assume that \( f : X \to X' \) is a map between \( G \)-sets, and that \( U' \) is an invariant entourage of \( X' \) such that \( f(U) \subseteq U' \).

**Corollary 2.12.** The adjunction (2.22) induces an adjunction

\[
(2.28) \quad L^U_G \tilde{f}^* : \mathbf{Sh}_C^{U'}(X') \rightleftarrows \mathbf{Sh}_C^U(X) : \tilde{f}^*.
\]

Moreover, \( L^U_G \tilde{f}^* \) preserves limits.

Let \( X \) be in \( \mathbf{GSet} \) and \( i : Y \to X \) be the inclusion of an invariant subset. Let \( U \) be an invariant entourage of \( X \) and set \( U_Y := (Y \times Y) \cap U \).

**Lemma 2.13.** We have an adjunction

\[
\tilde{g}^*: \mathbf{Sh}_C^{U_Y}(X) \rightleftarrows \mathbf{Sh}_C^{U_Y}(Y) : \tilde{g}^*
\]

and the relation \( \tilde{g}^* i^*_U \simeq \text{id} \).
Proof. We have $i(U_Y) \subseteq U$ and can therefore apply Corollary 2.12 to $i$ in place of $f$ and $U_Y$ in place of $U'$. In order to remove the application of the sheafification functor $L^U_{G}$ on the left adjoint side it then suffices to observe that $\hat{i}^{*} \iota_{*}$ obviously sends $U$-sheaves to $U_{Y}$-sheaves. Since $i^{-1}(-) \circ i(-) = \text{id}$ on $\mathcal{P}_{Y}$ we get the relation $\hat{i}^{*} \iota_{*} \simeq \text{id}$ on $\text{PSh}_{\mathcal{C}}$ which induces the desired relation $\hat{i}^{*} \iota_{*} \simeq \text{id}$ by applying $\lim_{BG}$ and restricting to sheaves. □

Let $X$ be in $G\text{Set}$ with invariant subsets $Y$ and $Z$ such that $Y \cup Z = X$. Then we have the following inclusions:

$$Y \cap Z \xrightarrow{l} Y \xrightarrow{m} Z \quad \xrightarrow{k} \quad \xrightarrow{j} X$$

Let $U$ be an invariant entourage of $X$ and consider $M$ in $\text{Sh}^U_{\mathcal{C}}(X)$. In the square below, the morphisms are the units of the adjunctions $(\hat{i}^{*}, \iota_{*})$ etc.

**Lemma 2.14** (Glueing Lemma). If $(Y, Z)$ is a $U$-covering family of $X$, then we have a cartesian square

$$M \quad \xrightarrow{\hat{j}_{*} \iota^{*} \mathcal{G}} \quad \xrightarrow{\hat{k}_{*} \iota^{*} \mathcal{G}} M$$

in $\text{Sh}^U_{\mathcal{C}}(X)$.

Proof. The filler of the square is obtained from the equality $im = k = jl$. The evaluation $\text{Sh}^U_{\mathcal{C}}(X) \to \text{Sh}^U_{\mathcal{C}}(X)$ detects cartesian squares by Lemma 7.13. So it suffices to check the assertion in the non-equivariant case.

Since (2.30) is a square of $U$-sheaves (this follows from Lemma 2.13) it suffices to check that the evaluation of the square (2.30) at every $U$-bounded $B$ in $\mathcal{P}_{X}$ is cartesian. Since $(Y, Z)$ is a $U$-covering, $B$ is contained in one of $Y$ or $Z$. We consider the case that $B \subseteq Y$ (the case $B \subseteq Z$ is analogous). The evaluation of (2.30) at $B$ is the square in $\mathcal{C}$

$$M(B) \xrightarrow{\simeq} M(B) \xrightarrow{i} M(Z \cap B) \xrightarrow{\simeq} M(Z \cap B)$$

which is obviously cartesian. □

2.3. Sheaves on $G\text{Coarse}$. In Section 2.2, we considered $\mathcal{C}$-valued sheaves as a functor on pairs of a $G$-set equipped with an invariant entourage. In the present section, we get rid of the explicit choice of an entourage by equipping the $G$-set with a whole collection of such entourages called a coarse structure and considering the union of the sheaf categories for all these coarse entourages. In this way we eventually obtain a functor of $\mathcal{C}$-valued sheaves on the category $G\text{Coarse}$ of $G$-coarse spaces.

Let $X$ be in $G\text{Set}$.
Definition 2.15. A $G$-coarse structure on $X$ is a subset $C_X$ of $P_{X \times X}$ satisfying the following conditions:

1. $C_X$ is $G$-invariant.
2. $\text{diag}(X) \in C_X$.
3. $C$ is closed under forming subsets, finite unions, inverses (see (2.23)) and compositions (see (2.24)).
4. The sub-poset of $G$-invariants $C^G_X$ is cofinal in $C_X$.

The elements of $C_X$ will be called coarse entourages of $X$. We will often use the notation $C^G_X, \Delta_X$ for the sub-poset of $C_X$ of invariant coarse entourages containing the diagonal.

Example 2.16. If $X$ is a $G$-set, then we can consider the minimal and the maximal $G$-coarse structures $X_{\text{min}}$ and $X_{\text{max}}$ on $X$. Here $C_{X_{\text{min}}}$ consists of the subsets of $\text{diag}(X)$ and $C_{X_{\text{max}}} = P_{X \times X}$.

Consider two $G$-sets with $G$-coarse structures $(X, C_X)$ and $(X', C_{X'})$ and a morphism $f : X \to X'$ in $G\text{Set}$.

Definition 2.17. The map $f$ is controlled if $f(C_X) \subseteq C_{X'}$.

Definition 2.18. A $G$-coarse space is a pair $(X, C_X)$ (usually denoted just by $X$) of a $G$-set with a $G$-coarse structure. A morphism between $G$-coarse spaces is an equivariant controlled map.

The category $G\text{Coarse}$ of $G$-coarse spaces and controlled maps is complete and cocomplete [BEKWa, Prop. 2.18 and 2.21], and the forgetful functor to $G\text{Set}$ preserves limits and colimits since it has a left adjoint $X \mapsto X_{\text{min}}$ and a right adjoint $X \mapsto X_{\text{max}}$.

Using precomposition with the forgetful functor $G\text{Coarse} \to G\text{Set}$, the functor $\text{PSh}^G$ from (2.12) induces a functor (denoted by the same symbol)

\[(2.31) \quad \text{PSh}^G : G\text{Coarse} \times \text{Fun}(BG, \text{Cat}_{\infty,*}^\text{LEX}) \to \text{Cat}_{\infty,*}^\text{LEX}.\]

Let $X$ be in $G\text{Coarse}$. Consider the invariant entourage

\[(2.32) \quad U(\pi_0(X)) := \bigcup_{U \in C_X} U.\]

This entourage is an invariant equivalence relation on $X$.

Definition 2.19. The $G$-set of equivalence classes $\pi_0(X)$ with respect to $U(\pi_0(X))$ is called the set of coarse components of $X$.

Example 2.20. For a $G$-set $X$ we have $\pi_0(X_{\text{min}}) \cong X$ and $\pi_0(X_{\text{max}}) \cong \ast$. ♦

Remark 2.21. If $Y$ is in $P_X$, we can consider $Y$ with the induced coarse structure, and then take $\pi_0(Y)$ in the sense of Definition 2.19 for the trivial group. Alternatively, we can consider the subset

\[\{Z \in \pi_0(X) \mid Z \cap Y \neq \emptyset\}\]

of $\pi_0(X)$. Both constructions give canonically isomorphic sets. The latter description shows that $\pi_0(Y)$ is a $G$-invariant subset of $\pi_0(X)$ if $Y$ is a $G$-invariant subset. ♦
For $\mathbf{C}$ in $\Fun(BG, \text{Cat}_{\infty,*}^{\text{LEX}})$ we will abbreviate $\text{Sh}_{\mathbf{C}}^{U(\pi_0(X)), G}$ (see (2.26)) by $\text{Sh}_{\mathbf{C}}^{\pi_0, G}(X)$ and $L^{U(\pi_0(X)), G}$ (see Corollary 2.9) by $L^{\pi_0, G}$.

If $f: X \to X'$ is a morphism in $G\text{Coarse}$, then $f(U(\pi_0(X)) \subseteq U(\pi_0(X'))$. As a consequence of Corollary 2.12, we get:

**Corollary 2.22.** We have an adjunction

$$L^{\pi_0, G}f^*: \text{Sh}_{\mathbf{C}}^{\pi_0, G}(X') \rightleftarrows \text{Sh}_{\mathbf{C}}^{\pi_0, G}(X): f^*_G$$

Moreover, $L^{\pi_0, G}f^*$ preserves small limits.

Moreover, using in addition the existence of the right adjoints in (2.27), we obtain a subfunctor

$$\text{Sh}^{\pi_0, G}: G\text{Coarse} \times \Fun(BG, \text{Cat}_{\infty,*}^{\text{LEX}}) \to \text{CAT}_{\infty,*}^{\text{LEX}}$$

of the functor $\text{PSh}^{G}$ from (2.31).

If $U$ and $U'$ are invariant entourages of $X$ such that $U \subseteq U'$, then every $U'$-covering family is a $U$-covering family (Definition 2.2). Consequently, we get an inclusion $\text{Sh}_{\mathbf{C}}^{U, G}(X) \to \text{Sh}_{\mathbf{C}}^{U', G}(X)$. We define the $\infty$-category

$$\text{Sh}_{\mathbf{C}}^{G}(X) := \text{colim}_{U \in \mathcal{C}^G_X} \text{Sh}_{\mathbf{C}}^{U, G}(X).$$

The objects of $\text{Sh}_{\mathbf{C}}^{G}(X)$ are called sheaves. The filtered colimit is interpreted in $\text{CAT}_{\infty}$ and actually defines an object of $\text{CAT}_{\infty,*}^{\text{LEX}}$ (Example 7.5).

As a consequence of Corollary 2.12 and Corollary 2.10, we get a subfunctor

$$\text{Sh}^{G}: G\text{Coarse} \times \Fun(BG, \text{Cat}_{\infty,*}^{\text{LEX}}) \to \text{CAT}_{\infty,*}^{\text{LEX}}$$

of the functor $\text{Sh}^{\pi_0, G}$ from (2.34).

In general we do not expect that for a morphism $f: X \to X'$ in $G\text{Coarse}$ the morphism $f^*: \text{Sh}_{\mathbf{C}}^{G}(X) \to \text{Sh}_{\mathbf{C}}^{G}(X')$ has a left adjoint. The reason is that the left adjoint in the adjunction (2.28) explicitly depends on the entourage $U$. But we have such left adjoints for a special sort of morphisms in $G\text{Coarse}$ called coarse coverings.

Let $f: X \to X'$ be a morphism in $G\text{Coarse}$.

**Definition 2.23.** The morphism $f$ is called a coarse covering if it satisfies the following conditions:

1. The restriction $f_Y: Y \to f(Y)$ to every coarse component $Y$ of $X$ is an isomorphism in $G\text{Coarse}$ of coarse components.

2. The $G$-coarse structure of $X$ is generated by the entourages $f^{-1}(U') \cap U(\pi_0(X))$ for all $U'$ in $\mathcal{C}_{X'}$.

**Example 2.24.** If $W$ is a $G$-set, then the projection $W_{\text{min}} \otimes X \to X$ is a coarse covering. Here $\otimes$ is the cartesian product in $G\text{Coarse}$ and $W_{\text{min}}$ is as in Example 2.16.

Since for every $U$ in $\mathcal{C}_{X}$ we have $U \subseteq U(\pi_0(X))$, we have an inclusion $\text{Sh}_{\mathbf{C}}^{U, G}(X) \subseteq \text{Sh}_{\mathbf{C}}^{\pi_0, G}(X)$. This explains the meaning of the word “restricts” in the following statement.

**Lemma 2.25.** If $f: X \to X'$ is a coarse covering, then the adjunction (2.33) restricts to an adjunction

$$L^{\pi_0, G}f^*: \text{Sh}_{\mathbf{C}}^{G}(X') \rightleftarrows \text{Sh}_{\mathbf{C}}^{G}(X): f^*_G.$$
Proof. First we observe that the non-equivariant case implies the equivariant case by passing to the limit over \( BG \). It then suffices to show that \( L^{\pi_0} \hat{f}^* \) preserves sheaves. Let \( U' \) be in \( C_{X'} \) and \( M \) be in \( Sh^U_{C}(X') \). We will show that \( L^{\pi_0} \hat{f}^* M \in Sh^U_{C}(X) \) for \( U := f^{-1}(U') \cap U(\pi_0(X)) \) which is a coarse entourage of \( X \) by Definition 2.23 (2). It suffices to show that \( (L^{\pi_0} \hat{f}^* M)|_Y \in Sh^U_{C}(Y) \) for every coarse component \( Y \) of \( X \), where \( U_Y := U \cap (Y \times Y) \). We first note that \( (L^{\pi_0} \hat{f}^* M)|_Y \simeq (\hat{f}^* M)|_Y \) by the formula (2.17) for \( L^{\pi_0} \). Since \( f \) restricts to isomorphisms between coarse components (as coarse spaces), the pullback is an equivalence \( \hat{f}^* Y : Sh^U_{C}(f(Y)) \xrightarrow{\simeq} Sh^U_{C}(Y) \). Since \( (\hat{f}^* M)|_Y \simeq \hat{f}^* Y (M_{f(Y)}) \) and \( f(U_Y) = U' \cap (f(Y) \times f(Y)) \), we have \( M_{f(Y)} \in Sh^U_{C}(f(Y)) \). We conclude that \( (\hat{f}^* M)|_Y \in Sh^U_{C}(Y) \). \( \square \)

We now consider a pullback square

\[
\begin{array}{ccc}
Y & \xrightarrow{f'} & Y' \\
\downarrow{g} & & \downarrow{g'} \\
X & \xrightarrow{f} & X'
\end{array}
\]

in \( G\text{Coarse} \).

**Lemma 2.26.** The canonical morphism of functors

\[
L^{\pi_0, G} \hat{g}^*, G \hat{f}^* : Sh^U_{C}(X) \rightarrow Sh^U_{C}(Y')
\]

is an equivalence.

**Proof.** The non-equivariant case implies the equivariant case by passing to the limit over \( BG \).

Using that the underlying square of sets is cartesian, we have for \( Z' \) in \( P_Y \), the relation \( f^{-1}(g'(Z')) = g(f'^{-1}(Z')) \). This immediately implies that the canonical morphism

\[
\hat{g}'^* \hat{f}_* \rightarrow \hat{f}_* \hat{g}^* : PSh_{C}(X) \rightarrow PSh_{C}(Y')
\]

is an equivalence. The morphism in question is given by

\[
L^{\pi_0} \hat{g}'^* \hat{f}_* \simeq L^{\pi_0} \hat{f}_* \hat{g}^* \rightarrow L^{\pi_0} \hat{f}_* L^{\pi_0} \hat{g}^* \leftarrow L^{\pi_0} \hat{f}_* L^{\pi_0} \hat{g}^* ,
\]

where for the last equivalence we employ the fact that \( \hat{f}_* \) preserves \( \pi_0 \)-sheaves by Lemma 2.6. So it remains to show that \( L^{\pi_0} \hat{f}_* \hat{g}^* M \rightarrow L^{\pi_0} \hat{f}_* L^{\pi_0} \hat{g}^* M \) is an equivalence for every \( M \) in \( Sh^U_{C}(Y) \). Using formula (2.17) for \( L^{\pi_0} \), its evaluation on a subset \( Z' \) in \( P_Y \) is given by the morphism

\[
\prod_{D' \in \pi_0(Z')} M(g(f'^{-1}(D')) \rightarrow \prod_{D' \in \pi_0(Z')} \prod_{D \in \pi_0(f'^{-1}(D'))} M(g(D)) ,
\]

which in the factor \( D' \) is induced by the restrictions along the embeddings \( g(D) \rightarrow g(f'^{-1}(D')) \) for all \( D \) in \( \pi_0(f'^{-1}(D')) \). Using that \( M \) is a \( \pi_0 \)-sheaf, we can rewrite the domain of the map in the form

\[
\prod_{D' \in \pi_0(Z')} \prod_{C \in \pi_0(g(f'^{-1}(D')))} M(C) \rightarrow \prod_{D' \in \pi_0(Z')} \prod_{D \in \pi_0(f'^{-1}(D'))} M(g(D)) .
\]

We now argue that for fixed \( D' \) in \( \pi_0(Z') \) the map

\[
\pi_0(f'^{-1}(D')) \rightarrow \pi_0(g(f'^{-1}(D'))), \quad D \mapsto g(D)
\]

is an equivalence.
is a well-defined bijection. Let us first show that \( g(D) \) is a coarse component of \( g(f'^{-1}(D')) \) whenever \( D \) is a coarse component of \( f'^{-1}(D') \). Suppose \( d \) is a point in \( D \) and \( x \) is a point in \( g(f'^{-1}(D')) \) such that \( \{(g(d), x)\} \in \mathcal{C}_X \). Then \( x = g(y) \) for some point \( y \) in \( f'^{-1}(D') \). Since \( D' \) is coarsely connected we have \( \{(f'(d), f'(y))\} \in \mathcal{C}_Y \). We conclude that \( \{(d, y)\} \in \mathcal{C}_Y \) by the characterisation of the entourages in a pullback in \( \mathbf{GCoarse} \). Hence \( y \in D \) and thus \( x \in g(D) \).

Since the map \((2.41)\) is surjective by definition, we only have to check injectivity. Let \( D_0 \) and \( D_1 \) be in \( \pi_0(f'^{-1}(D')) \) such that \( g(D_0) = g(D_1) \). Then there are points \( z_0 \) in \( D_0 \) and \( z_1 \) in \( D_1 \) such that \( \{(g(z_0), g(z_1))\} \in \mathcal{C}_X \). Since we also have \( \{(f'(z_0), f'(z_1))\} \in \mathcal{C}_Y \), it follows that \( \{(z_0, z_1)\} \in \mathcal{C}_Y \), again by the characterisation of entourages in a pullback in \( \mathbf{GCoarse} \).

This implies that the morphism \((2.40)\) is an equivalence. \(\square\)

Let \( i: Y \to X \) be the inclusion of a subspace in \( \mathbf{GCoarse} \). As a consequence of Lemma 2.13 we get:

**Corollary 2.27.** We have an adjunction

\[
(2.42) \quad \hat{\tau}^* G: \mathbf{Sh}^G_C(X) \leftrightarrows \mathbf{Sh}^G_C(Y): \check{\tau}^* G
\]

and the relation \( \hat{\tau}^* G \circ \check{\tau}^* G \cong \text{id} \).

We again consider the pullback square \((2.38)\). If \( g' \) is a coarse covering, then \( g \) is a coarse covering, too (see \[ \text{[BEKwd, Lem. 2.11]} \]). Similarly, if \( g' \) is the inclusion of a subspace, then so is \( g \).

**Corollary 2.28.** If \( g' \) is a coarse covering or an inclusion of a subspace, then the canonical morphism of functors

\[
L^{\pi_0 G} \hat{\tau}^* G f^*_G \to \hat{\tau}^* G L^{\pi_0 G} \hat{\tau}^* G: \mathbf{Sh}^G_C(X) \to \mathbf{Sh}^G_C(Y')
\]

is an equivalence.

**Proof.** If \( g \) is a coarse covering, then we use Lemma 2.26 together with Lemma 2.25. If \( g \) is an inclusion, then we use Corollary 2.27 and the fact that we can drop the application of \( L^{\pi_0} \). \(\square\)

Let \( X \) be in \( \mathbf{GCoarse} \), and let \( V \) be in \( \mathbf{C}_X \). Note that with \( U \) in \( \mathbf{C}_X \) we also have \( VUV^{-1} \in \mathbf{C}_X \) by Definition 2.15 (3). As a consequence of Corollary 2.11, we get:

**Corollary 2.29.** The functor \( V^*_G \) from \((2.13)\) restricts to an endofunctor of \( \mathbf{Sh}^G_C(X) \).

Let \( f: X \to X' \) be a morphism in \( \mathbf{GCoarse} \) and \( V' \) be in \( \mathbf{C}_X \). Then we define the entourage \( V := f^{-1}(V') \cap U(\pi_0(X)) \) of \( X \) (see \((2.32)\)). If \( f \) is a coarse covering, then we have \( V \in \mathbf{C}_X \) by Definition 2.23 (2).

**Lemma 2.30.** If \( f \) is a coarse covering, then we have an equivalence of functors

\[
L^{\pi_0 G} \hat{\tau}^* G V^*_G \cong V^*_G L^{\pi_0 G} \hat{\tau}^* G: \mathbf{Sh}^G_C(X) \to \mathbf{Sh}^G_C(X)
\]

**Proof.** Using that \( f \) is a coarse covering, we see that for every \( Y \) in \( \mathcal{P}_X \) we have \( f(V(Y)) = V'(f(Y)) \). Furthermore, using the explicit formulas, one checks that \( L^{\pi_0 G} \) and \( V^*_G \) commute. This implies the chain of equivalences

\[
L^{\pi_0 G} \hat{\tau}^* G V^*_G \cong L^{\pi_0 G} V^*_G \hat{\tau}^* G \cong V^*_G L^{\pi_0 G} \hat{\tau}^* G.
\]

\(\square\)
Let $i: Y \to X$ be an inclusion of a subspace in \textit{GCoarse}, and let $V$ be in $\mathcal{C}^{G, \Delta}_X$. Then we let $j: V(Y) \to X$ be the inclusion and set $V_Y := V \cap (Y \times Y)$.

**Lemma 2.31.** We have the relation

\begin{equation}
\hat{\gamma}^{*,G} V^G \simeq V_Y \hat{\gamma}^{*,G} \hat{j}^* G ; \, \text{Sh}_C^G(X) \to \text{Sh}_C^G(Y).
\end{equation}

**Proof.** The non-equivariant case implies the equivariant case by passing to the limit over $BG$. For every $Z$ in $\mathcal{P}_Y$ we have the relation $V(i(Z)) = i(V_Y(Z)) \cap V(Y)$. This implies the desired relation between operations on presheaves and therefore on sheaves.

\textbf{2.4. GBornCoarse and equivariantly small sheaves.} The category $\text{Sh}_C^G(X)$ is too large to have interesting $K$-theoretic invariants since it always admits Eilenberg swindles. To remedy this defect, we introduce the concept of equivariantly small sheaves. We equip the coarse spaces with an additional structure, a collection of subsets (called bounded subsets) satisfying the axioms of a bornology. The naive idea would then be to require that a $C$-valued sheaf is equivariantly small if its values on bounded subsets belong to $C^\omega$. This condition leads to a well-behaved coarse homology theory, but this theory does not take the “correct” values; more specifically, Proposition 5.3 would fail for this theory. However, a sufficiently equivariant version of this condition turns out to behave exactly as desired.

Let $X$ be in \textit{GSet}.

**Definition 2.32.** A $G$-bornology on $X$ is a subset $B_X$ of $\mathcal{P}_X$ satisfying the following properties:

1. $B_X$ is $G$-invariant.
2. $B_X$ is closed under forming subsets and finite unions.
3. $\bigcup_{B \in B_X} B = X$.

A $G$-bornological space is a pair $(X, B_X)$ of a $G$-set $X$ with a $G$-bornological structure $B_X$. The elements of $B_X$ will be called the bounded subsets of $X$.

Consider $G$-bornological spaces $(X, B_X)$ and $(X', B_{X'})$ and a morphism $f: X \to X'$ in \textit{GSet}.

**Definition 2.33.**

1. $f$ is proper if $f^{-1}(B_{X'}) \subseteq B_X$.
2. $f$ is bornological if $f(B_X) \subseteq B_{X'}$.

We denote by \textit{GBorn} the category of $G$-bornological spaces and proper maps.

Let $X$ be a $G$-set with a $G$-coarse structure $C_X$ and a $G$-bornology $B_X$.

**Definition 2.34.** $C_X$ and $B_X$ are compatible if for every $B$ in $B_X$ and $V$ in $C_X$ also $V[B] \in B_X$ (see (2.7)).

**Definition 2.35.** A $G$-bornological coarse space is a triple $(X, C_X, B_X)$ (usually abbreviated by the symbol $X$) of a $G$-set $X$ with a $G$-coarse structure $C_X$ and a $G$-bornological structure $B_X$ such that $C_X$ and $B_X$ are compatible. A morphism between $G$-bornological coarse spaces is a morphism of the underlying $G$-coarse spaces which is in addition proper.

In this way we get a category $\text{GBornCoarse}$ of $G$-bornological coarse spaces. It comes with a forgetful functor

\begin{equation}
\text{GBornCoarse} \to \text{GCoarse}.
\end{equation}
Remark 2.36. By dropping condition Definition 2.32 (3), one gets an even better category of generalised $G$-bornological coarse spaces $\text{GBornCoarse}$ which (in contrast to $\text{GBornCoarse}$, see the discussion in [BEKWa, Sec. 2.2]) is complete and cocomplete [Hei]. As shown in this reference, $\text{GBornCoarse}$ and $\text{GBornCoarse}$ yield equivalent categories of equivariant coarse homology theories. As the present paper builds on [BEKWa], [BEKW19] and [BEKWc] working with $\text{GBornCoarse}$, we keep using this category also in the present paper.

Remark 2.37. The category $\text{GBornCoarse}$ has a symmetric monoidal structure $\otimes$. For $X, X'$ in $\text{GBornCoarse}$ the underlying $G$-set of $X \otimes X'$ is $X \times X'$ with the diagonal action. Its $G$-coarse structure is generated by the entourages $U \times U'$ for all $U$ in $C_X$ and $U'$ in $C_{X'}$, and its $G$-bornology is generated by the sets $B \times B'$ for all $B$ in $B_X$ and $B'$ in $B'_{X'}$. The forgetful functor (2.44) is symmetric monoidal with respect to $\otimes$ on $\text{GBornCoarse}$ and the cartesian product on $G\text{Coarse}$ (which we also denote by $\otimes$). See also [BEKWa, Ex. 2.17].

Example 2.38. For $S$ in $G\text{Set}$ we can consider the objects $S_{\max, \max}$, $S_{\min, \max}$, and $S_{\min, \min}$ in $\text{GBornCoarse}$, where the first index $\min$ or $\max$ refers to the minimal or maximal $G$-coarse structure (Example 2.16), and the second $\min$ or $\max$ refers to the minimal bornology (all finite subsets) or the maximal bornology (all subsets) on $S$. We actually get a functor

$$i: G\text{Set} \to \text{GBornCoarse}, \quad S \mapsto S_{\min, \max}.$$ 

Example 2.39. The group $G$ has a canonical coarse structure $C_{\text{can}}$ generated by the entourages $U \times U'$ for all $U$ in $C_X$ and $U'$ in $C_{X'}$, and its $G$-bornology is generated by the sets $B \times B'$ for all $B$ in $B_X$ and $B'$ in $B'_{X'}$. The forgetful functor (2.44) is symmetric monoidal with respect to $\otimes$ on $\text{GBornCoarse}$ and the cartesian product on $G\text{Coarse}$ (which we also denote by $\otimes$). See also [BEKWa, Ex. 2.17].

Example 2.40. If $X$ is in $\text{GBornCoarse}$ and $W$ is in $C_X$, then write $X_W$ for the object of $\text{GBornCoarse}$ obtained from $X$ by replacing $C_X$ by $C_X, W := C((W))$, the $G$-coarse structure generated by $W$. The identity of the set $X$ is a morphism $X_W \to X$ in $\text{GBornCoarse}$.

By pulling back the functors from (2.31), (2.34) and (2.36) along the forgetful functor (2.44), we obtain functors

$$\text{PSh}^G, \text{Sh}^{\pi_0, G}: \text{GBornCoarse} \times \text{Fun}(BG, \text{Cat}_{\infty, \ast}^{\text{LEX}}) \to \text{Cat}_{\infty, \ast}^{\text{LEX}}$$

and

$$\text{Sh}^G: \text{GBornCoarse} \times \text{Fun}(BG, \text{Cat}_{\infty, \ast}^{\text{LEX}}) \to \text{CAT}_{\infty, \ast}^{\text{LEX}}.$$ 

Consider $X, Y$ in $\text{GBornCoarse}$ and a map $f: Y \to X$ between the underlying $G$-sets.

Definition 2.41. $f$ is a covering if it has the following properties:

1. $f$ is a coarse covering (Definition 2.23).
2. $f$ is bornological (Definition 2.33 (2)).
3. For every $B$ in $B_Y$ there exists a finite bound (depending on $B$) on the cardinality of the fibres of the map $\pi_0(B) \to \pi_0(X)$ (Remark 2.21).

Remark 2.42. Definition 2.41 is equivalent to [BEKWa, Def. 2.14]. The condition Definition 2.41 (3) is preserved by forming finite unions or taking subsets. So it suffices to check this condition on a set of generators of the bornology $B_Y$. 

*
We obtain a category $GBC^\dagger$ of $G$-bornological coarse spaces and coverings. It also comes with a forgetful functor

$$GBC^\dagger \to GCoarse.$$  

**Example 2.43.** Let $W$ be a $G$-set and $X$ be in $GBornCoarse$. Then the projection

$$W_{\text{min}, \text{min}} \otimes X \to X$$

is a covering (compare with Example 2.24).

We have a functor

$$GSet \to GBC^\dagger, \ S \mapsto S_{\text{min, min}}.$$  

Using the existence of the left adjoints asserted in Corollary 2.22 and Lemma 2.25, we can consider sheaves on $GBC^\dagger$ as functors

$$\text{Sh}^*_{\tau_G, G}: GBC^\dagger, \text{op} \times \text{Fun}(BG, \text{Cat}_{\infty, *}) \to \text{Cat}_{\infty, *},$$

(2.47) \hspace{1cm}

$$\text{Sh}^G: GBC^\dagger, \text{op} \times \text{Fun}(BG, \text{Cat}_{\infty, *}) \to \text{CAT}_{\text{Lex}}^\dagger.$$  

Let $X$ be in $GBorn$.

**Definition 2.44.** An element $Y$ in $P_X$ is called $G$-bounded if there exists $B$ in $B_X$ such that $Y = GB$.

**Remark 2.45.** Note that $G$-bounded subsets are $G$-invariant by definition. The $G$-bounded elements of $P_X$ generate a bornology denoted by $GB_X$. We have $B_X \subseteq GB_X$.

If $X$ is a bornological coarse space, then $GB_X$ is again compatible with the coarse structure.

If $H$ is a subgroup of $G$ and $C$ is a complete $\infty$-category, then we can consider the natural transformation

$$r^G_H: \lim_{BG} \to \lim_{BH} \circ \text{Res}^G_H: \text{Fun}(BG, C) \to \text{Fun}(BH, C).$$  

(2.48)

We write $\text{res}^G_H: GBornCoarse \to HBornCoarse$ for the functor which restricts the action from $G$ to $H$ (we use the same notation also for the restriction of actions on other sorts of spaces). For $X$ in $GBornCoarse$ and $C$ in $\text{Fun}(BG, \text{Cat}_{\infty, *})$ we have an equivalence $\text{Res}^G_H \text{Sh}_C(X) \simeq \text{Sh}_{\text{Res}^G_H C}(\text{res}^G_H X)$ in $\text{Fun}(BH, \text{CAT}_{\infty, *})$ (compare with (2.55)). We will usually drop the symbol $\text{Res}^G_H$ in front of the $\infty$-category $C$, but we will always write $\text{res}^G_H$ in front of space variables. The transformation (2.48) thus induces a left-exact functor

$$r^G_H: \text{Sh}_C^G(X) \to \text{Sh}_C^H(\text{res}^G_H X).$$  

(2.49)

Varying $X$ these functors give rise to a natural transformation

$$r^G_H: \text{Sh}_C^G \to \text{Sh}_C^H \circ \text{res}^G_H$$

of functors from $GBornCoarse$ to $\text{CAT}_{\infty, *}$.

Let $X$ be in $GBornCoarse$, and let $C$ be in $\text{Fun}(BG, \text{Cat}_{\infty, *})$. For a subgroup $H$ of $G$ and an $H$-invariant subset $Y$ of $X$ we have a left-exact evaluation functor

$$\text{ev}_Y: \text{Sh}_C^G(X) \xrightarrow{r^G_H} \text{Sh}_C^H(\text{res}^G_H X) \xrightarrow{\text{ev}_Y} \text{Sh}_C^H(Y) \xrightarrow{\text{ev}_Y} \text{Sh}_C^H(*) \simeq C^H,$$  

(2.50)
where $i : Y \to \text{res}_H^G X$ and $p : Y \to \ast$ denote the inclusion map and the projection, respectively. If $M$ is a sheaf on $X$, then we write $M(Y) := \text{ev}_Y(M)$ for the value of $M$ on $Y$ considered as an object of $C^H := \lim_{BH} \text{Res}_H^G C$.

Let $M$ be in $\text{Sh}_C^G(X)$.

**Definition 2.46.** We call $M$ equivariantly small if $M(Y) \in C^{H,\omega}$ for all subgroups $H$ of $G$ and all $H$-bounded subsets $Y$ of $X$.

Let $X$ be in $\text{GBornCoarse}$, and let $C$ be in $\text{Fun}(BG, \text{Cat}_{\infty,*,\omega}^\Lambda)$.

**Definition 2.47.** We denote by $\text{Sh}_C^{G,\text{eqsm}}(X)$ the full subcategory of $\text{Sh}_C^G(X)$ of equivariantly small objects.

**Lemma 2.48.** We have $\text{Sh}_C^{G,\text{eqsm}}(X) \in \text{Cat}_{\infty,*,\omega}^\Lambda$.

**Proof.** Since $\text{ev}_Y$ preserves finite limits and finite limits of cocompact objects are cocompact, $\text{Sh}_C^{G,\text{eqsm}}(X)$ is closed under taking finite limits. This implies that $\text{Sh}_C^{G,\text{eqsm}}(X) \in \text{CAT}_{\infty,*,\omega}^\Lambda$. It remains to show that $\text{Sh}_C^{G,\text{eqsm}}(X)$ is essentially small.

For every $Y$ in $\mathcal{P}_X$ the family $(B)_{B \in \mathcal{P}_X(Y)}$ is a $U$-covering family of $Y$ for every entourage $U$. It follows that the restriction functor $\text{Sh}_C^G(X) \to \text{Fun}^G(B_{\infty,*,\omega}^\Lambda, C)$ is fully faithful. Since the values of equivariantly small sheaves on bounded subsets are cocompact, this exhibits $\text{Sh}_C^{G,\text{eqsm}}(X)$ as a full subcategory of $\text{Fun}^G(B_{\infty,*,\omega}^\Lambda, C)$. The latter $\infty$-category is essentially small, so $\text{Sh}_C^{G,\text{eqsm}}(X)$ is essentially small, too. \[ \square \]

Let $f : X \to X'$ be a morphism in $\text{GBornCoarse}$.

**Lemma 2.49.** The morphism $\hat{f}_*^G$ from (2.37) preserves equivariantly small objects.

**Proof.** Let $M$ be in $\text{Sh}_C^{G,\text{eqsm}}(X)$. If $Y'$ is an $H$-bounded subset of $X'$, then $f^{-1}(Y')$ is an $H$-bounded subset of $X$. Consequently, $\hat{f}_*^G(M(Y')) \simeq M(f^{-1}(Y')) \in C^{H,\omega}$. \[ \square \]

Let $X$ be in $\text{GBornCoarse}$, and let $V$ be in $C_{X}^{G,\Delta}$. Recall the endofunctor $V_*^G$ of $\text{Sh}_C^G(X)$ from Corollary 2.29.

**Lemma 2.50.** The endofunctor $V_*^G$ preserves equivariantly small objects.

**Proof.** Let $M$ be in $\text{Sh}_C^{G,\text{eqsm}}(X)$ and assume that $Y$ is an $H$-bounded subset of $X$. We must show that $(V_*^G M)(Y) \in C^{H,\omega}$.

Note that $V(Y)$ is also $H$-bounded. Let $i : Y \to \text{res}_H^G X$, $j : V(Y) \to \text{res}_H^G X$ and $k : V(Y) \to Y$ denote the inclusions, and let $p : Y \to \ast$ and $q : V(Y) \to \ast$ denote the
projections. We have the following chain of equivalences

\[
(V^G_M)(Y) \overset{\text{def}}{=} \text{ev}_Y (V^G_M) \\
\overset{(2.50)}{=} \hat{\varphi}^H V^G_M \\
\overset{\text{Lemma 2.31}}{=} \hat{\varphi}^H V^G_M \\
\overset{j=\text{isk}}{=} \hat{\varphi}^H V^G_M \\
\overset{\text{Corollary 2.27}}{=} \hat{\varphi}^H V^G_M \\
\overset{V_Y(Y)\overset{\text{cor}}{=}V(Y)}{=} \hat{\varphi}^H V^G_M \\
\overset{(2.50)}{=} \text{ev}_Y (M) \\
\overset{\text{def}}{=} M (V(Y)) .
\]

Consequently, we have \((V^G_M)(Y) \simeq M (V(Y)) \in C^{H,\omega}\). \(\square\)

Let \(\phi: C \to C'\) be a morphism in \(\text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{LEX}})\).

**Lemma 2.51.** The morphism \(\tilde{\phi}^G: \text{Sh}^G_C (X) \to \text{Sh}^G_C (X)\) preserves equivariantly small objects.

**Proof.** Note that \(\phi^H: C^H \to C'^H\), being a morphism in \(\text{Cat}_{\infty,*}^{\text{LEX}}\), preserves cocompact objects. Let \(M\) be in \(\text{Sh}^G_C^{\text{eqsm}} (X)\). If \(Y\) is an \(H\)-bounded subset of \(X\), then \(\tilde{\phi}^G M (Y) \simeq \phi^H (M (Y)) \in C'^{H,\omega}\). \(\square\)

As a consequence of Lemmas 2.48, 2.49 and 2.51 we get:

**Corollary 2.52.** We have a subfunctor \(\text{Sh}^G_{\text{eqsm}}: G\text{BornCoarse} \times \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{LEX}}) \to \text{Cat}_{\infty,*}^{\text{Lex}}\) of \(\text{Sh}^G\) from (2.46).

Let \(X\) be in \(G\text{BornCoarse}\), and let \(i: Y \to X\) be the inclusion of a \(G\)-invariant subset.

**Lemma 2.53.** The morphism \(\tilde{i}^*G\) from (2.42) preserves equivariantly small objects.

**Proof.** Any \(H\)-bounded subset \(Z\) of \(Y\) is also an \(H\)-bounded subset of \(X\), so \(\tilde{i}^*G M (Z) \simeq M (i(Z)) \in C^{H,\omega}\). \(\square\)

Let \(p: Z \to X\) be a morphism in \(G\text{BornCoarse}^\dagger\), i.e., a covering.

**Proposition 2.54.** The morphism \(L_{\pi_0,G} \tilde{\varphi}^G: \text{Sh}^G_C (X) \to \text{Sh}^G_C (Z)\) preserves equivariantly small objects.

**Corollary 2.55.** We have a subfunctor \(\text{Sh}^{G,\dagger,\text{eqsm}}: G\text{Bc}^{\dagger,\text{op}} \times \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{LEX}}) \to \text{Cat}_{\infty,*}^{\text{Lex}}\) of \(\text{Sh}^{G,\dagger}\) from (2.47).
The proof of Proposition 2.54 requires some preparation. We will first describe the behaviour of the transfer in some concrete cases, and then observe that every covering is modelled locally on these special cases. This allows us to show that transfers along coverings also preserve equivariantly small sheaves.

Let $H$ be a subgroup of $G$. For any complete category $C$ the restriction and right Kan extension functors along the inclusion functor $BH \to BG$ are parts of an adjunction

$$\text{Res}_H^G : \text{Fun}(BG, C) \to \text{Fun}(BH, C) : \text{Coind}_H^G.$$ 

In the following, we provide an explicit model for the unit morphism

$$(2.51) \text{unit}: C(*_{BG}) \to \text{Coind}_H^G \text{Res}_H^G C(*_{BG}),$$

where $C$ is in $\text{Fun}(BG, C)$ and $*_{BG}$ is the unique object of $BG$. Let $s: G/H \to G$ be a section of the projection map $G \to G/H$ of sets. For $g$ in $G$ we let $C(g): C(*_{BG}) \to C(*_{BG})$ denote the action of the element $g$.

**Lemma 2.56.** The unit morphism $(2.51)$ is equivalent to the morphism

$$(C(s(gH)))_{gH}: C(*_{BG}) \to \prod_{G/H} C(*_{BG}).$$

**Proof.** By the pointwise formula for the right Kan extension, there is an equivalence

$$(2.52) \text{Coind}_H^G \text{Res}_H^G C(*_{BG}) \simeq \lim_{BH \times BG*_{BG}/} \text{pr}_{BH}^* j^* C.$$ 

The set of objects of the category $BH \times BG*_{BG}/$ can be identified with the set $G$ such that $g$ in $G$ corresponds to the object $o_g := (s_{BH}, g: *_{BG} \to *_{BG})$. Under the equivalence $(2.52)$ the unit $C(*_{BG}) \to \text{Coind}_H^G \text{Res}_H^G C(*_{BG})$ is the adjoint of the morphism

$$C(*_{BG}) \to \text{pr}_{BH}^* \text{Res}_H^G C$$

in $\text{Fun}(BH \times BG*_{BG}/, C)$ given at the object $o_g$ by the map

$$C(*_{BG})(o_g) \simeq C(*_{BG}) \xrightarrow{C(g)} C(*_{BG}) \simeq \text{pr}_{BH}^* \text{Res}_H^G C(o_g).$$

A morphism $o_g \to o_{g'}$ in $BH \times BG*_{BG}/$ is a pair $(h, g'')$ in $H \times G$ such that $j(h) = g''$ and $g' = g''g$. We conclude that $BH \times BG*_{BG}/$ is discrete, and that the functor

$$BH \times BG*_{BG}/ \to G/H, \quad o_g \mapsto gH$$

is an equivalence, where we consider the set $G/H$ as a discrete category. The section $s$ induces an inverse equivalence

$$e_s: G/H \to BH \times BG*_{BG}/, \quad gH \mapsto (*_{BH}, o_s(gH)).$$

The following square commutes:

$$\begin{array}{ccc}
C(*_{BG}) & \longrightarrow & \text{Coind}_H^G \text{Res}_H^G C(*_{BG}) \\
(C(s(gH)))_{gH} \downarrow & & \simeq \\
\prod_{G/H} C(*_{BG}) & \leftarrow & \lim_{BH \times BG*_{BG}/} \text{Res}_H^G C
\end{array}$$

□
Since $\text{Cat}_{\infty, *}^{\text{LEX}}$ is complete by Lemma 7.10, we can apply the preceding discussion and Lemma 2.56 to the case $C := \text{Cat}_{\infty, *}^{\text{LEX}}$.

In the following, we consider sets as discrete categories. For $C$ in $\text{Cat}_{\infty, *}^{\text{LEX}}$ and $X$ in $\text{Set}$ we have a canonical identification

$$(2.53) \quad \text{Fun}(X, C) \xrightarrow{\simeq} \prod_{X} C, \quad C \mapsto (C(x))_{x \in X}$$

in $\text{Cat}_{\infty, *}^{\text{LEX}}$. More generally, for $X$ in $G\text{Set}$ and $C$ in $\text{Fun}(BG, \text{Cat}_{\infty, *}^{\text{LEX}})$ we have the object $\text{Fun}(X, C)$ in $\text{Fun}(BG, \text{Cat}_{\infty, *}^{\text{LEX}})$ with the $G$-action by conjugation. Furthermore,

$$(2.54) \quad \text{Fun}^G(X, C) := \lim_{BG} \text{Fun}(X, C)$$

is the $\infty$-category of $G$-equivariant functors.

Let $X$ be in $G\text{Set}$, and let $C$ be in $\text{Fun}(BG, \text{Cat}_{\infty, *}^{\text{LEX}})$. Then the following diagram

$$(2.55) \quad \begin{array}{ccc}
G\text{Set}^{\text{op}} & \xrightarrow{\text{Fun}(-, C)} & \text{Fun}(BG, \text{Cat}_{\infty, *}^{\text{LEX}}) \\
\downarrow \text{res}_H^G & & \downarrow \text{Res}_H^G \\
H\text{Set}^{\text{op}} & \xrightarrow{\text{Fun}(-, \text{Res}_H^G C)} & \text{Fun}(BH, \text{Cat}_{\infty, *}^{\text{LEX}})
\end{array}$$

commutes.

Consider the inclusion

$$(2.56) \quad i : \text{res}_H^G X \to \text{res}_H^G (G/H \times X), \quad i(x) := (eH, x)$$

in $H\text{Set}$. The definition of the functor $i^*$ in the statement of the following lemma implicitly uses the square (2.55).

**Lemma 2.57.** The adjoint

$$\text{Fun}(G/H \times X, C) \to \text{Coind}_H^G \text{Res}_H^G \text{Fun}(X, C)$$

of

$$i^* : \text{Res}_H^G \text{Fun}(G/H \times X, C) \to \text{Res}_H^G \text{Fun}(X, C)$$

is an equivalence in $\text{Fun}(BG, \text{Cat}_{\infty, *}^{\text{LEX}})$.

**Proof.** The adjoint of $i^*$ is given by the composition

$$(2.57) \quad \begin{array}{ccc}
\text{Fun}(G/H \times X, C) & \xrightarrow{\text{unit}} & \text{Coind}_H^G \text{Res}_H^G \text{Fun}(G/H \times X, C) \\
\downarrow \text{Coind}_H^G (i^*) & & \downarrow \text{Coind}_H^G (i^* \circ \text{unit}) \\
\text{Fun}(G/H \times X, C) & \xrightarrow{i^* \circ \text{unit}} & \text{Coind}_H^G \text{Res}_H^G \text{Fun}(X, C).
\end{array}$$

The first morphism is the unit of the adjunction $(\text{Res}_H^G, \text{Coind}_H^G)$. We must show that the evaluation of (2.57) at $*_{BG}$ is an equivalence. The evaluation of the unit has been calculated in Lemma 2.56. Using this lemma (applied to $C := \text{Fun}(G/H \times X, C)$ in $\text{Fun}(BG, \text{Cat}_{\infty, *}^{\text{LEX}})$) the evaluation of (2.57) at $*_{BG}$ is equivalent to the morphism

$$(2.58) \quad \text{Fun}(G/H \times X, C)(*_{BG}) \xrightarrow{(i^* \circ \text{unit}(*_{G/H}))_{*_{BG}}} \prod_{G/H} \text{Fun}(X, C)(*_{BG})$$
in \( \mathbf{Cat}_{\infty, \ast}^{\text{LEX}} \), where we use the abbreviation \( \alpha(g') := \mathbf{Fun}(G/H \times X, C)(g') \) for the action of \( g' \) in \( G \) on \( \mathbf{Fun}(G/H \times X, C)(\ast_{BG}) \). Using the exponential law and the equivalence (2.53) in order to rewrite the domain in the form

\[
\mathbf{Fun}(G/H \times X, C)(\ast_{BG}) \simeq \mathbf{Fun}(G/H, \mathbf{Fun}(X, C)(\ast_{BG})) \simeq \prod_{G/H} \mathbf{Fun}(X, C)(\ast_{BG})
\]

it becomes obvious that (2.58) is an equivalence. \( \square \)

Let \( p: G/H \times X \to X \) be the \( G \)-equivariant projection map and recall the notation \( r^G_H \) from (2.48). Recall further the convention that we usually drop \( \text{Res}^G_H \) in front of \( C \).

Let

\[
\eta: \mathbf{Fun}(X, C) \to \text{Coind}^G_H \text{Res}^G_H \mathbf{Fun}(X, C)
\]

be the unit of the adjunction \( (\text{Res}^G_H, \text{Coind}^G_H) \). We then define the functor

\[
\text{coind}^G_H: \mathbf{Fun}^H(\text{res}^G_H X, C) \to \mathbf{Fun}^G(X, C)
\]

as the composition

\[
\mathbf{Fun}^H(\text{res}^G_H X, C) \simeq \lim_{BG} \text{Coind}^G_H \text{Res}^G_H \mathbf{Fun}(G/H \times X, C) \xrightarrow{\lim_{BG} \eta_*} \lim_{BG} \mathbf{Fun}(X, C) \simeq \mathbf{Fun}^G(X, C),
\]

where \( \eta_* \) is the right adjoint of \( \eta \) (its existence will be shown in the proof of Lemma 2.58 below).

The left vertical arrow in (2.59) implicitly uses the square (2.55).

**Lemma 2.58.** There exists a commutative diagram

\[
\begin{array}{ccc}
\mathbf{Fun}^G(G/H \times X, C) & \xrightarrow{p^*} & \mathbf{Fun}^G(X, C) \\
\downarrow r^G_H & & \downarrow \text{coind}^G_H \\
\mathbf{Fun}^H(\text{res}^G_H(G/H \times X), C) & \xrightarrow{\lim_{BG}(i^*)} & \mathbf{Fun}^H(\text{res}^G_H X, C)
\end{array}
\]

in \( \mathbf{Cat}_{\infty, \ast}^{\text{LEX}} \) whose diagonal is an equivalence. Furthermore, the functor \( \text{coind}^G_H \) restricts to a functor

\[
\text{coind}^G_H\omega: \mathbf{Fun}^H(\text{res}^G_H X, C)^\omega \to \mathbf{Fun}^G(X, C)^\omega
\]

whose essential image generates the target under finite limits and retracts.

**Proof.** By Lemma 2.57 (for the lower triangle and the diagonal equivalence) and the identity \( p \circ i = \text{id}_X \) (for the upper triangle), we have the following commutative diagram

\[
\begin{array}{ccc}
\mathbf{Fun}(G/H \times X, C) & \xrightarrow{p'} & \mathbf{Fun}(X, C) \\
\downarrow \text{unit} & & \downarrow \eta \\
\text{Coind}^G_H \text{Res}^G_H \mathbf{Fun}(G/H \times X, C) & \xrightarrow{\text{coind}^G_H(i^*)} & \text{Coind}^G_H \text{Res}^G_H \mathbf{Fun}(X, C)
\end{array}
\]

in \( \mathbf{Fun}(BG, \mathbf{Cat}_{\infty, \ast}^{\text{LEX}}) \). Since \( p_* \) has a right adjoint (given by the right Kan extension functor \( p_* \) along \( p \)), also \( \eta \) has one which will be denoted by \( \eta_* \). Passing to the right
adjoints in the upper triangle in (2.60) we get the diagram

\[
\begin{align*}
\text{Fun}(G/H \times X, C) & \xrightarrow{p_*} \text{Fun}(X, C) \\
\text{Coind}^G_H \text{Res}^G_H \text{Fun}(G/H \times X, C) & \xrightarrow{\cong} \text{Coind}^G_H \text{Res}^G_H \text{Fun}(X, C)
\end{align*}
\]

in \(\text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty,*})\). We now apply \(\lim_{BG}\). Use that \(r^G_{H}\) in (2.48) is given by

\[
(2.61) \quad \lim_{BG} \lim_{BG}(\text{unit}) \rightarrow \lim_{BG} \text{Coind}^G_H \text{Res}^G_H \simeq \lim_{BH} \text{Res}^G_H,
\]

and recall the square (2.55) in order to rewrite the lower left corner. Lastly, use the definition \(\text{coind}^G_H := \lim_{BG} \eta_*\) in order to get the square (2.59).

We now show the second assertion. Since the functor \(\text{coind}^G_H\) is a morphism in \(\text{Cat}^{\text{LEX}}_{\infty,*}\), it preserves cocompact objects and therefore gives restricts to \(\text{coind}^G_{H,\omega}\) as asserted. In order to see the essential image of \(\text{coind}^G_{H,\omega}\) generates the target under finite limits and retracts, we consider the diagram (2.62)

\[
\begin{align*}
\text{Fun}(X, C)(*_{BG})^\omega & \xrightarrow{a_H} \text{Fun}^H(\text{res}^G_H X, C)^\omega \xrightarrow{\text{coind}^G_H} \text{Fun}^G(X, C)^\omega \\text{(7.15)} & \xrightarrow{\simeq} \\
\text{Fun}(X, C)(*_{BG})^\omega & \xrightarrow{\text{can}_H} \text{colim}_{BH} \text{Fun}(\text{res}^G_H X, C)^\omega \xrightarrow{!} \text{colim}_{BG} \text{Fun}(X, C)^\omega
\end{align*}
\]

The functors \(a_H\) and \(a_G\) are the restrictions to cocompact objects of the right adjoints (which exist as seen in the proof of Lemma 7.17) of the canonical functors

\[
e_{*BH} : \text{Fun}^H(\text{res}^G_H X, C) \rightarrow \text{Fun}(X, C)(*_{BG}), \quad e_{*BG} : \text{Fun}^G(X, C) \rightarrow \text{Fun}(X, C)(*_{BG})
\]

(instances of (7.7)). The left square commutes by Lemma 7.17 (2), and the arrow marked by ! is defined such that the right square commutes. In order to show that
the upper triangle commutes, we first observe\(^2\) that

\[
\begin{array}{c}
\text{Fun}(X, C)(*_{BG}) \xleftarrow{e_{*BG}} \text{Fun}^H(\text{res}^G_H X, C) \xleftarrow{r^G_H} \text{Fun}^G(X, C)
\end{array}
\]

(implicitly we identify \(\text{Fun}(X, C)(*_{BG})\) with \(\text{Res}^G_H \text{Fun}(X, C)(*_{BH})\)) commutes in a canonical way. Then we take right adjoints, use (2.61) in order to identify the right adjoint of \(r^G_H\) with \(\text{coind}^G_H\), and restrict to cocompact objects.

By Lemma 7.17 (3) the essential images of can\(_H\) (this is can\((*_{BG})\)) in the notation of Lemma 7.17 and can\(_G\) generate their respective targets under retracts and finite limits. Hence the essential image of the restriction of \(\text{coind}^G_H\) to cocompact objects generates its target under finite limits and retracts.

\[\square\]

**Remark 2.59.** Let \(C\) be in \(\text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty,*})\). Applying Lemma 2.58 to the case \(X = *\), we get a morphism

\[(2.63) \quad \text{coind}^G_H : C^H \rightarrow C^G\]

in \(\text{Cat}^{\text{LEX}}_{\infty,*}\). This morphism restricts to a morphism

\[(2.64) \quad \text{coind}^G_{H,\omega} : C^{H,\omega} \rightarrow C^{G,\omega}\]

in \(\text{Cat}^{\text{Lex,perf}}_{\infty,*}\) whose essential image generates the target under retracts and finite limits.

\[\diamondsuit\]

The commutative diagram (2.59) can be realised as a diagram of categories of sheaves:

**Corollary 2.60.** There exists a commutative diagram

\[
\begin{array}{c}
\text{Sh}_C^{G}(((G/H)_{\text{min}} \otimes X_{\text{min}})) \xrightarrow{\bar{\rho}^G_H} \text{Sh}_C^{G}(X_{\text{min}}) \\
\text{Sh}_C^H(\text{res}^G_H((G/H)_{\text{min}} \otimes X_{\text{min}})) \xrightarrow{\bar{\tau}^H} \text{Sh}_C^H(\text{res}^G_H X_{\text{min}}) \downarrow_{\text{coind}^G_H}
\end{array}
\]

Furthermore, the functor \(\text{coind}^G_H\) restricts to a functor

\[
\text{coind}^G_{H,\omega} : \text{Sh}_C^H(\text{res}^G_H X_{\text{min}})^\omega \rightarrow \text{Sh}_C^G(X_{\text{min}})^\omega
\]

whose essential image generates its target under finite limits and retracts.

---

\(^2\)We consider the diagram

\[
\begin{array}{c}
BG \\
* \\
\text{BH}
\end{array}
\]

Then we must observe that the morphism \(v_{G,*} \rightarrow v_{G,*}u_G, u^*_G \simeq u^*_G\) induced by the unit of the \((u^*_G, u_G,*)\)-adjunction is equivalent to the morphism \(v_{G,*} \rightarrow v_{G,*}j_* j^* \simeq v_{H,*}j^* \rightarrow v_{H,*} u_{H,*} u^*_H j^* \simeq u^*_G\) induced by the units of the adjunctions \((u^*_H, u_{H,*})\) and \((j^*, j_*)\).
Proof. If $Y$ is in $G\text{Set}$, then $(\{y\})_{y \in Y}$ is a $\text{diag}(Y)$-covering family of $Y$. Consequently, in view of (2.54), the functor $M \mapsto (M(\{y\}))_{y \in Y}$ induces an equivalence $\text{Sh}_C^G(Y_{\text{min}}) \cong \text{Fun}^G(Y, C)$. Using these equivalences and (2.55) (in order to move $\text{Res}_H^G$ from outside to $\text{res}_H^G$ inside), we can replace the functor categories in the corners of (2.59) by sheaf categories. One further checks that the horizontal morphisms are given by the sheaf-theoretic operations as indicated. This yields the diagram (2.65). The second assertion of Corollary 2.60 follows from the second assertion of Lemma 2.58. \qed

Recall that $\otimes$ denotes the cartesian product in $G\text{Coarse}$. Let $H$ be a subgroup of $G$, and let $Y$ be in $H\text{Coarse}$. We form the $G$-coarse space $G_{\text{min}} \otimes \text{res}_{\{1\}}^H Y$ with the $G$-action $(g', (g, y)) \mapsto (g'g, y)$. The group $H$ acts by automorphisms on the $G$-coarse space $G_{\text{min}} \otimes \text{res}_{\{1\}}^H Y$ such that $(h, (g, y)) \mapsto (gh^{-1}, hy)$. The colimit in the following definition is interpreted in $G\text{Coarse}$.

**Definition 2.61.** We define the $G$-coarse space

$$G \otimes_H Y := \underset{B \in H}{\text{colim}}(G_{\text{min}} \otimes \text{res}_{\{1\}}^H Y).$$

**Remark 2.62.** The underlying set of $G \otimes_H Y$ can be identified with the set $G \times_H Y$ of equivalence classes $[g, y]$ with $[g, y] = [gh^{-1}, hy]$ for $h \in H$. It is equipped with the smallest $G$-coarse structure such that the canonical map from $G \otimes \text{res}_{\{1\}}^H Y$, $(g, y) \mapsto [g, y]$, is controlled.

We have an adjunction

$$G \otimes_H - : H\text{Coarse} \leftrightarrows G\text{Coarse} : \text{res}_H^G.$$

As in the corresponding adjunction between $G\text{Set}$ and $H\text{Set}$, the $H$-equivariant inclusions

$$Y \to \text{res}_H^G(G \otimes_H Y), \quad y \mapsto [e, y]$$

for $Y$ in $H\text{Coarse}$ define the unit, while the $G$-equivariant multiplication maps

$$\mu : G \otimes_H \text{res}_H^G X \to X, \quad [g, x] \mapsto gx$$

for $X$ in $G\text{Coarse}$ provide the counit of the adjunction. \hfill \dag

Let $X$ be in $G\text{Coarse}$, and let $Y$ be an $H$-invariant subspace for some subgroup $H$ of $G$. Denote by $i : Y \to \text{res}_H^G X$ the inclusion map. We consider the composition

$$i : G \otimes_H Y \xrightarrow{G \otimes_H i} G \otimes_H \text{res}_H^G X \xrightarrow{\mu} X$$

in $G\text{Coarse}$. Since $G \otimes_H i$ is an inclusion and $\mu$ is a coarse covering, both $G \otimes_H i^*,G$ and $L_{\pi_0}i^*,G$ define functors on sheaves by Lemma 2.25 and Corollary 2.27, respectively. We consider the composition\(^3\)

$$\hat{\tau}^*,G := G \otimes_H i^*,G \circ L_{\pi_0}i^*,G : \text{Sh}^G_C(X) \to \text{Sh}^G_C(G \otimes_H Y).$$

Recall the notation $ev_Y$ from (2.50).

\(^3\)Note that this is a slight abuse of notation since $\hat{\tau}^*,G$ is not directly obtained from $i$ by one of our previous constructions.
**Lemma 2.63.** There exists a commutative diagram

\[\begin{array}{c}
\text{Sh}_C^G(X) \xrightarrow{\tau^*G} \text{Sh}_C^G(G \otimes_H Y) \\
\downarrow\text{ev}_Y \xrightarrow{} \downarrow\text{ev}_{G \times_H Y} \\
C^H \xrightarrow{\text{coind}_H^G} C^G
\end{array}\]

**Proof.** Let \(q: Y \to \ast\) denote the projection map in \(H\text{Coarse}\). We have a canonical isomorphism \(G \otimes_H \ast \cong (G/H)_{\text{min}}\). By naturality of the transformation (2.49), we have a commutative diagram

\[\begin{array}{c}
\text{Sh}_C^G(X) \xrightarrow{L^G \circ \mu^*G} \text{Sh}_C^G((G \otimes_H X)_{\text{res}_H}) \xrightarrow{\text{ev}_{G \otimes_H \ast}^G} \text{Sh}_C^G(G \otimes_H Y) \xrightarrow{\text{ev}_{G \times_H Y}} \text{Sh}_C^G((G/H)_{\text{min}}) \\
\downarrow r_H^G \xrightarrow{} \downarrow r_H^G \xrightarrow{} \downarrow r_H^G \xrightarrow{} \downarrow r_H^G \\
\text{Sh}_C^H((\text{res}_H^G X)) \xrightarrow{\text{ev}_{G \otimes_H \ast}^G} \text{Sh}_C^H((\text{res}_H^G (G \otimes H Y))_{\text{res}_H}) \xrightarrow{\text{ev}_{G \times_H Y}} \text{Sh}_C^H((G/H)_{\text{min}})
\end{array}\]

Since the inclusion \(i\) can be factorized as the composition

\[Y \xrightarrow{j} \text{res}_H^G (G \otimes_H Y) \xrightarrow{\text{res}_H^G (G \otimes_H i)} \text{res}_H^G (G \otimes_H \text{res}_H^G X) \xrightarrow{\text{res}_H^G (\mu)} \text{res}_H^G X ,\]

where \(j\) is the canonical inclusion \(y \mapsto [e, y]\), we have an equivalence

\[\hat{i}^*H \simeq \hat{j}^*H \tau^*H .\]

Let \(k: \ast \to \text{res}_H^G (G/H)_{\text{min}}\) denote the inclusion of the point \(eH\). Since

\[\begin{array}{c}
Y \xrightarrow{j} \text{res}_H^G (G \otimes_H Y) \xrightarrow{\text{res}_H^G (G \otimes_H q \circ \hat{q}_s^H) \circ \hat{g}^*H} (G/H)_{\text{min}} \\
\downarrow k \xrightarrow{} \downarrow k
\end{array}\]

is a pullback in \(H\text{Coarse}\), by Corollary 2.28 we obtain an equivalence

\[\hat{k}^*H \circ \hat{g}^*H \circ \hat{q}_s^H \simeq \hat{q}_s^H \circ \hat{j}^*H .\]
Combining (2.69) and (2.70) with the commutative diagram (2.68), we have the solid part of the commutative diagram

\[
\begin{array}{c}
\text{Sh}\_C^G(X) \xrightarrow{i^G} \text{Sh}\_C^G(G \otimes_H Y) \xrightarrow{G \otimes_H \eta^G} \text{Sh}\_C^G((G/H)_{min}) \xrightarrow{j^G} \text{Sh}\_C^G(\ast) \simeq C^G \\
\text{Sh}\_C^H(\text{res}\_H^G X) \xrightarrow{i^H} \text{Sh}\_C^H(\text{res}\_H^G(G \otimes_H Y)) \xrightarrow{G \otimes_H \eta^H} \text{Sh}\_C^H(\text{res}\_H^G(G/H)_{min}) \xrightarrow{\coind\_H^G} \text{Sh}\_C^H(\ast) \simeq C^H
\end{array}
\]

We can extend the diagram by the dashed part by applying Corollary 2.60 (with \(X = \ast\) and \(i = k\)). The combination with the dotted part (reflecting the definition of the evaluation maps) then yields the asserted square.

\[\square\]

\textbf{Remark 2.64.} Note that the diagonal map in (2.67) sends \(M\) in \(\text{Sh}\_C^G(X)\) to \(L_{\pi_0(G,H)} M(G \times_H Y)\) in \(C^G\).

Let \(X\) be a \(G\)-coarse space, and let \(Y\) be in \(\pi_0(X)\). Denote by

\[G_Y := \{g \in G \mid gY = Y\}\]

the stabiliser of \(Y\). We consider \(Y\) as a \(G_Y\)-coarse subspace of \(\text{res}\_G^G X\).

\textbf{Lemma 2.65.} If \(X = GY\) (as sets), then the multiplication map \(\mu: G \otimes_{G_Y} Y \to X\) is an isomorphism in \(\mathbf{GCoarse}\).

\textit{Proof.} The morphism

\[G_{min} \otimes \text{res}\_G^G Y \to X, \quad (g, y) \mapsto gy\]

in \(\mathbf{GCoarse}\) is controlled and constant along \(G_Y\)-orbits for the \(G_Y\)-action

\[(h, (g, y)) \mapsto (gh^{-1}, hy)\]

on its domain. By the universal property of the colimit in Definition 2.61 it factorizes over the multiplication map \(\mu: G \otimes_{G_Y} Y \to X\) in \(\mathbf{GCoarse}\). The latter is surjective by our assumptions.

We now show that \(\mu\) is injective. Consider \([g, y], [g, y']\) in \(G \otimes_{G_Y} Y\) and suppose that \(\mu([g, y]) = \mu([g', y']).\) Then \(y = g^{-1}g' y'.\) Let \(y''\) be any point in \(Y.\) Since \(Y\) is coarsely connected, we have \(\{(y, y'')\} \in C_X\) and \(\{(y', y'')\} \in C_X.\) Since \(C_X\) is \(G\)-invariant, we also have \(g^{-1}g' \{(y, y'')\} = \{(y, g^{-1}g' y'')\} \in C_X\) and therefore \(g^{-1}g' y'' \in Y.\) Since \(y''\) is arbitrary, we can conclude that \(g^{-1}g' \in G_Y,\) and hence \([g, y] = [g', y']\).

If \(U\) is a \(G\)-invariant entourage of \(X,\) then we have \(U = G(U \cap (Y \times Y))\). This entourage is thus the image of \(\text{diag}(G) \times (U \cap (Y \times Y))\) under the composition of the projection \(G \times Y \to G \times_{G_Y} Y\) and the multiplication map and therefore the image of a coarse entourage of \(G \otimes_{G_Y} Y\) under the multiplication map \(\mu.\) This shows that the map \(\mu\) is an isomorphism of \(G\)-coarse spaces. \[\square\]
Let $X$ be in $G\text{Coarse}$, and let $A$ be a coarsely connected subspace of $X$. By $[A]$ in $\pi_0(X)$ we denote the coarse closure of $A$. We consider $GA$ as a $G$-coarse subspace of $X$ and $G[A]A$ as a $G[A]$-coarse subspace of $\text{res}^G_{G[A]}X$.

**Corollary 2.66.** The multiplication map induces an isomorphism

$$G \otimes_{G[A]} G[A]A \cong GA.$$  

**Proof.** The subspace $G[A]A$ of $GA$ is a coarse component of $GA$. Consequently, the corollary follows from Lemma 2.65 applied to $GA$ in place of $X$ and $G[A]A$ in place of $Y$. \hfill \Box

Let $p: Z \to X$ in $G\text{Coarse}$ be a coarse covering (Definition 2.23), let $M$ be in $\text{Sh}^G_{\text{coarse}}(X)$, and let $H$ be a subgroup of $G$.

**Proposition 2.67.** If $B$ is a coarsely connected subset of $Z$, then

$$L^{\pi_0, G}_{\tilde{p}^*G} M(HB) \cong \text{coind}_{H[B]}^H(M(p(H[B]))) \cdot$$  

**Proof.** We consider the $H[B]$-invariant subspace $H_B B$ of $Z$ with the induced coarse structure as an object of $H_B\text{Coarse}$. By Corollary 2.66, we have an isomorphism of $H$-coarse spaces $HB \cong H \otimes_{H[B]} H_B B$. Since $p$ induces an isomorphism $H[B] \cong p(H[B])$, we can identify $p(H[B])$ with the multiplication map

$$H \otimes_{H[B]} p(H[B]) \to H(p(H[B])).$$

The desired equivalence is now given by the composition

$$L^{\pi_0, G}_{\tilde{p}^*G} M(HB) \cong L^{\pi_0, H}_{\tilde{p}^*H} r_{H[B]}^G M(HB) \cong \text{coind}_{H[B]}^H(M(p(H[B]))) \cong \text{coind}_{H[B]}^H(M(p(H[B]))) \cdot$$

where the marked equivalence is given by the filler of the square (2.63). Here we apply Lemma 2.63 to $H$ in place of $G$, $H[B]$ in place of $H$, $p(H[B])$ in place of $Y$, and $\text{res}^G_{H[B]}X$ in place of $X$. We furthermore use Remark 2.64. \hfill \Box

Let $D$ be in $\text{CAT}^{\text{LEX}}_{\text{coarse}}$, and let $(D_i)_{i \in I}$ be a family of objects in $D$.

**Lemma 2.68.** If $\prod_{i \in I} D_i \in D^\omega$, then $D_i \simeq 0$ for all but finitely many $i$ in $I$.

**Proof.** Since $\prod_{i \in I} D_i \in D^\omega$ and $\prod_{i \in I} D_i \simeq \lim_F \prod_{i \in F} D_i$, with $F$ running over all finite subsets of $I$, the identity morphism factors through some finite product:

$$\text{id}: \prod_{i \in I} D_i \to \prod_{i \in F} D_i \to \prod_{i \in I} D_i .$$

This implies $\text{id}_{D_i} \simeq 0$ for all but finitely many $i$, and thus $D_i \simeq 0$ for all but finitely many $i$. \hfill \Box

Let $X$ be in $G\text{BornCoarse}$, $M$ be in $\text{Sh}^G_{\text{equiv}}(X)$, and let $B$ be a bounded subset of $X$.

**Corollary 2.69.** We have $M(B \cap C) \simeq 0$ for all but finitely many $C$ in $\pi_0(X)$.
Proof. Since \((B \cap C)_{C \in \pi_0(X)}\) is a \(\pi_0\)-covering family of \(B\) and \(M\) is in particular a \(\pi_0\)-sheaf, we have an equivalence

\[ M(B) \simeq \prod_{C \in \pi_0(X)} M(B \cap C). \]

By the equivariant smallness condition Definition 2.46 (applied to the trivial group), we have \(M(B) \in C^\infty\). Now apply Lemma 2.68. \(\square\)

Proof of Proposition 2.54. Recall that we consider a covering \(p: Z \to X\) of \(G\)-bornological coarse spaces. Let \(M\) be in \(\text{Sh}^G_{\text{cusp}}(X)\). Let \(H\) be a subgroup of \(G\) and \(Y\) be an \(H\)-bounded subset of \(Z\). Then we want to show that \(L^{\pi_0} \tilde{p}^{\ast, G} M(Y) \in C^H\).

We can choose a bounded subset \(B\) of \(Z\) such that \(Y = HB\). Since \(p\) is a covering (Definition 2.41) there exists a finite, coarsely disjoint family \((B_j)_{j \in J}\) of subsets of \(B\) such that \(B = \bigcup_{j \in J} B_j\) and such that the induced map on coarse closures \(p\vert_{[B_j]}: [B_j] \to [p(B_j)]\) is an isomorphism of coarse spaces for every \(j\) in \(J\). We fix \(j\) in \(J\) for the moment. Then for \(C\) in \(\pi_0(B_j)\) we have an equivalence

\[ L^{\pi_0} \tilde{p}^{\ast, G} M(B_j \cap C) \simeq M(p(B_j) \cap p(C)). \]

Since \(M(p(B_j)) \in C^\infty\), by Corollary 2.69 we have \(L^{\pi_0} \tilde{p}^{\ast, G} M(B_j \cap C) \simeq 0\) for all but finitely many \(C\) in \(\pi_0(B_j)\). Hence, without changing the value \(M(Y)\), we can replace \(B\) (and hence \(Y = HB\)) by a smaller subset which is contained in the union of finitely many coarse components of \(Z\). In addition we can then assume that \(B_j\) is coarsely connected for every \(j\) in \(J\).

Let \(\sim\) denote the equivalence relation on \(J\) such that \(j \sim j'\) if there exist \(h\) in \(H\) with \(h[B_j] = [B_{j'}]\). We choose a set \(S\) of representatives of \(J/\sim\). Note that \(S\) is finite. For every \(j\) in \(J\) we choose \(h_j\) in \(H\) such that \(h_j[B_j] = [B_s]\) with \(s\) in \(S\) and \(s \sim j\). For every \(s\) in \(S\) we then define the subset

\[ B'_s := \bigcup_{j \sim s} h_j B_j \]

of \(Y\). We observe that \((B'_s)_{s \in S}\) is a finite family of coarsely connected subsets, \((HB'_s)_{s \in S}\) is a coarsely disjoint family of subsets, and that \(Y = \bigcup_{s \in S} HB'_s\). We have the following chain of equivalences in \(C^H\)

\[ L^{\pi_0} \tilde{p}^{\ast, G} M(Y) \simeq \prod_{s \in S} L^{\pi_0} \tilde{p}^{\ast, G} M(HB'_s) \simeq \prod_{s \in S} \text{coin}_{H_{[B'_s]}}^H M(p(H_{[B'_s]}B'_s)), \]

where the first follows from the sheaf condition on \(M\), and the second is given by Proposition 2.67. Since \(\text{coin}_{H_{[B'_s]}}^H\) preserves cocompact objects by the second assertion of Lemma 2.58, we see that \(L^{\pi_0} \tilde{p}^{\ast, G} M(Y) \in C^H\). This finishes the proof of Proposition 2.54. \(\square\)

To close this section, we record some consequences of the equivariant smallness condition which will only come to bear in Section 5. The common theme among them is to translate the equivariant smallness condition into a more concrete cocompactness property for sheaves on some specific \(G\)-bornological coarse spaces.

Remark 2.70. If \(Y\) is a \(G\)-coarse space admitting a maximal entourage \(U\), e.g., \(Y = X_{\min}\) for a \(G\)-set \(X\), then by Lemma 2.8 we have \(\text{Sh}^G_C(Y) = \text{Sh}^G_C(Y) \subset \text{CAT}^\infty_{\text{LEX}}\). Consequently, we can consider cocompact objects in \(\text{Sh}^G_C(Y)\), see Definition 7.7. \(\diamond\)
Let $X$ be in $\text{Set}$. Recall the bornological coarse space $X_{\text{min, max}}$ from Example 2.38. For $x$ in $X$, let $i_x : \{x\} \to X$ denote the inclusion map.

**Lemma 2.71.** There exists a commutative diagram

$$
\begin{array}{ccc}
\text{Sh}_\text{eqsm}^G(X_{\text{min, max}}) & \longrightarrow & \text{Sh}_C(X_{\text{min, max}}) \\
\sum_{x \in X} i_x & \uparrow & M \mapsto (M(x))_x \\
\prod_{x \in X} C^\omega & \longrightarrow & \prod_{x \in X} C
\end{array}
$$

in which both vertical maps are equivalences. In particular,

$$
\text{Sh}_\text{eqsm}^G(X_{\text{min, max}}) \simeq \text{Sh}_C(X_{\text{min, max}})^\omega.
$$

**Proof.** The upper horizontal morphism is the canonical inclusion. The lower horizontal morphism is the composition

$$
\prod_{x \in X} C^\omega \overset{(\text{T.15})}{\simeq} \left(\prod_{x \in X} C\right)^\omega \hookrightarrow \prod_{x \in X} C.
$$

The right vertical morphism is an equivalence in view of the sheaf condition for $M$ since $(\{x\})_{x \in X}$ is a $\text{diag}(X)$-covering family of $X$.

Note that $X$ is a bounded subset of $X_{\text{min, max}}$. It follows from the definition of equivariant smallness and Corollary 2.69 that $M$ in $\text{Sh}_C(X_{\text{min, max}})$ is equivariantly small if and only if $M(\{x\}) \in C^\omega$ for all $x$ in $X$ and $M(\{x\}) \simeq 0$ for all but finitely many $x$ in $X$. This implies that the left vertical morphism is well-defined and an equivalence, too. The up-right-down composition in (2.71) is canonically equivalent to the lower horizontal map.

Since the lower horizontal morphism is equivalent to the inclusion of the cocompact objects of its target, the same also applies to the upper horizontal map (see also Remark 2.70). This shows the second assertion. \hfill \Box

Let $X$ be in $G\text{Set}$ and consider the embedding (2.56) for $H := \{1\}$. It induces an embedding of bornological coarse spaces

$$
\text{res}_{\{1\}}^G X_{\text{min, max}} \to \text{res}_{\{1\}}^G (G_{\text{min, min}} \otimes X_{\text{min, max}}).
$$

The diagonal map in (2.65) provides an equivalence

$$
\theta : \text{Sh}_C^G(G_{\text{min, min}} \otimes X_{\text{min, max}}) \cong \text{Sh}_C(\text{res}_{\{1\}}^G X_{\text{min, max}}).
$$

**Lemma 2.72.** The equivalence $\theta$ restricts to an equivalence

$$
\vartheta^\text{eqsm} : \text{Sh}_C^G(G_{\text{min, min}} \otimes X_{\text{min, max}}) \cong \text{Sh}_C^G(\text{res}_{\{1\}}^G X_{\text{min, max}}).
$$

In particular, we have an equivalence

$$
\text{Sh}_C^G(G_{\text{min, min}} \otimes X_{\text{min, max}}) \simeq \text{Sh}_C^G(G_{\text{min, min}} \otimes X_{\text{min, max}})^\omega.
$$

**Proof.** The second assertion of the Lemma follows from the first combined with the second assertion of Lemma 2.71.

We now show the first assertion. The equivalence $\theta$ has a decomposition

$$
\begin{array}{ccc}
\text{Sh}_C^G(G_{\text{min, min}} \otimes X_{\text{min, max}}) & \overset{r^G_{\{1\}}}{\longrightarrow} & \text{Sh}_C(\text{res}_{\{1\}}^G (G_{\text{min, min}} \otimes X_{\text{min, max}})) \\
& & \uparrow \quad \text{res}_{\{1\}}^G \\
\end{array}
\text{Sh}_C(\text{res}_{\{1\}}^G (G_{\text{min, min}} \otimes X_{\text{min, max}}))
$$
The morphism \( r^G_{\{1\}} \) obviously preserves equivariantly small objects, and the morphism \( \hat{r}^* \) preserves equivariantly small objects by Lemma 2.53. Hence the functor \( \theta^\text{eqsm} \) in (2.74) is well-defined. Since it is a restriction of an equivalence, it is clearly fully faithful. It remains to show that \( \theta^\text{eqsm} \) essentially surjective.

Let \( M \) be an object of \( \text{Sh}_C^H(G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}}) \) such that \( \theta(M) \) is equivariantly small in \( \text{Sh}_C^H(\text{res}^G_{\{1\}} X_{\text{min},\text{max}}) \). We claim that then \( M \) itself is equivariantly small. The claim (for all \( M \)) immediately implies that \( \theta^\text{eqsm} \) essentially surjective.

Let \( H \) be a subgroup of \( G \) and \( Y \) be an \( H \)-bounded subset of \( G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}} \). Then we must show that \( M(Y) \in C^{H,\omega} \).

We can choose a bounded subset \( B \) of \( G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}} \) such that \( Y = HB \). After replacing \( B \) by a smaller subset if necessary, we can assume that there exists a finite subset \( F \) of \( G \) and a family \( (X_f)_{f \in F} \) of \( B \)-subsets of \( X \) such that the projection \( F \to G/H \) is injective and \( B = \bigcup_{f \in F} f \times X_f \). Then \( (H(\{f\} \times X_f))_{f \in F} \) is a coarsely disjoint family of \( H \)-bounded subsets of \( G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}} \) and \( Y = \bigcup_{f \in F} H(H(\{f\} \times X_f)) \). Since \( M(Y) \approx \prod_{f \in F} M(H(\{f\} \times X_f)) \) by the sheaf condition on \( M \), it suffices to show that \( M(H(\{f\} \times X_f)) \in C^{H,\omega} \) for every \( f \in F \).

We now fix \( f \) in \( F \), set \( A := X_f \), and consider \( Z := H(\{f\} \times A) \) in \( \text{HBornCoarse} \) with the structures induced from the embedding \( Z \to \text{res}^H_{\{1\}}(G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}}) \).

We have a diagram in \( \text{HBornCoarse} \)

\[
\begin{array}{ccc}
Z & \xrightarrow{\text{incl}} & \text{res}^G_H(G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}}) \\
\cong & & \\
H \otimes \{1\} A_{\text{min},\text{max}} & &
\end{array}
\]

where \( \iota \) is as in (2.66) (for \( H \) in place of \( G \), \( \{1\} \) in place of \( H \), \( \text{res}^G_H(G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}}) \) in place of \( X \), and \( \{f\} \times A \) in place of \( Y \), and \( m \) is an isomorphism given by \( (h,a) \mapsto (hf,ha) \). By Lemma 2.63 we have the middle square of the following commutative diagram

\[
\begin{array}{ccc}
\text{Sh}_C^H(G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}}) & \xrightarrow{\sim} & \text{Sh}_C^H(\text{res}^G_{\{1\}} X_{\text{min},\text{max}}) \\
\downarrow r^G_H & & \downarrow \text{ev}_A \\
\text{Sh}_C^H(\text{res}^G_H(G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}})) & \xrightarrow{\text{ev}_{\{f\} \times A}} & C \\
\downarrow \tau^* & & \downarrow \text{coin}_{\{1\}}^H \\
\text{Sh}_C^H(H \otimes \{1\} A_{\text{min},\text{max}}) & \xrightarrow{\sim} & C^H \\
\downarrow \hat{m}^H & & \downarrow \text{ev}_Z \\
\text{Sh}_C^H(Z) & &
\end{array}
\]

The upper square and the lower triangle obviously commute. The down-up/right composition sends \( M \) in \( \text{Sh}_C^H(G_{\text{min},\text{min}} \otimes X_{\text{min},\text{max}}) \) to \( M(Z) \) in \( C^H \). The filler of the diagram provides an equivalence of \( M(Z) \cong \text{coin}^H_{\{1\}}(A) \). Since \( \theta(M)(A) \in C^H \) by assumption and \( \text{coin}^H_{\{1\}} \) preserves cocompact objects by the second assertion of Lemma 2.58, we conclude that \( M(Z) \in C^{H,\omega} \). This finishes the proof of the claim and hence of the lemma. \( \square \)
Let $X$ in $\mathbf{GSet}$, and let $C$ be in $\mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_\infty)$. The following proposition is an analogue of the second assertion of Lemma 2.72 in the case where we replaced the minimal coarse structure of $G$ by the canonical one (Example 2.39). If $G$ is infinite, then $\mathbf{Sh}_C^{G}(G_{\text{can}, \text{min}} \otimes X_{\text{min}, \text{max}})$ is only expected to admit finite limits so that it does not makes sense to consider cocompact objects therein (see Remark 2.70). But $\mathbf{PSh}_C^{G}(G_{\text{can}, \text{min}} \otimes X_{\text{min}, \text{max}})$ belongs to $\mathbf{Cat}^{\text{LEX}}_\infty$ and therefore admits a good notion of cocompact objects. The same applies to the functor category $\mathbf{Fun}^{G}(K, C)$ appearing in the statement of Lemma 2.74 below.

**Proposition 2.73.** We have an inclusion

$$\mathbf{Sh}_C^{G, \text{eqm}}(G_{\text{can}, \text{min}} \otimes X_{\text{min}, \text{max}}) \subseteq \mathbf{PSh}_C^{G}(G_{\text{can}, \text{min}} \otimes X_{\text{min}, \text{max}})^{\omega}.$$  

For $K$ be a $G$-finite $G$-simplicial set. By $K^0$ we denote the $G$-set of vertices of $K$. For $k$ in $K^0$ we let $G_k$ denote the stabilizer of $k$ in $G$. Let $C$ be in $\mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_\infty)$ and $M$ be in $\mathbf{Fun}^{G}(K, C)$. For $k$ in $K^0$ we consider $M(k)$ in $C^{G_k}$ in the natural way.

**Lemma 2.74.** If $M(k) \in C^{G_k, \omega}$ for all $k$ in $K^0$, then $M \in \mathbf{Fun}^{G}(K, C)^{\omega}$.

**Proof.** We first assume that $K$ is zero-dimensional and that $K^0 \in \mathbf{GOrb}$. We fix a point $k$ in $K^0$. The diagonal equivalence in (2.59) applied to $X := *$, $H := G_k$ (and using the identification $K = K^0 \cong G/G_k$) provides an equivalence

$$\mathbf{Fun}^{G}(K, C) \simeq \mathbf{Fun}^{G_k}(*, C) \simeq C^{G_k}, \quad M \mapsto M(k).$$

In this case, the lemma holds true for obvious reasons.

In the next step, we assume that $K = \Delta^n \times S$ for some $S$ in $\mathbf{GOrb}$ and $n \in \mathbb{N}$. We have an equivalence

$$\mathbf{Fun}^{G}(S \times \Delta^n, C) \simeq \mathbf{Fun}(\Delta^n, \mathbf{Fun}^{G}(S, C)).$$

By [Lur09, Prop. 5.3.4.13] applied to the right-hand side of this equivalence, we have $M$ in $\mathbf{Fun}^{G}(S \times \Delta^n, C)^{\omega}$ if and only if the evaluation of $M$ at every vertex of $\Delta^n$ belongs to $\mathbf{Fun}^{G}(S, C)^{\omega}$. By the case considered in the first paragraph, the latter condition is implied by our assumption on $M$.

Suppose now that we are given a pushout square

$$\begin{array} {ccc} L' & \xrightarrow{f'} & K' \\
\downarrow{i} & & \downarrow{j} \\
L & \xrightarrow{f} & K 
\end{array}$$

of $G$-simplicial sets where $i$ is a cofibration. Then the induced diagram of $\infty$-categories

$$\begin{array} {ccc} \mathbf{Fun}(K, C) & \xrightarrow{j^*} & \mathbf{Fun}(K', C) \\
\downarrow{f^*} & & \downarrow{j'^*} \\
\mathbf{Fun}(L, C) & \xrightarrow{i^*} & \mathbf{Fun}(L', C) 
\end{array}$$

is a pullback, and hence also a pullback in $\mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_\infty)$. The restriction functors in this diagram all have right adjoints given by right Kan extension functors.
These right adjoints are also morphisms in $\text{Fun}(BG, \text{Cat}_{\text{LEX}}^\infty)$. Applying $\lim_{BG}$ yields the pullback square

$$
\begin{array}{ccc}
\text{Fun}^G(K, C) & \xrightarrow{j^*G} & \text{Fun}^G(K', C) \\
j'^*G & & j'^*G \\
\text{Fun}^G(L, C) & \xrightarrow{i'^*G} & \text{Fun}^G(L', C)
\end{array}
$$

in $\text{Cat}_{\text{LEX}}^\infty$. Furthermore, all the functors in this square again have right adjoint morphisms in $\text{Cat}_{\text{LEX}}^\infty$. This implies that they all preserve cofiltered limits. Therefore, we can apply [Lur09, Lem. 5.4.5.7] to see that $M$ in $\text{Fun}^G(K, C)$ is cocomplete if $j^*G M$, $j'^*G M$, and $i'^*G f^*G M$ are all cocomplete.

The general case follows from this observation by induction on the number of equivariant simplices in $K$. 

**Proof of Proposition 2.73.** Let $M$ be in $\text{Sh}^G_{\text{eqm}}(G_{\text{can}, \text{min}} \otimes X_{\text{min}, \text{max}})$. Let $U$ be an entourage such that $M$ is a $U$-sheaf. After enlarging $U$ if necessary we may assume that $U = G(F \times F) \times \text{diag}(X)$ for some finite subset $F$ of $G$.

Let $F'$ be a subset of $F$. Then $F' \times X$ is a bounded subset of $G_{\text{can}, \text{min}} \otimes X_{\text{min}, \text{max}}$. By Corollary 2.69 and since $X_{\text{min}, \text{max}}$ has the discrete coarse structure, there exists a finite subset $X_{F'}$ of $X$ such that $M(F' \times \{x\}) = 0$ for all $x$ in $X \setminus X_{F'}$. Since $F$ is finite, the subset $X_0 := \bigcup_{F' \in P(F)} X_{F'}$ of $X$ is again finite. Moreover, for every subset $F''$ of $F$ we have $M(F'' \times \{x\}) = 0$ for all $x$ in $X \setminus X_0$.

We consider the $G$-invariant subset $G(F \times X_0)$ of $G \times X$. Furthermore, by $i: P_{U\text{bd}}^{G(F \times X_0)} \rightarrow P_{G \times X}$ we denote the inclusion of the $G$-subposet of $U$-bounded subsets of $G(F \times X_0)$. Then we have an adjunction

$$i^*G: \text{PSh}^G_{\text{can}}(G \times X) \rightleftarrows \text{Fun}^G(P_{U\text{bd}, \text{op}}^{G(F \times X_0)}, C) : i_*^G.$$  

We can consider $M$ as an object of $\text{PSh}^G_{\text{can}}(G \times X)$ and claim that the unit $M \rightarrow i_!^G i^*G M$ is an equivalence. By Lemma 7.12 and since $M$ is a $U$-sheaf it suffices to show that $M(B) \rightarrow i_!^G i^*G M(B)$ is an equivalence for every $U$-bounded subset $B$ of $G_{\text{can}, \text{min}} \otimes X_{\text{min}, \text{max}}$. We thus must show that the canonical morphism

$$(2.76)\quad M(B) \rightarrow \lim_{B' \in (P_{U\text{bd}}^{G(F \times X_0)})_{/B}} M(B')$$  

is an equivalence.

Since $B$ is $U$-bounded, there exists $g$ in $G$ and a subset $F'$ of $F$ such that $B = g(F' \times \{x\})$. Any other subset of $B$ is then of the form $B' = g(F'' \times \{x\})$ for some subset $F''$ of $F'$. Since $M$ is a $G$-invariant sheaf, we have an equivalence $M(B') \simeq M(F'' \times \{x\})$. We distinguish two cases:

1. If $B \notin P_{U\text{bd}}^{G(F \times X_0)}$, then $x \notin X_0$ and hence $M(B') \simeq 0$ for all $B'$ in $(P_{U\text{bd}}^{G(F \times X_0)})_{/B}$. In this case $M(B) \simeq 0$ and $\lim_{B' \in (P_{U\text{bd}}^{G(F \times X_0)})_{/B}} M(B') \simeq 0$.
2. If $B \in P_{U\text{bd}}^{G(F \times X_0)}$, then $B$ is final in $(P_{U\text{bd}}^{G(F \times X_0)})_{/B}$

Thus in both cases (2.76) is obviously an equivalence.

Above we have seen that $M \simeq i_!^G i^*G M$. Since the restriction functor $i^*G$ preserves cofiltered limits, $i^*G$ preserves cocomplete objects. Hence in order to show that $M \in \text{PSh}^G_{\text{can}}(G \times X)$, it suffices to check that $i^*G M \in \text{Fun}^G(P_{U\text{bd}, \text{op}}^{G(F \times X_0)}, C)$. Since
If \( F \) and \( X_0 \) are finite, the poset \( \mathcal{P}^{\text{bd}}_{\mathcal{G}(F \times X_0)} \) is \( G \)-finite. Since \( M(B) \in \mathcal{C}^{G_B} \) for every bounded subset \( B \) by assumption, we indeed have \( i^*G M \in \text{Fun}^G(\mathcal{P}^{\text{bd},\text{op}}_{\mathcal{G}(F \times X_0)}, \mathcal{C})^\omega \) by Lemma 2.74.

### 2.5. Localisation

The functor \( \text{Sh}_C^{G,\text{eqm}} \) is not coarsely invariant. The reason is that morphisms between sheaves are local. In the present section we introduce a localisation of \( \text{Sh}_C^{G,\text{eqm}}(X) \) which adds morphisms which may propagate in a way which is controlled by coarse entourages. The resulting left-exact \( \infty \)-category is the desired category of equivariant \( X \)-controlled objects in \( \mathcal{C} \). We prove that the mapping spaces in the localisation can be calculated in terms of a calculus of fractions formula. The dependence on the data \( X \) and \( \mathcal{C} \) will be discussed in Section 2.6.

Consider \( X \) in \( G\text{Coarse} \). If \( U \) is in \( \mathcal{C}_X^{G,\Delta} \), and \( Y \) is in \( \mathcal{P}_X \), then we have \( U(Y) \subseteq Y \) (see (2.8)). We consider the family of these inclusions for all \( Y \) as a transformation \( U(\_\_) \to \text{id} \) of endofunctors of \( \mathcal{P}_X \). It induces a transformation \( \text{id} \to U^G \) of endofunctors of \( \text{PSh}_C^{G}(X) \) which by Corollary 2.29 restricts to \( \text{Sh}_C^{G}(X) \).

We form the labelled left-exact \( \infty \)-category \((\text{Sh}_C^{G}(X), W_X)\), where the labelling \( W_X \) is generated (Definition 7.35 and Remark 7.36) by the set of morphisms

\[
\{ M \to U^G_* M \mid U \in \mathcal{C}_X^{G,\Delta}, M \in \text{Sh}_C^{G}(X) \}.
\]

Recall Definition 7.37 of the left-exact localisation.

**Definition 2.75.** We define \( \hat{\text{V}}_C^G(X) := W_X^{-1} \text{Sh}_C^{G}(X) \) in \( \text{CAT}^{\text{Lex}}_{\infty,*} \).

Note that \( \hat{\text{V}}_C^G(X) \) is defined by a localisation in \( \text{CAT}^{\text{Lex}}_{\infty,*} \). By the following result, it is actually equivalent to the Dwyer-Kan localisation (see (7.20)) in \( \text{CAT}_{\infty} \).

**Proposition 2.76.**

1. The canonical left-exact functor \( \text{Sh}_C^{G}(X) \to \hat{\text{V}}_C^G(X) \) presents its target as the Dwyer-Kan localisation \( \text{Sh}_C^{G}(X)[W_X^{-1}] \).

2. For \( M, N \) in \( \text{Sh}_C^{G}(X) \) there is a natural equivalence of mapping spaces:

\[
\text{colim}_{U \in \mathcal{C}_X^{G,\Delta}} \text{Map}_{\text{Sh}_C^{G}(X)}(M,U^G_* N) \overset{\sim}{\to} \text{Map}_{\hat{\text{V}}_C^G(X)}(\ell M, \ell N).
\]

**Proof.** Let \( \ell : \text{Sh}_C^{G}(X) \to \text{Sh}_C^{G}(X)[W_X^{-1}] \) denote the Dwyer-Kan localisation in \( \infty \)-categories. By Lemma 7.40, for assertion (1) it suffices to show that \( \ell \) is left-exact.

In order to verify this condition, we first establish the formula for mapping spaces in \( \text{Sh}_C^{G}(X)[W_X^{-1}] \). Let \( N \) be in \( \text{Sh}_C^{G}(X) \). Then we consider the category \( W(N) := \mathcal{C}_X^{G,\Delta,\text{op}} \) and the functor \( \pi : W(N) \to \text{Sh}_C^{G}(X) \) given by \( U \mapsto U^G_* N \). This pair \((W(N), \pi)\) is a putative left calculus of fractions at \( N \) (Definition 7.27) for the pair \((\text{Sh}_C^{G}(X), W_X)\). Indeed, the diagonal entourage is the final object of \( W(N) \) and \( \pi(\text{diag}(X)) \simeq N \). Furthermore, the morphism \( U \to \text{diag}(X) \) in \( W(N) \) is sent to the morphism \( N \to U^G_* N \) in \( W_X \).

We now show that this putative left calculus of fractions is actually a left calculus of fractions. By Definition 7.28, this amounts to showing that for every \( V \) in \( \mathcal{C}_X^{G,\Delta} \) the morphism \( M \to V^G_* M \) induces an equivalence

\[
\text{colim}_{U \in W(N)^{\text{op}}} \text{Map}_{\text{Sh}_C^{G}(X)}(V^G_* M,U^G_* N) \to \text{colim}_{U \in W(N)^{\text{op}}} \text{Map}_{\text{Sh}_C^{G}(X)}(M,U^G_* N).
\]
Using the adjunction (2.13) we can rewrite this morphism in the form
\[(2.78)\]
\[
\colim_{U \in W(N)^{op}} \text{Map}_{\mathcal{P}sh\mathcal{C}(X)}(U^{*,G}V^{G}M,N) \to \colim_{U \in W(N)^{op}} \text{Map}_{\mathcal{P}sh\mathcal{C}(X)}(U^{*,G}M,N).
\]

We will now see by a cofinality argument that this morphism is an equivalence. For every Y in \(\mathcal{P}X\) we have inclusions
\[
V(U[Y]) \subseteq U[Y] \subseteq V(VU[Y]) \subseteq VU[Y].
\]

We thus get transformations
\[(2.79)\]
\[
(UV)^{*,G}M \to (UV)^{*,G}V^{G}M \to U^{*,G}M \to U^{*,G}V^{G}M.
\]

Since the functor \(C^{G,\Delta}_X \to C^{G,\Delta}_X\) given by \(U \mapsto VU\) is cofinal, and in view of (2.79), we can replace \(U^{*,G}V^{G}M\) in the domain of (2.78) by \(U^{*,G}M\) and (2.78) is obviously an equivalence.

The calculus of fractions provides the following formula for the mapping space:
\[(2.80)\]
\[
\colim_{U \in C^{G,\Delta}_X} \text{Map}_{\mathcal{P}sh\mathcal{C}(X)}(M,U^{G} N) \xrightarrow{\simeq} \text{Map}_{\mathcal{P}sh\mathcal{C}(X)}(\ell M,\ell N).
\]

The comparison map is natural in \(M\) and \(N\).

Since \(C^{G,\Delta}_X\) is filtered, we can conclude from (2.80) that the functor \(\ell\) preserves finite limits. \(\square\)

Let now \(X\) be in \(G\text{BornCoarse}\). In view of Lemma 2.50 we can consider the labelled left-exact \(\infty\)-category \((\text{Sh}^{G,\text{eqsm}}_C(X), W^{\text{eqsm}}_X)\), where \(W^{\text{eqsm}}_X\) is generated by the set
\[
\{ M \to U^{G}M \mid U \in C^{G,\Delta}_X, M \in \text{Sh}^{G,\text{eqsm}}_C(X) \}.
\]

Definition 2.77. We define \(V^{G}_C(X) := W^{\text{eqsm},-1}_X \text{Sh}^{G,\text{eqsm}}_C(X)\) in \(\text{Cat}^{\text{Lex}}_{\infty, *}\). \(\diamond\)

Proposition 2.78.

1. The canonical left-exact functor \(\text{Sh}^{G,\text{eqsm}}_C(X) \to V^{G}_C(X)\) presents its target as the Dwyer-Kan localisation \(\text{Sh}^{G,\text{eqsm}}_C(X)[W^{\text{eqsm},-1}_X]\).

2. For \(M, N\) in \(\text{Sh}^{G,\text{eqsm}}_C(X)\) there is a natural equivalence of mapping spaces:
\[(2.81)\]
\[
\colim_{U \in C^{G,\Delta}_X} \text{Map}_{\mathcal{P}sh\mathcal{C}(X)}(M,U^{G} N) \xrightarrow{\simeq} \text{Map}_{\mathcal{P}sh\mathcal{C}(X)}(\ell M,\ell N).
\]

Proof. This is shown by the same argument as for Proposition 2.76. \(\square\)

The universal property of the localisation provides the marked arrow in
\[(2.82)\]
\[
\begin{array}{ccc}
\text{Sh}^{G,\text{eqsm}}_C(X) & \longrightarrow & \text{Sh}^{G}_C(X) \\
\ell & \downarrow & \ell \\
V^{G}_C(X) & \longrightarrow & \hat{V}^{G}_C(X)
\end{array}
\]

Corollary 2.79. The marked arrow in (2.82) is a fully faithful inclusion.

Proof. We use that the upper horizontal arrow in (2.82) is fully faithful, and that the mapping spaces in \(V^{G}_C(X)\) and \(\hat{V}^{G}_C(X)\) are given by the coinciding formulas (2.81) and (2.77), respectively. \(\square\)
Example 2.80. Assume that $X$ in $G\text{Coarse}$ has the discrete coarse structure. Then $c_X^{G,\Delta} = \{ \text{diag}_X \}$. Since for $M$ in $\text{Sh}_C^G(X)$ the morphism $M \to \text{diag}_X^G M$ is equivalent to the identity of $M$, the class $W_X$ consists of equivalences. Consequently the canonical morphism $\ell : \text{Sh}_C^G(X) \to \hat{V}_C^G(X)$ is an equivalence.

Similarly, if $X$ in $G\text{BornCoarse}$ has the minimal coarse structure, then $W_X^{\text{eqsm}}$ consists of equivalences and the canonical morphism $\ell : \text{Sh}_C^{G,\text{eqsm}}(X) \to V_C^G(X)$ is an equivalence.

2.6. Functoriality of $V_C^G$. In this section we show that $V_C^G(X)$ depends functorially on $X$ and $C$.

If $f : X \to X'$ is a morphism in $G\text{Coarse}$ and $C$ is in $\text{Fun}(BG, \text{Cat}_\infty^{\text{LEX}})$, then we have the bold part of the following diagram

$$
\begin{array}{ccc}
\text{Sh}_C^G(X) & \xrightarrow{f^*_s} & \text{Sh}_C^G(X') \\
\ell_X & & \ell_{X'} \\
\hat{V}_C^G(X) & \xrightarrow{f^*_s} & \hat{V}_C^G(X')
\end{array}
$$

(2.83)

Lemma 2.81. The morphism $\widehat{f}_s^G$ descends essentially uniquely to a morphism $f_* : \hat{V}_C^G(X) \to \hat{V}_C^G(X')$ in $\text{CAT}_{\infty, s}^{\text{LEX}}$ completing the square (2.83).

Proof. In view of the universal property of the left-exact localisation $\ell_X$, it suffices to show that $\ell_X f^*_s$ sends the generators of $W_X$ to equivalences in $\hat{V}_C^G(X')$.

Let $V$ be in $c_X^{G,\Delta}$ and set $f(V)_\Delta := f(V) \cup \text{diag}(X)$ in $c_{X'}^{G,\Delta}$. For $Y'$ in $\mathcal{P}_{X'}$, we have an inclusion

$$
f^{-1}(f(V)_\Delta(Y')) \subseteq V(f^{-1}(Y'))
$$

of subsets of $X$. The family of these inclusions for all $Y'$ in $\mathcal{P}_{X'}$ induces a transformation

$$
t : \widehat{f}_s^G \circ V_s^G \to f(V)_\Delta^G \circ f^*_s : \text{Sh}_C^G(X) \to \text{Sh}_C^G(X')
$$

in the diagram below. If $M$ is in $\text{Sh}_C^G(X)$, then we consider $t(M) : M \to V_s^G M$ in $W_X$. The morphism $\widehat{f}_s^G t(M)$ fits into the following sequence of morphisms in $\text{Sh}_C^G(X')$:

$$
\begin{array}{ccc}
\widehat{f}_s^G M & \xrightarrow{t} & f(V)_\Delta^G \circ f^*_s M \\
\ell_{f(V)_\Delta}(\widehat{f}_s^G M) & & \ell_{f(V)_\Delta}(f(V)_\Delta^G \circ f^*_s M) \\
\widehat{f}_s^G V_s^G M & \xrightarrow{t} & f(V)_\Delta^G \circ f^*_s V_s^G M \\
\ell_{f(V)_\Delta}(\widehat{f}_s^G V_s^G M) & & \ell_{f(V)_\Delta}(f(V)_\Delta^G \circ f^*_s V_s^G M)
\end{array}
$$

The morphisms marked by $!$ belong to $W_X$, and induce equivalences in $\hat{V}_C^G(X')$. By the two-out-of-six property we conclude that all morphisms in this diagram are sent to equivalences in $\hat{V}_C^G(X')$. This shows that $\ell_X f^*_s W_X$ consists of equivalences in $\hat{V}_C^G(X')$.

Let $f : X \to X'$ be a morphism in $G\text{BornCoarse}$. As a consequence of Lemma 2.49, Corollary 2.79 and Proposition 2.78 we get:
Corollary 2.82. The morphism $f_*$ from (2.83) restricts to a morphism
\[ f_* : V^G_C(X) \rightarrow V^G_C(X') \]
in $\text{Cat}^{\text{L Lex}}_{\infty,*}$.

Let $X$ be in $G\text{Coarse}$, and let $\phi : C \rightarrow C'$ be a morphism in $\text{Fun}(BG, \text{Cat}^{\text{L Lex}}_{\infty,*})$. Then we obtain the following diagram:

(2.84)
\[
\begin{array}{ccc}
\sh^G_C(X) & \xrightarrow{\phi^G_*} & \sh^G_C(X') \\
\ell_X & \downarrow & \ell_X \\
\hat{V}^G_C(X) & \xrightarrow{\phi_*^*} & \hat{V}^G_C(X)
\end{array}
\]

Lemma 2.83. The morphism $\hat{\phi}^G_*$ descends essentially uniquely to a morphism $\phi_* : \hat{V}^G_C(X) \rightarrow \hat{V}^G_C(X)$ in $\text{CAT}^{\text{L Lex}}_{\infty,*}$ completing the square (2.84).

Proof. The morphism $\hat{\phi}^G_*$ preserves the generators of the localisation. \hfill \square

If $X$ is in $G\text{BornCoarse}$, then Lemma 2.51 implies:

Corollary 2.84. The morphism $\phi_*$ from (2.84) restricts to a morphism
\[ \phi_* : V^G_C(X) \rightarrow V^G_C(X) . \]

Let $\tilde{W}_X$ denote the labelling of $\sh^G_C(X)$ generated by the morphisms which are sent to equivalences by $\ell_X$. By Lemma 2.81 and Lemma 2.83, the functor $(X, C) \mapsto \sh^G_C(X)$ can be promoted to a functor with values in labelled left-exact $\infty$-categories
\[ (\ell \sh^G : G\text{Coarse} \times \text{Fun}(BG, \text{Cat}^{\text{L Lex}}_{\infty,*}) \rightarrow \text{CAT}^{\text{L Lex}}_{\infty,*} , \ X \mapsto (\sh^G_C(X), \tilde{W}_X) \]
(see (7.22) in the version for $\text{CAT}^{\text{L Lex}}_{\infty,*}$). We now invert the labelled maps by postcomposing with the localisation functor $\text{Loc}$ (the $\text{CAT}^{\text{L Lex}}_{\infty,*}$-version of (7.23)) and obtain the functor
\[ (\ell \sh^G) : G\text{Coarse} \times \text{Fun}(BG, \text{Cat}^{\text{L Lex}}_{\infty,*}) \rightarrow \text{CAT}^{\text{L Lex}}_{\infty,*} \]
(2.86) \[ \xrightarrow{\ell \sh^G} \xrightarrow{\ell \text{CAT}^{\text{L Lex}}_{\infty,*}} \xrightarrow{\text{Loc}} \text{CAT}^{\text{L Lex}}_{\infty,*} . \]

As a consequence of Corollary 2.82 and Corollary 2.84, we obtain a subfunctor (the restriction along (2.44) is hidden)
\[ (\ell \sh^G) : G\text{BornCoarse} \times \text{Fun}(BG, \text{Cat}^{\text{L Lex}}_{\infty,*}) \rightarrow \text{CAT}^{\text{L Lex}}_{\infty,*} . \]

(2.87) \[ \xrightarrow{\ell \sh^G_{\text{equiv}}} \xrightarrow{\ell \text{Cat}^{\text{L Lex}}_{\infty,*}} \xrightarrow{\text{Loc}} \text{Cat}^{\text{L Lex}}_{\infty,*} . \]

Finally, consider a coarse covering $f : X' \rightarrow X$ (Definition 2.23) and the square
\[ (2.88) \]
\[
\begin{array}{ccc}
\sh^G_C(X) & \xrightarrow{L_{\gamma_0}^G \hat{f}^*} & \sh^G_C(X') \\
\ell_X & \downarrow & \ell_{X'} \\
\hat{V}^G_C(X) & \xrightarrow{f'^*} & \hat{V}^G_C(X')
\end{array}
\]

Lemma 2.85.

1. The morphism $L_{\gamma_0}^G \hat{f}^*$ descends essentially uniquely to a morphism $f^*$ completing the square (2.88).
(2) If \( f \) is a covering (Definition 2.41), then the morphism \( f^* \) (in (1)) restricts to a morphism

\[ f^*: V^G_C(X) \to V^G_C(X') . \]

Proof. It follows from Lemma 2.30 that \( L^{*,G} \tilde{f}^{*,G} \) sends the generators of \( W_X \) to generators of \( W_{X'} \). This implies (1). (2) is now an immediate consequence of Proposition 2.54 and Corollary 2.79. \( \square \)

3. Properties of \( V^G_C \)

We now study the behavior of \( V^G_C \) as a functor on \( G \)-bornological coarse spaces. The results of this section are instrumental in showing that \( V^G_C \) gives rise to a coarse homology theory upon application of a finitary localising invariant, which will be the subject of Section 6.

3.1. Coarse invariance. Below we consider the set \( \{0, 1\} \) with the trivial \( G \)-action. Recall Example 2.38 and the monoidal structure from Remark 2.37. For every \( X \) in \( G \text{BornCoarse} \) the projection

\[ \{0, 1\}_{\text{max, max}} \otimes X \to X \]

is a morphism in \( G \text{BornCoarse} \).

Let \( M \) be some \( \infty \)-category and consider a functor \( E: G \text{BornCoarse} \to M \).

Definition 3.1. \( E \) is called coarsely invariant if the projection (3.1) induces an equivalence \( E(\{0, 1\}_{\text{max, max}} \otimes X) \to E(X) \) for every \( X \) in \( G \text{BornCoarse} \).

Remark 3.2. We say that two morphisms \( f, g: X \to X' \) in \( G \text{BornCoarse} \) are close to each other if \( (f \times g)(\text{diag}(X)) \in C_{X'} \). Closeness is an equivalence relation on \( \text{Hom}_{G \text{BornCoarse}}(X, X') \) for all \( X, X' \) in \( G \text{BornCoarse} \). A morphism in \( G \text{BornCoarse} \) is a coarse equivalence if it can be inverted up to closeness.

It is easy to see that the following assertions on \( E \) are equivalent:

(1) \( E \) is coarsely invariant.

(2) For every pair of close morphisms \( f, g \) we have \( E(f) \sim E(g) \).

(3) \( E \) sends coarse equivalences to equivalences.

\( \diamond \)

Lemma 3.3. The functor

\[ V^G_C: G \text{BornCoarse} \to \text{Cat}^{\text{Lex}}_{\infty, *} \]

is coarsely invariant.

Proof. We consider a pair \( f, g: X \to X' \) of morphisms which are close to each other. By Remark 3.2 it suffices to show that \( V^G_C(f) \simeq V^G_C(g) \).

Let \( Y' \) be in \( \mathcal{P}_X \), and assume that \( V' \in C^{G, \Delta}_{X'} \) is symmetric and such that \( (f \times g)(\text{diag}(X)) \subseteq V' \). Then we have the following chain of inclusions of subsets of \( X \):

\[ f^{-1}(Y') \supseteq g^{-1}(V'(Y')) \supseteq f^{-1}(V'^{\triangle}(Y')) \supseteq g^{-1}(V'^{\triangle, 2}(Y')) \supseteq g^{-1}(V'^{\triangle, 3}(Y')) . \]

For \( M \) in \( \text{Sh}^{G, \text{exm}}_{\infty}(X) \) we then get induced morphisms

\[ \tilde{f}^{*, G}_s M \to V^{*, G}_s \tilde{g}^{*, G}_s M \to V'^{\triangle, 2, G}_s \tilde{f}^{*, G}_s M \to V'^{\triangle, 3, G}_s \tilde{g}^{*, G}_s M . \]
The marked morphisms are sent to equivalences in $V_C^G(X')$ since they belong to $W_{X'}$. By the two-out-of-six property for equivalences, all morphisms in (3.2) are sent to equivalences, in particular, the first one. Consequently, the second map in the zig-zag

$$\tilde{g}_* M \xrightarrow{\iota_V(\bar{G}_* M)} V'_* \tilde{g}_* M \xleftarrow{f_*}$$

is an equivalence. The morphism $\iota_V(\bar{G}_* M)$ is also an equivalence because it belongs to $W_{X'}$. We conclude that $f_*$ and $g_*$ are naturally equivalent functors $V_C^G(X) \rightarrow V_C^G(X')$. □

3.2. Flasques. Let $X$ be in $G\text{BornCoarse}$.

**Definition 3.4** ([BEKWa, Def. 3.8]). $X$ is flasque if it admits an endomorphism $f: X \rightarrow X$ with the following properties:

1. $f$ is close to $\text{id}_X$.
2. For every $U$ in $C_X$ we have $\bigcup_{n \in \mathbb{N}} f^n(U) \in C_X$.
3. For every $B$ in $B_X$ there exists $n$ in $\mathbb{N}$ such that $f^n(X) \cap B = \emptyset$.

We say that $f$ implements the flasqueness of $X$.

**Definition 3.5.** $X$ is pre-flasque if it admits an endomorphism $f: X \rightarrow X$ with properties **Definition 3.4** (2) and **Definition 3.4** (3). ♦

We say that $f$ implements the pre-flasqueness of $X$.

Let $M$ be a semi-additive $\infty$-category (**Definition 7.20**). A semi-additive category is enriched in commutative monoids, and we use the symbol $+$ in order to denote the sum of morphisms.

Let $M$ be an object of $M$.

**Definition 3.6.** $M$ is flasque if it admits an endomorphism $S$ such that

$$S \simeq \text{id}_M + S.$$ ♦

We again say that $S$ implements the flasqueness of $M$. By $M^0$ we denote the smallest full subcategory of $M$ which is closed under filtered colimits and contains all flasque objects.

Let $E: G\text{BornCoarse} \rightarrow M$ be a functor.

**Definition 3.7.** $E$ preserves flasqueness if it sends flasque objects of $G\text{BornCoarse}$ to objects in $M^0$. ♦

**Remark 3.8.** If $M$ is additive (**Definition 7.52**), then a flasque object is a zero object. Consequently, if $M$ is additive, then $M^0$ consists of zero objects. In this case, the following conditions are equivalent:

1. $E$ preserves flasqueness.
2. $E$ vanishes on flasques, i.e., sends flasque $G$-bornological coarse spaces to zero objects.

Below in **Lemma 3.15** (3) we will show that the functor $V_C^G$ is flasqueness preserving. But note that $V_C^G$ is not the final object of consideration. We will also need to show that certain functors derived from $V_C^G$ by auxiliary constructions in **Section 4** are flasqueness preserving, too. To this end we introduce the stronger notions of a factorially flasqueness and functorially pre-flasqueness preserving functor, and we verify that $V_C^G$ has these properties.
Definition 3.9.  
(1) Let $\text{Fl}^{\text{pre}}(G\text{BornCoarse})$ be the category of pairs $(X, f)$ consisting of $X$ in $G\text{BornCoarse}$ and an endomorphism $f$ implementing pre-flasqueness of $X$.

(2) Let $\text{Fl}(G\text{BornCoarse})$ be the full subcategory of $\text{Fl}^{\text{pre}}(G\text{BornCoarse})$ of those pairs $(X, f)$, where $f$ implements flasqueness of $X$. ♦

Let $M$ be a pre-additive $\infty$-category.

Definition 3.10.  
(1) Let $\text{End}(M)$ be the full subcategory of the arrow category of $M$ spanned by endomorphisms of the objects of $M$.

(2) Let $\text{Fl}(M)$ be the full subcategory of $\text{End}(M)$ of those pairs $(M, S)$, where $S: M \to M$ implements flasqueness of $M$. ♦

Let $P$ be some auxiliary $\infty$-category (e.g., $\text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, *})$ in our application), and let $E: G\text{BornCoarse} \times P \to M$ be a functor.

Definition 3.11. $E$ functorially preserves pre-flasqueness if it refines to a functor $\text{Fl}^{\text{pre}}(E): \text{Fl}^{\text{pre}}(G\text{BornCoarse}) \times P \to \text{End}(M)$ such that for every $(X, f)$ in $\text{Fl}^{\text{pre}}(G\text{BornCoarse})$ and $P$ in $P$ we have an equivalence

$$S \simeq \text{id}_M + E(f, P) \circ S,$$

where $(M, S) := \text{Fl}^{\text{pre}}(E)((X, f), P)$. \footnote{Warning: We do not require that $S$ is equivalent to $E(f)$.}

In detail, the word \textit{refines} means that there is a commutative diagram

$$
\begin{array}{ccc}
\text{Fl}^{\text{pre}}(G\text{BornCoarse}) \times P & \xrightarrow{\text{Fl}^{\text{pre}}(E)} & \text{End}(M) \\
q \times \text{id}_P & \downarrow & p \\
G\text{BornCoarse} \times P & \xrightarrow{E} & M \\
\end{array}
$$

filled by an equivalence, where $p$ and $q$ are the functors forgetting the respective endomorphisms. ♦

Definition 3.12. $E$ functorially preserves flasqueness if it refines to a functor $\text{Fl}(E): \text{Fl}(G\text{BornCoarse}) \times P \to \text{Fl}(M)$.

The word \textit{refines} again means that there is a commutative diagram

$$
\begin{array}{ccc}
\text{Fl}(G\text{BornCoarse}) \times P & \xrightarrow{\text{Fl}(E)} & \text{Fl}(M) \\
\downarrow & & \downarrow \\
G\text{BornCoarse} \times P & \xrightarrow{E} & M \\
\end{array}
$$

filled by an equivalence, where the vertical functors forget the respective endomorphisms.

Lemma 3.13. Assume:
(1) $E$ functorially preserves pre-flasqueness.

\footnote{Warning: We do not require that $S$ is equivalent to $E(f)$.}
(2) For every $P$ in $\mathbf{P}$ the functor $E(-, P) : \mathbf{G BornCoarse} \to \mathbf{M}$ is coarsely invariant.

Then $E$ functorially preserves flasqueness.

Proof. Let $(X, f)$ be in $\mathbf{Fl}(\mathbf{G BornCoarse})$, $P$ in $\mathbf{P}$, and consider the object

$$(M, S) := \mathbf{Fl}^{pre}(E)((X, f), P)$$

in $\text{End}(\mathbf{M})$. Since $f$ is close to the identity of $X$ and $E(-, P)$ is coarsely invariant, we have $E(f, P) \simeq \text{id}_M$. Hence the equivalence (3.4) reduces to (3.3), and we have $(M, S) \in \mathbf{Fl}(\mathbf{M})$.

The following is obvious from the definitions.

**Corollary 3.14.** If $E$ functorially preserves flasqueness, then $E(-, P)$ preserves flasqueness for every object $P$ in $\mathbf{P}$.

Let $\mathbf{C}$ be in $\mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_{\infty, *})$. Note that $\mathbf{Cat}^{\text{LEX}}_{\infty, *}$ is semi-additive by Lemma 7.21. So Definitions 3.7, 3.11 and 3.12 apply to the functor

$$V^G : \mathbf{G BornCoarse} \times \mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_{\infty, *}) \to \mathbf{Cat}^{\text{LEX}}_{\infty, *},$$

from (2.87). In this case we use the more natural symbol $\times$ instead of $+$ for the sum of morphisms.

**Lemma 3.15.**

1. The functor $V^G$ functorially preserves pre-flasqueness.
2. The functor $V^G$ functorially preserves flasqueness.
3. The functor $V^G$ preserves flasqueness for every $\mathbf{C}$ in $\mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_{\infty, *})$.

Proof. We start with (1). We consider the functor $\mathbf{Sh}$ from (2.36). We extend it to a functor

$$\mathbf{Fl}^{pre}(\mathbf{Sh}) : \mathbf{Fl}^{pre}(\mathbf{G BornCoarse}) \times \mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_{\infty, *}) \to \text{End}(\mathbf{Fun}(BG, \mathbf{CAT}^{\text{LEX}}_{\infty, *}))$$

such that

$$\mathbf{Fl}^{pre}(\mathbf{Sh})((X, f), C) := (\mathbf{Sh}_C(X), \prod_{n \in \mathbb{N}} \hat{f}_n) .$$

We must argue that the infinite product $\hat{S}(X, f) := \prod_{n \in \mathbb{N}} \hat{f}_n$ of morphisms in (3.7) exists. For $M$ in $\mathbf{Sh}_C(X)$, by Definition 3.4 (2) there exists $U$ in $C^G$ such that $\hat{f}_n(M)$ in $\mathbf{Sh}_C^U(X)$ for all $n \in \mathbb{N}$. Since $\mathbf{Sh}_C^U(X)$ belongs to $\mathbf{CAT}^{\text{LEX}}_{\infty, *}$, the product $\prod_{n \in \mathbb{N}} \hat{f}_n(M)$ exists in $\mathbf{Sh}_C(X)$ and hence in $\mathbf{Sh}_C(X)$.

By composing $\mathbf{Fl}^{pre}(\mathbf{Sh})$ with $\lim_{BG}$ we obtain a functor

$$\mathbf{Fl}^{pre}(\mathbf{Sh})^G : \mathbf{Fl}^{pre}(\mathbf{G BornCoarse}) \times \mathbf{Fun}(BG, \mathbf{Cat}^{\text{LEX}}_{\infty, *}) \to \text{End}(\mathbf{CAT}^{\text{LEX}}_{\infty, *})$$

$$(X, f, C) \mapsto (\mathbf{Sh}_C^G(X), \hat{S}(X, f)^G) .$$

Consider the square

\[
\begin{array}{ccc}
\mathbf{Sh}_C^G(X) & \xrightarrow{\hat{S}(X, f)^G} & \mathbf{Sh}_C^G(X) \\
\downarrow \xi_X & & \downarrow \xi_X \\
\mathbf{\hat{V}}_C^G(X) & \xrightarrow{\hat{S}(X, f)} & \mathbf{\hat{V}}_C^G(X)
\end{array}
\]
Similarly as in the proof of Lemma 2.81, using Definition 3.4 (2) on \( f \), we check that \( \ell_X \tilde{S}(X, f)^G \) sends the generators of \( W_X \) to equivalences in \( \hat{V}_C^G(X) \). It follows that \( \tilde{S}(X, f)^G \) descends to the desired functor
\[
S(X, f) : \hat{V}_C^G(X) \to \hat{V}_C^G(X).
\]
The obvious equivalence
\[
\tilde{S}(X, f)^G \simeq \mathrm{id}_{\hat{V}_C^G(X)} \times (f_*^G \circ \tilde{S}(X, f)^G)
\]
descends to the desired equivalence (see (3.4))
\[
S(X, f) \simeq \mathrm{id}_{\hat{V}_C^G(X)} \times (f_* \circ S(X, f)).
\]
It remains to show that \( S(X, f) \) preserves the full subcategory \( V_C^G(X) \) of \( \hat{V}_C^G(X) \).

To this end, we show that \( \tilde{S}(X, f)^G \) preserves the subcategory \( \mathrm{Sh}_C^{G, \text{eqsm}}(X) \) of \( \mathrm{Sh}_C^G(X) \).

Let \( M \) be in \( \mathrm{Sh}_C^{G, \text{eqsm}}(X) \). Consider an \( H \)-bounded subset \( Y \) of \( X \). By Definition 3.4 (3), there exists \( n_0 \) in \( \mathbb{N} \) such that \( f^{n_0}(X) \cap Y = \emptyset \). Then we have
\[
(\tilde{S}(X, f)^G(M))(Y) \simeq \prod_{n \leq n_0} \hat{f}^n_* G M(Y).
\]

By Lemma 2.49, \( \hat{f}^n_* G M \) is equivariantly small for all \( n \). Thus every factor of the product belongs to \( C^{H, \omega} \), and therefore the finite product, too. We conclude that \( \tilde{S}(X, f)^G(M) \in \mathrm{Sh}_C^{G, \text{eqsm}}(X) \). This finishes the proof of assertion (1).

(2) follows from (1) by Lemma 3.13 since \( V_C^G \) is coarsely invariant by Lemma 3.3. (3) follows from (2) and Corollary 3.14.

3.3. \( u \)-continuity. Let \( X \) be in \( G_{\text{BornCoarse}} \). If \( U \) is in \( C_C^G \), then \( X_U \) denotes the \( G \)-bornological coarse space obtained from \( X \) by replacing the original coarse structure of \( X \) by the \( G \)-coarse structure \( C(\{U\}) \) generated by \( U \). There is a canonical morphism \( X_U \to X \) given by the identity of the underlying sets.

Let \( M \) be an \( \infty \)-category which admits all small filtered colimits. Consider a functor \( E : G_{\text{BornCoarse}} \to M \).

**Definition 3.16.** \( E \) is \( u \)-continuous if the natural morphism
\[
\colim_{U \in C_C^G} E(X_U) \to E(X)
\]
is an equivalence for every \( X \) in \( G_{\text{BornCoarse}} \).

Let \( C \) be in \( \text{Fun}(BG, \text{Cat}_{\infty, \ast}^{\text{LEX}}) \) and note that \( \text{CAT}_{\infty, \ast}^{\text{Lex}} \) admits small filtered colimits by (the large version of) Proposition 7.9. Recall the functor \( \hat{V}_C^G \) from (2.86).

**Proposition 3.17.** The functor \( \hat{V}_C^G : G_{\text{BornCoarse}} \to \text{CAT}_{\infty, \ast}^{\text{Lex}} \) is \( u \)-continuous.

**Proof.** Let \( X \) be in \( G_{\text{BornCoarse}} \). Filtered colimits and limits in \( \text{CAT}_{\infty, \ast}^{\text{Lex}} \) may by computed in \( \text{CAT}_{\infty} \). Moreover, the localisation involved in the definition of \( \hat{V}_C^G(X) \) may also be taken in \( \text{CAT}_{\infty} \) by Proposition 2.76. Consider the following
chain of functors:
\[
\operatorname{colim}_{U \in \mathcal{C}_X^G} \hat{V}_C^G(X_U) \simeq \operatorname{colim}_{U \in \mathcal{C}_X^G} \left( \operatorname{colim}_{U \in \mathcal{C}_X^G} \left( \left( \lim_{BG} \mathbf{Sh}_C(X_U) \right) \left[ W_{X_U}^{-1} \right] \right) \right)
\]
\[
\overset{i}{\rightarrow} \left( \operatorname{colim}_{U \in \mathcal{C}_X^G} \mathbf{Sh}_C(X_U) \right) \left[ \operatorname{colim}_{U \in \mathcal{C}_X^G} W_{X_U}^{-1} \right]
\]
\[
\overset{ii}{\rightarrow} \left( \lim_{BG} \operatorname{colim}_{U \in \mathcal{C}_X^G} \mathbf{Sh}_C(X_U) \right) \left[ \operatorname{colim}_{U \in \mathcal{C}_X^G} W_{X_U}^{-1} \right]
\]
\[
\overset{iii}{\rightarrow} \left( \lim_{BG} \mathbf{Sh}_C(X) \right) \left[ W_X^{-1} \right]
\]
\[
\simeq \hat{V}_C^G(X)
\]

The morphism marked by $i$ is an equivalence since the localisation Loc is a left adjoint (see (7.23)), and hence commutes with filtered colimits. The morphism marked by $ii$ is an equivalence by Lemma 7.23 since $BG$ has only a single object, $\mathcal{C}_X^G$ is filtered, and $\mathbf{Sh}(X_U) \to \mathbf{Sh}(X_{U'})$ is fully faithful for all $U, U'$ in $\mathcal{C}_X^G$ with $U \subseteq U'$. The morphism marked by $iii$ is an equivalence since the map
\[
\mathcal{C}_X^G \to \{(U, V) \in \mathcal{C}_X^G \times \mathcal{C}_X^G \mid U \in \mathcal{C}_X^G, \ V \in \mathcal{C}_X^G, \ U \to (U, U)\}
\]
is cofinal. \hfill \Box

**Lemma 3.18.** The functor $V_C^G : \mathbf{GBornCoarse} \to \mathbf{Cat}_{\infty,*}^{\text{Lex}}$ is $u$-continuous.

**Proof.** Lemma 2.49 applied to the morphisms $X_U \to X$, Proposition 3.17, and the fact that a filtered colimit of fully faithful functors is fully faithful implies that we have an inclusion $\operatorname{colim}_{U \in \mathcal{C}_X^G} \hat{V}_C^G(X_U) \to \hat{V}_C^G(X)$ of full subcategories of $\hat{V}_C^G(X)$. It is also essentially surjective since the notion of equivariant smallness is independent of the coarse structure. \hfill \Box

3.4. **Subspace inclusions.** In this section we derive some technical results which will enter the discussion of excision for $V_C^G$ in Section 3.5.

Let $i : Z \to X$ be the inclusion of a subspace in $\mathbf{GBornCoarse}$.

**Proposition 3.19.** The functors $i_* : \hat{V}_C^G(Z) \to \hat{V}_C^G(X)$ and $i_* : V_C^G(Z) \to V_C^G(X)$ are fully faithful.

The proof requires a little preparation.

We fix $U$ in $\mathcal{C}^{G, \Delta}_X$ and consider the sub-poset
\[
\mathcal{P}^U_X := \{ A \in \mathcal{P}_X \mid U_Z(A \cap Z) = U(A) \cap Z \}
\]
of $\mathcal{P}_X$, where $U_Z := U \cap (Z \times Z)$ is considered as an entourage in $\mathcal{C}^{G, \Delta}_Z$.

**Lemma 3.20.**

1. If $A, A'$ are in $\mathcal{P}^U_X$, then we also have $A \cap A' \in \mathcal{P}^U_X$.
2. For every $Y$ in $\mathcal{P}_X$ the coslice $(\mathcal{P}^U_X)_Y$ has a unique minimal element.

**Proof.** (1) follows from the fact that thinning (see (2.8)) preserves intersections.

We now show (2). We will show that the unique minimal element of $(\mathcal{P}^U_X)_Y$ is given by
\[
U[Y, Z] := Y \cup \bigcup_{x \in Y \cap Z, U_Z(x) \subseteq Y \cap Z} U[x] \setminus Z.
\]
We first check that \( U[Y, Z] \) belongs to the set \((\mathcal{P}_U^U)_Y \).

By definition, we have \( Y \subseteq U[Y, Z] \). It remains to check that

\[
U_Z(U[Y, Z] \cap Z) = U(U[Y, Z]) \cap Z.
\]

The inclusion

\[
U(U[Y, Z]) \cap Z \subseteq U_Z(U[Y, Z] \cap Z)
\]

is easy. We discuss the other inclusion

\[
U_Z(U[Y, Z] \cap Z) \subseteq U(U[Y, Z]) \cap Z.
\]

Let \( x \) be in \( U_Z(U[Y, Z] \cap Z) \). Then we have the relations \( x \in Y \cap Z \) and \( U_Z[x] \subseteq Y \cap Z \).

This implies

\[
U[x] = U_Z[x] \cup (U[x] \setminus Z) \subseteq (Y \cap Z) \cup U[Y, Z] = U[Y, Z],
\]

and hence \( x \in U(U[Y, Z]) \cap Z \).

We now show that \( U[Y, Z] \) is the unique minimal element in \((\mathcal{P}_X^U)_Y \). Suppose that \( A \) is any other element of \((\mathcal{P}_X^U)_Y \). Assume that \( x \) is in \( U[Y, Z] \). We consider two cases:

1. \( x \in Y \): Then \( x \in A \).
2. \( x \notin Y \): Then \( x \in U[x] \setminus Z \) for some \( x' \) in \( Y \cap Z \) satisfying \( U_Z[x'] \subseteq Y \cap Z \subseteq A \cap Z \). Since \( U_Z(A \cap Z) = U(A) \cap Z \), we also have \( U[x'] \subseteq A \). Because of \( x \in U[x'] \), this implies \( x \in A \).

□

We define a map \( R : \mathcal{P}_X \to \mathcal{P}_X^U \) by \( R(Y) := U[Y, Z] \). The following assertions are then clear:

**Corollary 3.21.**

1. \( R \) is a poset morphism.
2. We have a natural transformation \( \text{id} \to R \).
3. If \( Y \in \mathcal{P}_X^U \), then \( Y = R(Y) \).
4. For all \( Y \) in \( \mathcal{P}_X \) we have \( Z \cap R(Y) = Y \).

We now consider a poset \( P \) (later it will be \( \mathcal{P}_X^\text{op} \)). To \( P \) we associate the poset \( \text{Tw}(P) \) whose elements are pairs \( (p, q) \) with \( p \leq q \), and with the relation \( (p, q) \leq (p', q') \) if and only if \( p \leq p' \leq q' \leq q \). Note that we have a projection \( \text{Tw}(P) \to P \times P^{\text{op}} \). Let \( Q \) be a sub-poset of \( P \). Then define the sub-poset

\[
\text{Tw}(Q, P) := \{(q, p) \in \text{Tw}(P) \mid q \in Q \}
\]

of \( \text{Tw}(P) \).

**Lemma 3.22.** The canonical inclusion \( \text{Tw}(Q) \subseteq \text{Tw}(Q, P) \) is cofinal.

**Proof.** We need to show that the slice \( \text{Tw}(Q)_{(q, p)} \) is weakly contractible for all \((q, p)\) in \( \text{Tw}(Q, P) \). Let \( c : \text{Tw}(Q)_{(q, p)} \to \text{Tw}(Q)_{(q, q)} \) denote the functor mapping \((q, p) \leq (q_1, q_2)\) to \((q, p) \leq (q, q)\), and let \( \text{const} : \text{Tw}(Q)_{(q, p)} \to \text{Tw}(Q)_{(q, q)} \) denote the functor with constant value \((q, p) \leq (q, q)\). Since for every \((q, p) \leq (q_1, q_2)\) we have \((q, q) \leq (q_1, q_2)\) and \((q, q) \leq (q, q)\), we obtain a zig-zag of natural transformations \( c \to \text{const} \). Consequently, \( \text{Tw}(Q)_{(q, p)} \) is weakly contractible. □

Let \( C \) be some \( \infty \)-category and consider two functors \( M, N : P \to C \). Then we form the functor

\[
\text{Map}(M, N) : \text{Tw}(P)^{\text{op}} \to P^{\text{op}} \times P \xrightarrow{M^{\text{op}} \times N} C^{\text{op}} \times C \xrightarrow{\text{Map}_{\text{op}}} \text{Spc}.
\]
Note that (see e.g. [Gla16, Prop. 2.3])
\[(3.9) \quad \text{Map}_{\text{Fun}(P, C)}(M, N) \simeq \lim_{\text{Tw}(P)^{\text{op}}} \text{Map}(M, N).\]

**Lemma 3.23.** Assume that there exists a poset morphism \(R: P \to Q\) such that:

1. For all \(q\) in \(Q\) we have \(R(q) = q\).
2. For every \(p\) in \(P\) we have \(R(p) \leq p\).
3. For every \(p\) in \(P\) the morphism \(M(R(p)) \to M(p)\) is an equivalence.

Then the commutative diagram
\[
\begin{array}{ccc}
\text{Tw}(Q, P)^{\text{op}} & \xrightarrow{\text{Map}(M, N)|_{\text{Tw}(Q, P)^{\text{op}}}} & \text{Spc} \\
\downarrow & & \\
\text{Tw}(P)^{\text{op}} & \xrightarrow{\text{Map}(M, N)} & \text{Spc}
\end{array}
\]

exhibits \(\text{Map}(M, N)\) as a right Kan extension of \(\text{Map}(M, N)|_{\text{Tw}(Q, P)^{\text{op}}}\).

**Proof.** We have to check that for every \((p, p')\) in \(\text{Tw}(P)\) the canonical map
\[(3.10) \quad \text{Map}_C(M(p), N(p')) \to \lim_{(p, p') \leq (q, q') \in \text{Tw}(Q, P)^{\text{op}}(p, q')/} \text{Map}_C(M(q), N(q'))
\]
is an equivalence. By assumption, the poset \(\text{Tw}(Q, P)^{\text{op}}(p, q')/\) has an initial element \((R(p), p')\). So we replace the limit by the evaluation at this initial element and use the equivalence \(M(R(p)) \to M(p)\) in order to conclude that \((3.10)\) is an equivalence. \(\square\)

**Corollary 3.24.** Under the assumptions of Lemma 3.23, the canonical map
\[
\lim_{\text{Tw}(P)^{\text{op}}} \text{Map}(M, N) \to \lim_{\text{Tw}(Q)^{\text{op}}} \text{Map}(M, N)
\]
is an equivalence.

**Proof.** This follows from combining Lemma 3.22 and Lemma 3.23 with the observation that the limit of a right Kan extension is equivalent to the limit of the extended functor. \(\square\)

**Proof of Proposition 3.19.** In view of Corollary 2.79, it suffices to show that the functor \(i_*: \hat{\text{V}}^G_C(Z) \to \hat{\text{V}}^G_C(X)\) is fully faithful.

Let \(M\) and \(N\) in \(\text{Sh}^G_C(Z)\). We will use the formula for mapping spaces in the localisation provided by Proposition 2.76. The map of posets \(C^G_X \to C^G_Z\) given by \(U \mapsto U_Z := U \cap (Z \times Z)\) is cofinal. Note that \(\hat{\text{V}}^G_*\) is fully faithful by the second assertion of Corollary 2.27. The map
\[
\text{Map}_{\hat{\text{V}}^G_C(Z)}(\ell Z M, \ell Z N) \to \text{Map}_{\hat{\text{V}}^G_C(X)}(i_* \ell X M, i_* \ell X N)
\]
induced by \(i\) can be identified with the map
\[
\text{colim}_{U \in C^G_X} \text{Map}_{\text{Sh}^G_C(X)}(\hat{\text{V}}^G_* U Z M, N) \to \text{colim}_{U \in C^G_X} \text{Map}_{\text{Sh}^G_C(X)}(\hat{\text{V}}^G_* U Z M, \hat{\text{V}}^G_* U Z N)
\]
induced by the canonical map \(\hat{\text{V}}^G_* U Z M \to U Z i_* M\). We claim that the map
\[
\text{Map}_{\text{Sh}^G_C(X)}(\hat{\text{V}}^G_* U Z M, N) \to \text{Map}_{\text{Sh}^G_C(X)}(\hat{\text{V}}^G_* U Z M, \hat{\text{V}}^G_* U Z N)
\]
is an equivalence for every \(U\) in \(C^G_X\). Using the equivalence \((3.9)\) in order to calculate the mapping space in \(\text{PSh}_C\) (and therefore in \(\text{Sh}_C\)), and that \(\text{Sh}^G_C(X) \simeq \text{Sh}^G_C(X)\).
\[ \text{lim}_{BG} \text{Sh}_C(X), \text{this map can be identified with the image under } \text{lim}_{BG} \text{ of the following natural map in } \text{Fun}(BG, \text{Spc}) \]

\[ \lim_{\text{Tw}(\mathcal{P}_X^{op})^{op}} \text{Map}_C(M(\cap Z), N(UZ(\cap Z))) \to \lim_{\text{Tw}(\mathcal{P}_X^{op})^{op}} \text{Map}_C(M(\cap Z), N(U(\cap Z))). \]

By Corollary 3.21, the inclusion of posets

\[ \mathcal{P}_X^{U, op} \to \mathcal{P}_X^{op} \]

(see (3.8) for the definition of \( \mathcal{P}_X^{U} \)) satisfies the assumptions of Lemma 3.23.

Then Corollary 3.24 implies that (3.11) is canonically equivalent to the map

\[ \lim_{\text{Tw}(\mathcal{P}_X^{op})^{op}} \text{Map}_C(M(\cap Z), N(U(\cap Z))). \]

By definition of \( \mathcal{P}_X^{U} \), we can identify the arguments of \( N \) on both sides and see that this morphism is an equivalence. □

3.5. Excision. Let \( \mathcal{Y} := (Y_\ell)_{\ell \in L} \) be a filtered family of invariant subsets of an object \( X \) in \( \text{GBornCoarse} \). The members \( Y_\ell \) will be considered as objects of \( \text{GBornCoarse} \) with the coarse and bornological structures induced from \( X \).

Definition 3.25. \( \mathcal{Y} \) is a big family if for every \( U \) in \( C_X \) and \( \ell \) in \( L \) there exists \( \ell' \) in \( L \) such that \( U[Y_\ell] \subseteq Y_{\ell'} \) (see (2.7) for the thickening construction).

For a functor \( E: \text{GBornCoarse} \rightarrow M \) to a cocomplete target we set

\[ E(\mathcal{Y}) := \text{colim}_{\ell \in L} E(Y_\ell). \]

The family of inclusions \( (Y_\ell \rightarrow X)_{\ell \in L} \) induces a canonical morphism

\[ E(\mathcal{Y}) \rightarrow E(X). \]

Definition 3.26. A complementary pair on \( X \) is a pair \( (Z, \mathcal{Y}) \) of an invariant subset \( Z \) and a big family \( \mathcal{Y} \) such that there exists \( \ell \) in \( L \) with \( Z \cup Y_\ell = X \).

We can form the big family \( Z \cap \mathcal{Y} := (Z \cap Y_\ell)_{\ell \in L} \) on \( Z \).

Definition 3.27. \( E \) is called excisive if for every \( X \) in \( \text{GBornCoarse} \) with a complementary pair \( (Z, \mathcal{Y}) \) the commutative square

\[ \begin{array}{c}
E(Z \cap \mathcal{Y}) \longrightarrow E(Z) \\
\downarrow \quad \downarrow \\
E(\mathcal{Y}) \longrightarrow E(X)
\end{array} \]

is a pushout square.

Remark 3.28. We do not expect that \( V_G^{U} \) is excisive in the sense of Definition 3.27. Proposition 3.30 below is the appropriate statement in this case using a suitable notion of excisive square for left-exact categories introduced in Definition 7.49.

Let \( E: \text{GBornCoarse} \rightarrow \text{Cat}^{\text{Lex}}_{\text{co}, \infty} \) be a functor.
Definition 3.29. \( E \) is called \( l \)-excisive if for every \( X \) in \( \text{GBornCoarse} \) with a complementary pair \((Z,Y)\) the commutative square

\[
\begin{array}{ccc}
E(Z \cap Y) & \longrightarrow & E(Z) \\
\downarrow & & \downarrow \\
E(Y) & \longrightarrow & E(X)
\end{array}
\]

is an excisive square in \( \text{Cat}^{\text{Lex}}_{\infty,*} \) (Definition 7.49). ♦

Let \( C \) be in \( \text{Fun}(BG, \text{Cat}^{\text{Lex}}_{\infty,*}) \).

Proposition 3.30. The functor \( V^G_C : \text{GBornCoarse} \rightarrow \text{Cat}^{\text{Lex}}_{\infty,*} \) is \( l \)-excisive.

The proof of Proposition 3.30 occupies the remainder of this section.

By Lemma 2.49, there is a commutative square

\[
\begin{array}{ccc}
\text{Sh}_C^{G,\text{eqsm}}(Z \cap Y) & \xrightarrow{i^*} & \text{Sh}_C^{G,\text{eqsm}}(Z) \\
\downarrow g^G_{\ast} & & \downarrow g^G_{\ast} \\
\text{Sh}_C^{G,\text{eqsm}}(Y) & \xrightarrow{i^*} & \text{Sh}_C^{G,\text{eqsm}}(X)
\end{array}
\]

in \( \text{Cat}^{\text{Lex}}_{\infty,*} \), where \( g^G_{\ast} \) is induced by the family of morphisms \((i_{\ell} : Z \cap Y_{\ell} \rightarrow Y_{\ell})_{\ell \in L}\) using Lemma 2.49, and \( i^*_Z \) and \( i^*_X \) are instances of the morphism \((3.13)\). By Corollary 2.82, the square \((3.14)\) induces a square

\[
\begin{array}{ccc}
V^G_C(Z \cap Y) & \xrightarrow{i^*} & V^G_C(Z) \\
\downarrow g^* & & \downarrow g^* \\
V^G_C(Y) & \xrightarrow{i^*_X} & V^G_C(X)
\end{array}
\]

in \( \text{Cat}^{\text{Lex}}_{\infty,*} \). In order to show Proposition 3.30, we must show that the square \((3.15)\) in \( \text{Cat}^{\text{Lex}}_{\infty,*} \) is excisive. This amounts to showing that the horizontal functors are fully faithful, and that the induced morphism between their stable cofibres (see Lemma 7.48) is an equivalence.

Note that \( i_X \) is the colimit of the family of functors \((V^G_C(Y) \rightarrow V^G_C(X))_{\ell \in L}\).

Since the members of this family are fully faithful by Proposition 3.19, and a filtered colimit of fully faithful functors is again fully faithful, we conclude that \( i_X \) is fully faithful. Similarly, \( i_Z \) is fully faithful.

The rest of the argument is devoted to the comparison of the stable cofibres. We define the labelling \( W_{i_X} \) of \( V^G_C(X) \) to be generated by all morphisms with fibres in the essential image of \( i_X \). Then the stable cofibre of \( i_X \) is defined (Definition 7.45) by stabilisation (Definition 7.42) and localisation (Definition 7.37) as follows:

\[
\text{Cofib}^s(i_X) := \text{Sp}(W_{i_X}^{-1}V^G_C(X)).
\]

We define the labelling \( W_{i_Z} \) of \( V^G_C(Z) \) and \( \text{Cofib}^s(i_Z) \) similarly.
We consider the following diagram (using the notation introduced in (7.28))

\[
\begin{array}{c}
\text{Sh}_C^{G,\text{eqsm}}(X) \xrightarrow{\ell_X} V_C^G(X) \xrightarrow{C_{\text{fib}}(\iota_X)} \\
\downarrow \gamma^* \circ G \\
\text{Sh}_C^{G,\text{eqsm}}(Z) \xrightarrow{\ell_Z} V_C^G(Z) \xrightarrow{C_{\text{fib}}(\iota_Z)}
\end{array}
\]

in $\text{Cat}^{\text{Lex}}_{\infty,*}$. At the moment forget the dashed part.

**Lemma 3.31.** The universal property of $\ell_X$ provides the dotted arrow $\tilde{\iota}^*$.

**Proof.** It suffices to show that the composition

\[
\tilde{\gamma}^* \circ G : \text{Sh}_C^{G,\text{eqsm}}(X) \xrightarrow{\gamma^* \circ G} \text{Sh}_C^{G,\text{eqsm}}(Z) \xrightarrow{\ell_Z} V_C^G(Z) \xrightarrow{C_{\text{fib}}(\iota_Z)}
\]

sends the morphisms in $W_C^{\text{eqsm}}$ (see the text before Definition 2.77) to equivalences.

Let $M$ be in $\text{Sh}_C^{G,\text{eqsm}}(X)$ and $V$ be in $C_{\infty,*}^{G,\Delta}$. Then we consider the generator $\iota_V(M) : M \to V^G M$ of $W_C^{\text{eqsm}}$. It suffices to show that $\ell^*_{\nu} \nu \times G \iota_V(M)$ is an equivalence.

Let $U$ be in $C_{\infty,*}^{G,\Delta}$ such that $M \in \text{Sh}_C^{U,G}(X)$. Using that the family $\mathcal{Y}$ is big, we can choose a member $Y$ of $\mathcal{Y}$ such that $(Z,Y)$ is a $VUV^{-1}$-covering family of $X$ (see (22.4) and Definition 2.2). Since $U \subseteq VUV^{-1}$, the pair $(Z,Y)$ is also a $U$-covering family, and we have $V^G_M \in \text{Sh}_C^{VUV^{-1, G}(X)}$ by Corollary 2.11.

We now use the notation introduced in connection with the Glueing Lemma 2.14. By Lemma 2.14, the morphism $\nu \times G$ is equivalent in $\text{Sh}_C^G(X)$ to the morphism

\[
\begin{array}{c}
\gamma^G_{\text{fib}} \times \gamma^G_{\text{fib}} \times G \times M \xrightarrow{\ell^G_{\text{fib}} \times G \times M} \gamma^G_{\text{fib}} \times G \times V^G M \times \gamma^G_{\text{fib}} \times G \times V^G M.
\end{array}
\]

The functor $\tilde{\gamma}^* \circ G$ is left-exact and therefore preserves the fibre products in (3.17). The functor $\ell^G_{\nu} \times G$ sends objects in the image of $\kappa^G_G$ or $\gamma^G_{\text{fib}}$ to objects in the image of $\iota_Z$ by the commutativity (justified by Corollary 2.28 and Lemma 2.53) of the diagram

\[
\begin{array}{c}
\text{Sh}_C^{G,\text{eqsm}}(Z \cap \mathcal{Y}) \xrightarrow{\gamma^G} \text{Sh}_C^{G,\text{eqsm}}(Z) \\
\downarrow \gamma^* \circ G \\
\text{Sh}_C^{G,\text{eqsm}}(\mathcal{Y}) \xrightarrow{\gamma^G} \text{Sh}_C^{G,\text{eqsm}}(X) \\
\downarrow \gamma^G \\
\text{Sh}_C^{G,\text{eqsm}}(Y)
\end{array}
\]

Finally by definition, the functor $\ell^G_{\nu}$ sends objects in the image of $\iota_Z$ to zero objects. Using the above observations, in $C_{\infty,*}^{\text{fib}}(\iota_Z)$ we get an equivalence

\[
\ell^G_{\nu} \nu \times G \iota_V(M) \simeq \ell^G_{\nu} \nu \times G [G \iota \times G M \to \iota \times G V^G M].
\]

Using Lemma 2.31, setting $M' := \tilde{\gamma}^* \times G \iota \times G M$ for the inclusion $h : V(Z) \to X$ and $V_Z := V \cap (Z \times Z)$, we have an equivalence $\tilde{\gamma}^* \times G V^G M \simeq V^G_{Z,*}M'$. 

Lemma 2.14

Again and that (\ref{equivalence}). We now show that it is an equivalence. Our candidate for the inverse functor is the map

\[ \ell^*_x \bar{\ell}^*_Z \iota_{V}(M) \rightarrow M' \tag{3.19} \]

where \( \iota_{V} \) is induced by the unit \( \text{id} \).

We first observe that \( \ell^*_x \bar{\ell}^*_Z \iota_{V}(M') \) is an equivalence since \( \iota_{V}(M') \in W^\text{eqsm}. \)

We then argue that \( \ell^*_x \bar{\ell}^*_Z \iota_{V} \) is an equivalence, too. To this end we show that the map

\[ \ell^*_x \bar{\ell}^*_Z \iota_{V}(M) \rightarrow M' \]

is equivalent to the one induced by the unit \( \text{id} \). Arguing similarly as above using Lemma 2.14 again and that \( (V(Z), Y \cap Z) \) is a \( U_{Z} \)-covering family of \( Z \), we have an equivalence

\[ \text{Fib}(\tilde{\iota}^*_x \iota_{V}(M) \rightarrow \tilde{\iota}^*_x \iota_{V}(M')) \simeq \text{Fib}(\tilde{\iota}^*_x \iota_{V}(M) \rightarrow \tilde{\iota}^*_x \iota_{V}(M')) , \]

where \( n: Y \cap V(Z) \rightarrow Z \) is the inclusion map. The latter object obviously lies in the essential image of \( \iota_{V} \).

Since both maps in (3.19) are equivalences, we conclude that \( \tilde{\iota}^*_x \iota_{V}(M) \) is an equivalence.

\[ \square \]

Equation (3.15) induces a morphism between stable cofibres

\[ \tilde{\iota}: \text{Cofib}^s(\iota_{V}) \rightarrow \text{Cofib}^s(\iota_{X}) . \]

We now show that it is an equivalence. Our candidate for the inverse functor is the map \( \tilde{\iota}^* \) in diagram (3.16). We will obtain \( \tilde{\iota}^* \) using the universal property of the morphism \( \ell^*_x \). To this end, we note that its target is stable. In order to construct \( \tilde{\iota}^* \), it therefore suffices to show the following result.

**Lemma 3.32.** The morphism \( \tilde{\iota}^* \) from (3.16) sends the morphisms in \( W_{i_{X}} \) to equivalences.

**Proof.** Let \( f \) be a morphism in \( V^G_C(X) \) such that \( \text{Fib}(f) \) belongs to the essential image of \( \iota_{X} \). As any morphism in a localisation, it can be lifted to a morphism \( \tilde{f} \) in \( \text{Sh}_{C}^{G, \text{eqsm}}(X) \) such that \( \ell_{X}(\tilde{f}) \) is equivalent to \( f \). In the following, we use the commutativity of the established part of the diagram (3.16). We have an equivalence

\[ \tilde{\iota}^*(f) \simeq \ell^*_x \bar{\ell}^*_Z \iota_{V}(\tilde{f}) . \]

By the assumption on \( f \), there exists an object \( P \) of \( \text{Sh}_{C}^{G, \text{eqsm}}(Y) \) such that

\[ \iota_{X} \ell_{Y}(P) \simeq \text{Fib}(f) . \]

On the one hand, since \( \tilde{\iota}^* \) is left-exact, we have the equivalence

\[ \tilde{\iota}^* \iota_{X} \ell_{Y}(P) \simeq \tilde{\iota}^* \text{Fib}(f) \simeq \text{Fib}(\tilde{\iota}^* f) . \]
On the other hand, we have an equivalence

\[ \bar{\ell}^* \ell_X \ell_Y (P) \overset{(\ref{2.83})}{=} \bar{\ell}^* \ell_X \pi_X^G (P) \]

\[ \overset{(\ref{3.16})}{=} \ell^*_z \ell_Z \pi^G \pi_X^G (P) \]

\[ \overset{(\ref{3.18})}{=} \ell^*_z \ell_Z \tilde{\pi}^G \pi_X^G (P) \]

\[ \overset{(\ref{2.83})}{=} \ell^*_z \ell_Z \tilde{\pi}^G \pi_X^G (P) \]

\[ \simeq 0. \]

We now use that Cofib\(^s(\iota_Z)\) is stable in order to conclude that \( \bar{\ell}^*(f) \) is an equivalence.

\( \square \)

Proof of Proposition 3.30. From Lemma 3.32 we get a further factorisation

(3.20) \( \bar{\iota}^*: \text{Cofib}^s(\iota_X) \rightarrow \text{Cofib}^s(\iota_Z) \),

of \( \bar{\iota}^* \), namely the dashed arrow in (3.16). By construction, we have the equivalence

\[ \bar{\iota}^* \circ \bar{\iota}^*_s \simeq \text{id}_{\text{Cofib}^s(\iota_Z)}. \]

The transformation \( \text{id}_{\text{Sh}_C(X)} \rightarrow \tilde{\iota}^*_s \bar{\iota}^* \) furthermore induces a transformation

\[ \text{id}_{\text{Cofib}^s(\iota_Z)} \rightarrow \bar{\iota}^* \bar{\iota}^*_s. \]

It remains to show that the latter is an equivalence. Let \( M \) be an object of \( \text{Sh}_C^G(X) \). We can choose \( U \) in \( C_C^G(X, \Delta) \) such that \( M \in \text{Sh}_C^{U,G}(X) \) and a member \( Y \) of \( Y \) such that \( (Y, Z) \) is a \( U \)-covering family of \( X \). By Lemma 2.14 we have an equivalence

\[ M \simeq \tilde{\iota}^*_s \tilde{\iota}^*_G(M) \times_{\bar{\iota}^*_s \bar{\iota}^*_G \pi_X^G(M)} \tilde{\iota}^*_s \tilde{\iota}^*_G \pi_X^G(M). \]

We must show that \( \ell^*_z \ell^*_X \ell^*_Y \) sends the projection \( \tilde{\iota}^*_s \tilde{\iota}^*_G(M) \times_{\bar{\iota}^*_s \bar{\iota}^*_G \pi_X^G(M)} \tilde{\iota}^*_s \tilde{\iota}^*_G \pi_X^G(M) \rightarrow \tilde{\iota}^*_s \tilde{\iota}^*_G \pi_X^G(M) \)

to an equivalence in Cofib\(^s(\iota_X)\). This is clear since \( \ell^*_z \ell^*_X \ell^*_Y \) preserves fibre products and sends the objects \( \tilde{\iota}^*_s \tilde{\iota}^*_G \pi_X^G(M) \) and \( \bar{\iota}^*_s \pi_X^G \pi_X^G(M) \) (which belong to the essential image of \( \bar{\iota}^*_s \)) to zero objects in Cofib\(^s(\iota_X)\). \( \square \)

3.6. Strong additivity. Let \( X \) be in \( G\text{BornCoarse} \), and let \((Y, Z)\) be a partition of \( X \) into invariant subsets. We say that \((Y, Z)\) is a coarsely disjoint decomposition if \([Y] \cap [Z] = \emptyset\). Here \([Y]\) and \([Z]\) denote the closures of \( Y \) and \( Z \) with respect to the equivalence relation \( U(\pi_0(X)) \) (see (2.32)).

Let \( E: G\text{BornCoarse} \rightarrow M \) be a functor with target an \( \infty \)-category admitting an initial object \( \emptyset \).

Definition 3.33. \( E \) is \( \pi_0^\ast \)-excisive if for every \( X \) in \( G\text{BornCoarse} \) with a coarsely disjoint decomposition \((Y, Z)\) the square

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & E(Z) \\
\downarrow & & \downarrow \\
E(Y) & \longrightarrow & E(X)
\end{array}
\]

is a pushout square. ♦
Note that \( \text{CAT}^{\text{Lex}}_{\infty, *} \) and \( \text{Cat}^{\text{Lex}}_{\infty, *} \) are pointed and therefore admit initial objects, namely the category \( 0 := \Delta^0 \). Let \( C \) be in \( \text{Fun}(BG, \text{CAT}^{\text{Lex}}_{\infty, *}) \).

**Lemma 3.34.** The functors \( \text{Sh}^G_C, \text{Sh}^G_{C, \text{eqsm}} \), and \( V^G_C \) are \( \pi_0 \)-excisive.

**Proof.** We consider \( X \) in \( G \text{BornCoarse} \) with a coarsely disjoint decomposition \((Y,Z)\).

We first consider the case of \( \text{Sh}^G_C \). We have an isomorphism of posets \( s : P_X \xrightarrow{\cong} P_Y \times P_Z \), \( B \mapsto (B \cap Y, B \cap Z) \).

Since \( C \) is left-exact, we have a product functor \( \times : C \times C \to C \). It induces the functor denoted by the same symbol in

\[
\text{Sh}^G_C(Y) \times \text{Sh}^G_C(Z) \xrightarrow{m} \text{Sh}^G_C(X),
\]

where the dotted arrow \( m \) is obtained by checking that the down-right composition takes values in the subcategory of sheaves. As a consequence of **Lemma 2.14**, we see that the functor \( m \) is an equivalence with inverse induced by the restrictions of sheaves along \( P_Y \to P_X \) and \( P_Z \to P_X \). We now use that \( \text{CAT}^{\text{Lex}}_{\infty, *} \) is semi-additive (the \( \text{CAT}^{\text{Lex}}_{\infty, *-version} \) of **Lemma 7.21**) in order to get the first equivalence in

\[
\text{Sh}^G_C(Y) \cup \text{Sh}^G_C(Z) \xrightarrow{\cong} \text{Sh}^G_C(Y) \times \text{Sh}^G_C(Z) \xrightarrow{m, \cong} \text{Sh}^G_C(X),
\]

showing that \( \text{Sh}^G_C \) is \( \pi_0 \)-excisive.

In order to get the result for \( \text{Sh}^G_{C, \text{eqsm}} \) we must check that \( m \) preserves equivariantly small sheaves. But this is clear since (by an inspection of **Definition 2.46**) the square

\[
\begin{array}{ccc}
\text{Sh}^G_{C, \text{eqsm}}(Y) & \longrightarrow & \text{Sh}^G_{C, \text{eqsm}}(X) \\
\downarrow & & \downarrow \\
\text{Sh}^G_C(Y) & \longrightarrow & \text{Sh}^G_C(X)
\end{array}
\]

is a pullback.

The family \((Y)\) consisting of the single member \( Y \) is a big family (**Definition 3.25**). Then \((Z, (Y))\) is a complementary pair (**Definition 3.26**). By **Proposition 3.30** and using that \( V^G_C(\emptyset) \simeq 0 \) and \( V^G_C(Y) \simeq V^G_C((Y)) \), we obtain an excisive square

\[
\begin{array}{ccc}
0 & \rightarrow & V^G_C(Z) \\
\downarrow & & \downarrow \text{id}_Z \circ \cdot \\
V^G_C(Y) & \xrightarrow{i_Y \cdot} & V^G_C(X)
\end{array}
\]

in \( \text{Cat}^{\text{Lex}}_{\infty, *-} \), where \( i_Y : Y \to X \) and \( i_Z : Z \to X \) are the inclusions. It induces a morphism

\[
V^G_C(Y) \cup V^G_C(Z) \rightarrow V^G_C(X).
\]

We want to show that this morphism is an equivalence in \( \text{Cat}^{\text{Lex}}_{\infty, *-} \). To this end we show that this morphism is fully faithful and essentially surjective.
The map $(3.23)$ is the localisation of $(3.22)$. This shows that $(3.23)$ is essentially surjective.

We already know from Proposition 3.19 that the components $i_{Y,*}$ and $i_{Z,*}$ of the morphism $(3.23)$ are fully faithful. It remains to show that if $M$ is in $\text{Sh}^E_G(Y)$ and $N$ is in $\text{Sh}^E_G(Z)$, then

$$\text{Map}_{\text{V}^E_G(X)}(i_{Y,*}E, i_{Z,*}E) \simeq *.$$  

This is obvious from the formula (2.81), noting that

$$\text{Map}_{\text{sh}^E_G(X)}(i_{Y,*}E, i_{Z,*}E) \simeq *$$

for all $V$ in $C^E_{X,Y}$, since the sheaves $i_{Y,*}E$ and $V^E_{Y,*}E$ have disjoint support. □

Consider a $\pi_0$-excisive functor $E: \text{GBornCoarse} \to M$. Let $X$ be in $\text{GBornCoarse}$ with a coarsely disjoint partition $(Y, Z)$. Then we can define a projection map as the composition

$$(3.24) \quad p_Y: E(X) \xrightarrow{\pi_0 \text{-exc."}} E(Y) \sqcup E(Z) \xrightarrow{q_Y} E(Y),$$

where $q_Y$ is classified by the morphisms $\text{id}_{E(Y)}: E(Y) \to E(Y)$ and $0: E(Z) \to E(Y)$.

Let $(X_i)_{i \in I}$ be a family in $\text{GBornCoarse}$.

**Definition 3.35** ([BEKWa, Ex.2.16]). We define the free union $\bigsqcup^\text{free} X_i$ to be the following $G$-bornological coarse space:

1. The underlying $G$-set is the disjoint union of $G$-sets $\bigsqcup_{i \in I} X_i$.
2. The coarse structure is generated by entourages $\bigsqcup_{i \in I} U_i$ for all families $(U_i)_{i \in I}$, where $U_i$ is in $C_{X_i}$ for every $i$ in $I$.
3. The bornology is generated by the set $\{B \mid B \in B_{X_i}, i \in I\}$ of subsets of $\bigsqcup_{i \in I} X_i$.

**Remark 3.36.** Note that in general the free union $\bigsqcup^\text{free}_{i \in I} X_i$ of the family $(X_i)_{i \in I}$ differs from the coproduct $\bigsqcup^\text{free}_{i \in I} X_i$ in $\text{GBornCoarse}$. The underlying $G$-sets of the coproduct and the free union coincide. But the coarse structure of the coproduct is generated by the set $\{U \mid U \in C_{X_i}, i \in I\}$. It is in general smaller than the coarse structure of the free union. Furthermore, the bornology of the coproduct is generated by the sets $\bigsqcup_{i \in I} B_i$ for all families $(B_i)_{i \in I}$ with $B_i$ in $B_{X_i}$. This bornology is bigger than the one of the free union. The identity of the underlying sets is a morphism $\bigsqcup_{i \in I} X_i \to \bigsqcup^\text{free}_{i \in I} X_i$ in $\text{GBornCoarse}$. □

If $E: \text{GBornCoarse} \to M$ is $\pi_0$-excisive, then for every $i_0$ in $I$ we have the coarsely disjoint partition of $\bigsqcup^\text{free}_{i \in I} X_i$ into the invariant subsets $X_{i_0}$ and $\bigsqcup_{i \in I \setminus \{i_0\}} X_i$, and therefore a projection (see (3.24))

$$p_{i_0}: E(\bigsqcup_{i \in I} X_i) \to E(X_{i_0}).$$

The family of projections $(p_i)_{i \in I}$ provides a map

$$(3.25) \quad E(\bigsqcup_{i \in I} X_i) \xrightarrow{(p_i)_{i \in I}} \prod_{i \in I} E(X_i).$$
Consider a pointed ∞-category $M$ admitting all products indexed by sets and a $π_0$-excusive functor $E : \text{GBornCoarse} → M$.

**Definition 3.37** ([BEKWa, Ex. 3.12]). $E$ is strongly additive if the maps (3.25) are equivalences for all families $(X_i)_{i ∈ I}$ in $\text{GBornCoarse}$.

Let $C$ be in $\text{Fun}(BG, \text{Cat}^{\text{Lex}}_\infty)$). By Lemma 3.34 and Proposition 7.9 (and its $\text{CAT}^{\text{Lex},*}_\infty$-version), Definition 3.37 applies to the functors $\text{Sh}^G_C$, $\text{Sh}^G_{C,\text{eqsm}}$, and $\mathcal{V}^G_C$.

**Proposition 3.38.** The functors $\text{Sh}^G_C$, $\text{Sh}^G_{C,\text{eqsm}}$, and $\mathcal{V}^G_C$ are strongly additive.

**Proof.** Let $(X_i)_{i ∈ I}$ be a family in $\text{GBornCoarse}$ and $X := \bigsqcup_{i ∈ I} X_i$.

We start with the functor $\text{Sh}^G_C$. We consider the entourage $U := \bigsqcup_{i ∈ I} X_i × X_i$ (note that $U$ does not belong to $C_X$ in general). Then we have an equivalence

$$\text{Sh}^G_{C,U}(X) \simeq \prod_{i ∈ I} \text{PSh}^G_C(X_i)$$

given by the family of restrictions along the family of inclusions $(X_i → X)_{i ∈ I}$. By an inspection of the definition of $\text{Sh}^G_C$ and Definition 3.35 (2) we conclude that this equivalence restricts to an equivalence

$$\text{Sh}^G_C(X) \simeq \prod_{i ∈ I} \text{Sh}^G_C(X_i).$$

Next we consider $\text{Sh}^G_{C,\text{eqsm}}$. Using Lemma 2.53, we get a commutative square

$$\text{Sh}^G_{C,\text{eqsm}}(X) \longrightarrow \prod_{i ∈ I} \text{Sh}^G_{C,\text{eqsm}}(X_i) \quad \text{and} \quad \text{Sh}^G_C(X) \sim \longrightarrow \prod_{i ∈ I} \text{Sh}^G_C(X_i).$$

We must show the the upper horizontal arrow is an equivalence. Since the vertical functors are also fully faithful, it suffices to show that it is essentially surjective.

Let $M$ be in $\text{Sh}^G_C(X)$ such that $M_{|X_i} ∈ \text{Sh}^G_{C,\text{eqsm}}(X_i)$ for every $i$ in $I$. Then we must show that $M ∈ \text{Sh}^G_{C,\text{eqsm}}(X)$. Let $H$ be a subgroup of $G$, and let $Y$ be an $H$-bounded subset of $X$. By Definition 2.44 we can find a bounded subset $B$ of $X$ such that $Y = HB$. In view of Definition 3.35 (3) the set $I_B := \{i ∈ I \mid X_i ∩ B ≠ ∅\}$ is finite. Note that $(Y ∩ X_i)_{i ∈ I_B}$ is a coarsely disjoint family (see Definition 3.35 (2)) of $H$-bounded subsets such that $Y = \bigcup_{i ∈ I_B} Y ∩ X_i$. We conclude that

$$M(Y) ≃ \prod_{i ∈ I_B} M_{|X_i}(Y ∩ X_i) ∈ C^{H,ω}$$

since the product is finite and each factor belongs to $C^{H,ω}$ by assumption.

We conclude that

$$(3.26) \quad \text{Sh}^G_{C,\text{eqsm}}(X) \simeq \prod_{i ∈ I} \text{Sh}^G_{C,\text{eqsm}}(X_i).$$

Finally, we consider $\mathcal{V}^G_C$. In order to show that

$$(3.27) \quad \mathcal{V}^G_C(X) → \prod_{i ∈ I} \mathcal{V}^G_C(X_i)$$
is an equivalence, we show that this functor is fully faithful and essentially surjective. In fact, essential surjectivity immediately follows from the case of $\operatorname{Sh}^G_{C, \text{eqsm}}$. In order to prove fully faithfulness, we use the formula for mapping spaces provided by Proposition 2.78.

Let $M, N$ be objects of $\operatorname{Sh}^G_{C, \text{eqsm}}(X)$. Then we write $M, N$ for the restrictions of $M, N$ to $X$. For an entourage $V$ in $C^G_{X, \Delta}$ we set $V_i := V \cap (X_i \times X_i)$ in $C^G_{X, \Delta}$. Then we have the chain of equivalences

\[
\operatorname{Map}_{\operatorname{Sh}^G_{C, \text{eqsm}}(X)}(\ell_X M, \ell_X N) \\
\cong \colim_{V \in C^G_{X, \Delta}} \operatorname{Map}_{\operatorname{Sh}^G_{C, \text{eqsm}}(X)}(M, V^G N) \\
\cong \colim_{V \in C^G_{X, \Delta}} \prod_{i \in I} \operatorname{Map}_{\operatorname{Sh}^G_{C, \text{eqsm}}(X_i)}(M_i, V^G_{i,*} N_i) \\
\cong \prod_{i \in I} \colim_{V_i \in C^G_{X_i, \Delta}} \operatorname{Map}_{\operatorname{Sh}^G_{C, \text{eqsm}}(X_i)}(M_i, V^G_{i,*} N_i) \\
\cong \prod_{i \in I} \operatorname{Map}_{\operatorname{Sh}^G_{C, \text{eqsm}}(X_i)}(\ell_{X_i} M_i, \ell_{X_i} N_i),
\]

where for the marked equivalence we use the definition of the coarse structure of the free union (Definition 3.35 (2)) and the fact that filtered colimits distribute over products in spaces (see Definition 7.24 and Example 7.25). This equivalence shows that (3.27) is fully faithful.

4. FURTHER CONSTRUCTIONS

4.1. Forcing continuity. Let $X$ be in $\mathcal{GBornCoarse}$, and let $F$ be a subset of $X$.

Definition 4.1 ([BEKWa, Def. 5.1]). $F$ is locally finite if for every $B$ in $B_X$ the set $B \cap F$ is finite.

We let $\mathcal{F}(X)$ denote the poset of invariant locally finite subsets of $X$.

Let $E: \mathcal{GBornCoarse} \to \mathcal{M}$ be a functor with a target admitting filtered colimits.

Definition 4.2. $E$ is continuous if the canonical morphism

\[
\colim_{F \in \mathcal{F}(X)} E(F) \to E(X)
\]

is an equivalence for every $X$ in $\mathcal{GBornCoarse}$.

Let

\[
i: \mathcal{GBornCoarse}^{\text{mb}} \to \mathcal{GBornCoarse}
\]

be the inclusion of the full subcategory of $G$-bornological coarse spaces which have the minimal bornology.

Lemma 4.3. The left Kan extension $E^c$ of $E \circ i$ along $i$
exists and is a continuous functor. The functor $E$ is continuous if and only if the canonical transformation $E^c \to E$ is an equivalence.

**Proof.** The left Kan extension of $E \circ i$ exists if and only if the colimit
$$\operatorname{colim}_{\mathcal{G} \text{-BornCoarse}_{mb}/X} E(Y)$$
exists in $\mathcal{M}$ for all $G$-bornological coarse spaces $X$. It is easy to see that the functor
$$F \mapsto \left(\mathcal{G} \text{-BornCoarse}_{mb}/X, F \mapsto (F \to X)\right)$$
is cofinal. Since $\mathcal{F}(X)$ is filtered and filtered colimits in $\mathcal{M}$ exist by assumption, we conclude that $E^c$ exists. Furthermore, we have an equivalence
$$E^c(X) \simeq \operatorname{colim}_{F \in \mathcal{F}(X)} E(F) \ .$$
The canonical transformation $E^c \to E$ is given pointwise by the canonical morphism
$$E^c(X) \simeq \operatorname{colim}_{F \in \mathcal{F}(X)} E(F) \to E(X) \ .$$
Hence $E$ is continuous if and only if $E^c \to E$ is an equivalence.

In order to show that $E^c$ is continuous, we apply the above to the functor $E^c$ in place of $E$. The assertion then follows from the fact that $(E^c)^c \to E^c$ is an equivalence. The latter follows from the chain of equivalences:
$$\begin{align*}
(E^c)^c(X) & \simeq \operatorname{colim}_{F \in \mathcal{F}(X)} E^c(F) \\
& \simeq \operatorname{colim}_{F \in \mathcal{F}(X)} \operatorname{colim}_{F' \in \mathcal{F}(F)} E(F') \\
& \simeq \operatorname{colim}_{F \in \mathcal{F}(X)} E(F) \\
& \simeq E^c(X) ,
\end{align*}$$
where the marked equivalence uses that $F$ is final in $\mathcal{F}(F)$.

**Definition 4.4.** We say that the functor $E^c$ is obtained from $E$ by forcing continuity.

In the following, we show that if a functor $E^c$ is obtained from $E$ by forcing continuity, then it inherits various properties from $E$.

Recall **Definition 3.1** of coarse invariance.

**Lemma 4.5.** If $E$ is coarsely invariant, then $E^c$ is coarsely invariant.

**Proof.** Let $X$ be in $\mathcal{G} \text{-BornCoarse}$. We must show that the projection $\{0,1\}_{\max,\max} \otimes X \to X$ induces an equivalence
$$E^c(\{0,1\}_{\max,\max} \otimes X) \to E^c(X) \ .$$
The collection of subsets $\{0,1\} \times F$ of $\{0,1\} \times X$ for $F$ in $\mathcal{F}(X)$ is cofinal in $\mathcal{F}(\{0,1\}_{\max,\max} \otimes X)$. Therefore, we get the second equivalence in the following
chain:
\[
\begin{align*}
E^c(\{0,1\}_{\text{max, max}} \otimes X) & \overset{(4.1)}{=} \colim_{F' \in F(\{0,1\}_{\text{max, max}} \otimes X)} E(F') \\
& \simeq \colim_{F \in F(X)} E(\{0,1\}_{\text{max, max}} \otimes F) \\
& \overset{!}{\simeq} \colim_{F \in F(X)} E(F) \\
& \overset{(4.1)}{=} E^c(X) .
\end{align*}
\]

The equivalence marked by ! follows from the coarse invariance of \(E^c\).

Recall Definition 3.16 of \(u\)-continuity.

**Lemma 4.6.** If \(E\) is \(u\)-continuous, then \(E^c\) is \(u\)-continuous.

**Proof.** Let \(X\) be in \(G\text{BornCoarse}\). We must show that the canonical morphism
\[
\colim_{U \in C^G_X} E^c(X_U) \to E^c(X)
\]
is an equivalence. This follows from the following chain of equivalences:
\[
\begin{align*}
\colim_{U \in C^G_X} E^c(X_U) & \overset{(4.1)}{=} \colim_{U \in C^G_X} \colim_{F \in \mathcal{F}(X)} E(F_X) \\
& \simeq \colim_{F \in \mathcal{F}(X)} \colim_{U \in C^G_X} E(F_X) \\
& \overset{!}{\simeq} \colim_{F \in \mathcal{F}(X)} \colim_{V \in C^G_{F_X}} E(F_V) \\
& \overset{!!}{\simeq} \colim_{F \in \mathcal{F}(X)} E(F_X) \\
& \overset{(4.1)}{=} E^c(X) .
\end{align*}
\]

Here \(F_X\) denotes the bornological coarse space \(F\) with the structure induced from \(X_U\). For the equivalences marked with ! we use that \(E\) is \(u\)-continuous. For !! we use that
\[
\{(U,V) \mid U \in C^G_X, V \in C^G_{F_X} \} \to C^G_{F_X}, \quad (U,V) \mapsto V
\]
is cofinal.

Let \(\mathcal{M}\) be a semi-additive \(\infty\)-category which in addition admits all small filtered colimits, and consider a functor \(E : G\text{BornCoarse} \to \mathcal{M}\). Recall Definition 3.7 of a flasqueness-preserving functor.

**Lemma 4.7.** If \(E\) preserves flasqueness, then so does \(E^c\).

**Proof.** Assume that \(X\) is a flasque object in \(G\text{BornCoarse}\) with flasqueness implemented by \(f : X \to X\). If \(F\) is in \(\mathcal{F}(X)\), then also \(\tilde{F} := \bigcup_{n \in \mathbb{N}} f^n(F) \in \mathcal{F}(X)\). Furthermore, \(\tilde{F}\) is flasque with flasqueness implemented by the restriction \(f|_{\tilde{F}}\). The map \(\mathcal{F}(X) \to \mathcal{F}(X), F \mapsto \tilde{F}\) is cofinal. This fact provides the second equivalence in
\[
E^c(X) \overset{(4.1)}{=} \colim_{F \in \mathcal{F}(X)} E(F) \simeq \colim_{F \in \mathcal{F}(X)} E(\tilde{F}) .
\]
Since $E(\tilde{F})$ belongs to $M^g$ for every $F$ in $\mathcal{F}(X)$ and $\mathcal{F}(X)$ is filtered, also $E^c(X)$ belongs to $M^g$, since the latter is by definition closed under filtered colimits. □

Let $P$ be some $\infty$-category and $E: \text{GBornCoarse} \times P \to M$ be a functor with $M$ as above. We let $E^c$ denote the functor obtained from $E$ by applying the construction of forcing continuity to the variable in $\text{GBornCoarse}$. Recall Definition 3.11 and Definition 3.12 of a functorially (pre-)flasqueness preserving functor.

Lemma 4.8.

1. If $E$ functorially preserves pre-flasqueness, then so does $E^c$.
2. If $E$ functorially preserves flasqueness, then so does $E^c$.

Proof. We give the argument for (1). We consider the following diagram

$$
\begin{array}{ccc}
\text{Fl}^\text{pre}(\text{GBornCoarse}^\text{mb}) \times P & \xrightarrow{p} & \text{Fl}^\text{pre}(M) \\
\text{Fl}^\text{pre}(\text{GBornCoarse}) \times P & \xrightarrow{q} & \text{Fl}^\text{pre}(M) \\
\text{GBornCoarse}^\text{mb} \times P & \xrightarrow{E^c} & M
\end{array}
$$

where the dotted arrows are defined as left Kan extensions, respectively. The universal property of the left Kan extensions provides a natural transformation

$$p \circ \text{Fl}^\text{pre}(E) \to E^c \circ q .$$

We must check that this transformation is an equivalence. In view of the pointwise formula for the left Kan extensions, this amounts to showing that for every $P$ in $P$ and $(X,f)$ in $\text{Fl}^\text{pre}(\text{GBornCoarse})$ the morphism

$$
\colim_{((F,g) \to (X,f)) \in \text{Fl}^\text{pre}(\text{GBornCoarse}^\text{mb})/(X,f)} p(\text{Fl}^\text{pre}(E)((F,g),P))
\to
\colim_{(F \to X) \in (\text{GBornCoarse}^\text{mb})/X} E(F,P)
$$

is an equivalence. The argument given in the proof of Lemma 4.7 shows that the functor $q$ induces a cofinal functor

$$\text{Fl}^\text{pre}(\text{GBornCoarse}^\text{mb})/(X,f) \to (\text{GBornCoarse}^\text{mb})/X .$$

Consequently, we can rewrite the morphism in question in the form

$$
\colim_{((F,g) \to (X,f)) \in \text{Fl}^\text{pre}(\text{GBornCoarse}^\text{mb})/(X,f)} p(\text{Fl}^\text{pre}(E)((F,g),P))
\to
\colim_{((F,g) \to (X,f)) \in \text{Fl}^\text{pre}(\text{GBornCoarse}^\text{mb})/(X,f)} E(q(F,g),P) .
$$

This is an equivalence since

$$(p \times \text{id}_P) \circ \text{Fl}^\text{pre}(E) \simeq E \circ (q \times \text{id}_P)$$

in view of (3.5).
Let \((X,f)\) be in \(\m{Fl}^{pre}(\m{GBornCoarse})\) and write \((M,S) := \m{Fl}^{pre}(E)((X,f),P)\) for \(P \in \m{P}\). We must check that \(S \simeq \id_M + E^{c}(f,P) \circ S\). This relation is equivalent to the colimit over the relations \(S(F,g) \simeq \id_{M(F)} + E(f,P) \circ S(F,g)\) indexed by \(((F,g) \rightarrow (X,f))\) in \(\m{Fl}^{pre}(\m{GBornCoarse}^{mb})/\langle X,f \rangle\), where \((M(F),S(F,g)) := \m{Fl}^{pre}(E)((F,g),P)\).

Assertion (2) is shown analogously. \(\Box\)

Let \(\m{M}\) be a pointed \(\infty\)-category admitting finite coproducts, small filtered colimits and small products. Recall \textit{Definition 3.33} and \textit{Definition 3.37} of \(\pi_0\)-excisiveness and strong additivity.

\textbf{Lemma 4.9.}

(1) If \(E\) is \(\pi_0\)-excisive, then so is \(E^{c}\).

(2) If \(E\) is strongly additive and filtered colimits distribute over products in \(\m{M}\) \hspace{1em} \textit{(see Definition 7.24)}, then \(E^{c}\) is also strongly additive.

\textit{Proof.} We first show (1). Let \(X\) be in \(\m{GBornCoarse}\) with a coarsely disjoint partition \((Y,Z)\) into invariant subsets. For every \(F \in \mathcal{F}(X)\) we get a partition \((F \cap Y, F \cap Z)\) of \(F\) into coarsely disjoint invariant subsets. Since \(E\) is \(\pi_0\)-excisive, we conclude that

\[ E(F) \simeq E(F \cap Y) \sqcup E(F \cap Z). \]

The formula (4.1) implies the equivalence

\[ E^c(X) \simeq E^c(Y) \sqcup E^c(Z) \]

by taking the colimit over \(\mathcal{F}(X)\). Here we use that the projection \(\mathcal{F}(X) \rightarrow \mathcal{F}(Y)\), \(F \mapsto F \cap Y\) is cofinal in order to get the equivalence

\[ \colim_{F \in \mathcal{F}(X)} E(F \cap Y) \simeq \colim_{F' \in \mathcal{F}(Y)} E(F') \]

(and similarly for \(Z\)).

We now show (2). Let \((X_i)_{i \in I}\) be a family in \(\m{GBornCoarse}\) and set \(X := \bigsqcup_{i \in I} X_i\) \hspace{1em} \textit{(Definition 3.35)}. Then \(F \in \mathcal{F}(X)\) if and only if \(F \cap X_i \in \mathcal{F}(X_i)\) for every \(i \in I\). Hence we have an isomorphism of posets \(\mathcal{F}(X) \cong \prod_{i \in I} \mathcal{F}(X_i)\) given by \(F \mapsto (F \cap X_i)_{i \in I}\). It gives the first equivalence in the following chain

\[ \colim_{F \in \mathcal{F}(X)} \prod_{i \in I} E(F \cap X_i) \simeq \prod_{i \in I} \colim_{F_i \in \mathcal{F}(X_i)} E(F_i). \]

while the second follows from the assumption that filtered colimits distribute over products in \(\m{M}\).

We have an isomorphism \(F \cong \bigsqcup_{i \in I}^{\text{free}} (F \cap X_i)\) in \(\m{GBornCoarse}\). Using the assumption that \(E\) is strongly additive, we get the marked equivalence in the following commutative diagram

\[ \begin{array}{ccc}
\colim_{F \in \mathcal{F}(X)} E(F) & \xrightarrow{\simeq} & E^c(X) \\
\downarrow \simeq \hspace{1em} & & \hspace{1em} \downarrow \simeq \\
\prod_{i \in I} \colim_{F_i \in \mathcal{F}(X_i)} E(F_i) & \xrightarrow{\simeq} & \prod_{i \in I} E^c(X_i)
\end{array} \]
We conclude that the right vertical morphism is an equivalence as desired. This implies Assertion (2).

We consider a functor $E: \mathbf{GBornCoarse} \to \mathbf{M}$. Recall the notion of excisiveness from Definition 3.27 and the notion of $l$-excisiveness (for $\mathbf{M} = \mathbf{Cat}_{\infty,\ast}^{\text{lex}}$) from Definition 3.29.

**Lemma 4.10.** If $E$ is excisive or $l$-excisive, then the same is true for $E^c$.

**Proof.** Let $X$ be in $\mathbf{GBornCoarse}$ with a complementary pair $(Z, Y)$. Then for every $F$ in $\mathcal{F}(X)$ we get a complementary pair $(F \cap Z, F \cap Y)$ on $F$. Hence

\[(4.3) \quad E^c(Z \cap Y) \longrightarrow E^c(Y) \quad E^c(Z) \longrightarrow E^c(X) \]

is the colimit over $F$ in $\mathcal{F}(X)$ of the following pushout squares in $\mathbf{M}$ (or excisive squares in $\mathbf{Cat}_{\infty,\ast}^{\text{lex}}$, see Definition 7.49):

\[
\begin{array}{ccc}
E(F \cap Z \cap Y) & \longrightarrow & E(F \cap Y) \\
\downarrow & & \downarrow \\
E(F \cap Z) & \longrightarrow & E(F_X)
\end{array}
\]

Here we use the assumption that $E$ is excisive or $l$-excisive, respectively. Since a filtered colimit of pushout squares (or excisive squares in $\mathbf{Cat}_{\infty,\ast}^{\text{lex}}$, see Lemma 7.50) we conclude that (4.3) is a pushout square (or an excisive square in $\mathbf{Cat}_{\infty,\ast}^{\text{lex}}$).

**4.2. Colimits.** Let $\mathbf{M}$ be a cocomplete $\infty$-category, $\mathbf{P}$ some auxiliary $\infty$-category, and consider a functor $E: \mathbf{GBornCoarse} \times \mathbf{P} \to \mathbf{M}$.

**Definition 4.11.** We define the functor $E_{G}$ as the composition

\[(4.4) \quad \mathbf{GBornCoarse} \times \mathbf{Fun}(BG, \mathbf{P}) \to \mathbf{Fun}(BG, \mathbf{GBornCoarse}) \times \mathbf{Fun}(BG, \mathbf{P})
\]

\[\xrightarrow{E} \mathbf{Fun}(BG \times BG, M) \xrightarrow{\text{diag}_{BG}} \mathbf{Fun}(BG, M) \xrightarrow{\text{colim}_{BG}} \mathbf{M} \quad \diamond\]

Let $\text{ev}: \mathbf{Fun}(BG, \mathbf{M}) \to \mathbf{M}$ denote the evaluation functor (see (7.6)). For $P$ in $\mathbf{Fun}(BG, \mathbf{P})$ write $E_{P,G}$ for the specialisation of $E_{G}$ at $P$. The functor $E_{P,G}$ inherits various coarse properties from $E(-, \text{ev}(P))$.

Recall Definition 3.1 of the notion of coarse invariance.

**Lemma 4.12.** If $E(-, \text{ev}(P))$ is coarsely invariant, then $E_{P,G}$ is coarsely invariant.

**Proof.** Let $X$ be in $\mathbf{GBornCoarse}$. We must show that the projection

$\{0, 1\}_{\text{max},\text{max}} \otimes X \to X$

induces an equivalence

$E_{P,G}(\{0, 1\}_{\text{max},\text{max}} \otimes X) \to E_{P,G}(X)$.

Since $E$ is coarsely invariant, and equivalences in $\mathbf{Fun}(BG, M)$ are detected by the evaluation functor $\text{ev}: \mathbf{Fun}(BG, M) \to M$, the projection induces an equivalence

$E(\{0, 1\}_{\text{max},\text{max}} \otimes X, P) \to E(X, P)$.
in $\text{Fun}(BG, M)$. Applying $\text{colim}_{BG}$, we get the desired equivalence.

Recall Definition 3.16 of $u$-continuity.

**Lemma 4.13.** If $E(-, ev(P))$ is $u$-continuous, then $E_{P,G}$ is $u$-continuous.

**Proof.** Let $X$ be in $\text{GBornCoarse}$. We must show that the canonical morphism

$$\text{colim}_{U \in C_{X}^{G}} E_{P,G}(X_U) \to E_{P,G}(X)$$

is an equivalence. Since $E(-, ev(P))$ is $u$-continuous, $C_{X}^{G}$ is cofinal in $C_{X}$ by Definition 2.15 (4), and colimits in $\text{Fun}(BG, M)$ are formed pointwise, we have an equivalence

$$\text{colim}_{U \in C_{X}^{G}} E(X_U, P) \to E(X, P)$$

in $\text{Fun}(BG, M)$. Applying $\text{colim}_{BG}$, we get the desired equivalence. □

Recall Definition 3.27 of excisiveness and Definition 3.29 of $l$-excisiveness (for $M = \text{Cat}_{\infty,*}^{\text{Lex}}$).

**Lemma 4.14.** If $E(-, ev(P))$ is excisive or $l$-excisive, then so is $E_{P,G}$.

**Proof.** Let $X$ be in $\text{GBornCoarse}$ with a complementary pair $(Z, Y)$. By the assumption on $E$, the square

$$
\begin{array}{ccc}
E(Y \cap Z, P) & \to & E(Z, P) \\
\downarrow & & \downarrow \\
E(Y, P) & \to & E(X, P)
\end{array}
$$

is a pushout square (or excisive square in the case $M = \text{Cat}_{\infty,*}^{\text{Lex}}$) in $\text{Fun}(BG, M)$. Applying $\text{colim}_{BG}$ produces the desired pushout square (or excisive square in the case $M = \text{Cat}_{\infty,*}^{\text{Lex}}$ by Lemma 7.50)

$$
\begin{array}{ccc}
E_{P,G}(Y \cap Z) & \to & E_{P,G}(Z) \\
\downarrow & & \downarrow \\
E_{P,G}(Y) & \to & E_{P,G}(X)
\end{array}
$$

in $M$. □

Recall Definition 3.7 of a flasqueness preserving functor and Definition 3.12 of a functorially flasqueness preserving functor.

**Lemma 4.15.** If $E$ is functorially flasqueness preserving, then $E_{G}$ is functorially flasqueness preserving.

**Proof.** By assumption, $E$ has an extension $\text{Fl}(E): \text{Fl}(\text{BornCoarse}) \times P \to \text{Fl}(M)$. Let

$$E_{eq}: \text{GBornCoarse} \times \text{Fun}(BG, P) \to \text{Fun}(BG, M)$$

be the composition of the first three morphisms in (4.4) such that $E_{G} \simeq \text{colim}_{BG} E_{eq}$. We define $\text{Fl}(E)_{eq}$ similarly. By functoriality, this extension gives rise to the following
commutative diagram:

\[
\begin{array}{c}
\text{Fl}(\text{GBornCoarse}) \times \text{Fun}(BG, P) \\
\downarrow \\
\text{GBornCoarse} \times \text{Fun}(BG, P)
\end{array}
\begin{array}{c}
\xrightarrow{\text{Fl}(E)_{eq}} \\
\xrightarrow{E_{eq}} \\
\xrightarrow{\text{colim}_{BG}}
\end{array}
\begin{array}{c}
\text{Fun}(BG, \text{Fl}(M)) \\
\downarrow \\
\text{Fun}(BG, \text{Fl}(M)) \\
\text{colim}_{BG} \\
M
\end{array}
\]

Hence \( E_G \) is functorially flasqueness preserving. \( \Box \)

Recall the functor \( i: \text{GSet} \to \text{GBornCoarse} \) from (2.45) sending \( S \) to \( S_{\min, \max} \).
Note that \( X \) in \( \text{GBornCoarse} \) is called bounded if \( X \in B_X \). Consider a functor \( E': \text{GBornCoarse} \to M \) with a cocomplete target.

**Definition 4.16.** \( E' \) is called hyperexcisive if for every \( W \) in \( \text{GSet} \) and bounded \( X \) in \( \text{GBornCoarse} \) the morphism

\[
\text{colim}_{(S \to W) \in G\text{Orb}/W} E'(S_{\min, \max} \otimes X) \to E'(W_{\min, \max} \otimes X)
\]

is an equivalence.

**Remark 4.17.** Equivalence (4.5) can be rewritten as

\[
E'(W_{\min, \max} \otimes X) \simeq \bigoplus_{S \in W/G} E'(S_{\min, \max} \otimes X),
\]

i.e., a hyperexcisive functor is excisive for certain infinite coarsely disjoint decompositions. This property is interesting if \( W/G \) is infinite, otherwise it follows from \( \pi_0 \)-excisiveness (Definition 3.33).

Consider again the situation that \( E \) is a functor \( \text{BornCoarse} \times P \to M \) and that \( P \) is in \( \text{Fun}(BG, P) \).

**Lemma 4.18.** Assume:

1. \( E(-, ev(P)) \) is continuous (Definition 4.2).
2. \( E(-, ev(P)) \) is \( \pi_0 \)-excisive (Definition 3.33).

Then \( E_{P,G} \) is hyperexcisive.

**Proof.** Let \( X \) in \( \text{GBornCoarse} \) be bounded, and let \( W \) be in \( \text{GSet} \). Let \( \text{GSet}^f \) be the full subcategory of \( \text{GSet} \) of \( G \)-finite \( G \)-sets. Then we have a commutative diagram

\[
\begin{array}{c}
\text{colim}_{(R \to W) \in \text{GSet}^f/W} E(R_{\min, \max} \otimes X, ev(P)) \\
\downarrow \simeq \\
\text{colim}_{(R \to W) \in \text{GSet}^f/W} \text{colim}_{F \in \mathcal{F}} E(R_{\min, \max} \otimes X, ev(P))
\end{array}
\begin{array}{c}
\xrightarrow{\simeq} \\
\xrightarrow{\text{colim}_{F \in \mathcal{F}}}
\end{array}
\begin{array}{c}
E(W_{\min, \max} \otimes X, ev(P)) \\
\text{colim}_{F \in \mathcal{F}} E(F, ev(P)) \\
E(F)
\end{array}
\]

in which both vertical maps are equivalences by continuity of \( E(-, ev(P)) \). Every locally finite subset of \( W_{\min, \max} \otimes X \) is contained in a subset of the form \( R \times X \) for some \( R \) in \( \text{GSet}^f \). Hence the lower horizontal arrow is induced by a cofinal
functor, and is thus also an equivalence. It follows that the top horizontal arrow is an equivalence. We have the sequence of equivalences

\[
\colim_{(S \to W) \in \text{GOrb}/W} E_G(S_{\text{min,max}} \otimes X, \text{ev}(P)) \\
\simeq \colim_{BG} \colim_{(S \to W) \in \text{GOrb}/W} E(S_{\text{min,max}} \otimes X, \text{ev}(P)) \\
\simeq \colim_{BG} \colim_{(R \to W) \in \text{GSet}/W} \colim_{(S \to R) \in \text{GOrb}/R} E(S_{\text{min,max}} \otimes X, \text{ev}(P)) \\
\simeq \colim_{BG} E(R_{\text{min,max}} \otimes X) \\
\simeq E_G(W_{\text{min,max}} \otimes X, \text{ev}(P)),
\]

where the equivalence marked by ! follows from a cofinality consideration, the equivalence marked by !! uses $\pi_0$-excisiveness of $E(\_ , \text{ev}(P))$, and the equivalence marked by !!! is the upper horizontal equivalence in (4.6). □

4.3. $V_{G,c}$ and $V_c$. In this subsection we introduce the main objects of the present paper. Recall the functor $V^G$ from (2.87).

We omit the superscript $G$ when we consider the case of a trivial group.

**Definition 4.19.**

(1) We define

\[
V^{G,c} : \text{GBornCoarse} \times \text{Fun}(BG, \text{Cat}^{\text{LEX} \infty,\_}) \to \text{Cat}^{\text{LEX} \infty,\_}
\]

as the functor obtained from $V^G$ by forcing continuity (in the first variable), see Definition 4.4.

(2) We define

\[
V^c_G : \text{GBornCoarse} \times \text{Fun}(BG, \text{Cat}^{\text{LEX} \infty,\_}) \to \text{Cat}^{\text{LEX} \infty,\_}
\]

by applying Definition 4.11 to $V^c$ (the functor from (1) in the case of trivial $G$).

We write $V^{G,c}_C$ and $V^c_{C,G}$ for the evaluation of these functors at a fixed object $C$ in $\text{Fun}(BG, \text{Cat}^{\text{LEX} \infty,\_})$.

**Corollary 4.20.** The functor $V^{G,c}_C$ is

(1) coarsely invariant (Definition 3.1),
(2) $u$-continuous (Definition 3.16),
(3) $l$-excisive (Definition 3.29),
(4) (a) flasqueness preserving (Definition 3.7),
  (b) functorially pre-flasqueness preserving (Definition 3.11),
  (c) functorially flasqueness preserving (Definition 3.12),
(5) (a) $\pi_0$-excisive (Definition 3.33),
  (b) strongly additive (Definition 3.37) and
(6) continuous (Definition 4.2).

**Proof.** Coarse invariance holds by Lemmas 3.3 and 4.5, $u$-continuity by Lemmas 3.18 and 4.6, and excision by Proposition 3.30 and Lemma 4.10. The claims about the preservation of flasqueness are contained in Lemmas 3.15 and 4.7 to 4.8. $\pi_0$-excision
and strong additivity follow from Proposition 3.38 and Lemmas 3.34 and 4.9. Finally, continuity follows from Lemma 4.3. \qed

Corollary 4.21. The functor $V_{C,G}$ is
(1) coarsely invariant,
(2) $u$-continuous,
(3) $l$-excisive,
(4) flasqueness preserving and
(5) hyperexcisive.

Proof. Coarse invariance holds by Corollary 4.20 (1) and Lemma 4.12, $u$-continuity by Corollary 4.20 (2) and Lemma 4.13, and excision by Corollary 4.20 (3) and Lemma 4.14. The functor $V_{C,G}$ is flasqueness preserving by Corollary 4.20 (4) and Lemma 4.15, and hyperexcisive by Lemmas 3.34 and 4.18. \qed

Let $X$ be in $G\text{BornCoarse}$, and let $C$ be in $\text{Fun}(BG, \text{Cat}^{\text{LEX}})$. Lemma 4.22. The canonical morphism $V_{C,G}^{E,c}(X) \to V_{C,G}^G(X)$ is fully faithful.

Proof. Let $F$ be $\mathcal{F}(X)$, i.e., an invariant, locally finite subset of $X$ (see Definition 4.1). By Proposition 3.19, the inclusion $F \to X$ induces a fully faithful functor $V_{C,G}^F(F) \to V_{C,G}^G(X)$. Since a filtered colimit of fully faithful functors is fully faithful, we conclude that
$$V_{C,G}^{G,c}(X) \overset{\text{(4.1)}}{=} \colim_{F \in \mathcal{F}(X)} V_{C,G}^F(F) \to V_{C,G}^G(X)$$
is fully faithful. \qed

4.4. Transfers. In this section, we extend the construction of $V_{C,G}^G$ in order to capture not only its covariant functoriality for morphisms in $G\text{BornCoarse}$ but also the contravariant functoriality for coverings, and the compatibility among these operations. Technically this is accomplished by extending the functor to the $\infty$-category $G\text{BornCoarse}_{tr}$ of bornological coarse spaces with transfers along the inclusion
$$\iota: G\text{BornCoarse} \to G\text{BornCoarse}_{tr} \ .$$
The $\infty$-category $G\text{BornCoarse}_{tr}$ was introduced in [BEKWd, Def. 2.27]. We will use the equivalent description of this category given in Definition 4.40 below (see also Remark 4.41).

Let $E: G\text{BornCoarse} \to M$ be a functor.

Definition 4.23. $E$ admits transfers if there exists a functor $E_{tr}: G\text{BornCoarse}_{tr} \to M$ and an equivalence $E_{tr} \circ \iota \simeq E$. \dagger

Remark 4.24. Assuming that $M$ is cocomplete, one could consider the left Kan extension $\iota E: G\text{BornCoarse}_{tr} \to M$ of $E$ along $\iota$. But in general the morphism $E \to \iota^* \iota E$ is not an equivalence since $\iota$ is not fully faithful. So the problem of showing that $E$ admits transfers does not have such a trivial solution. \dagger

Recall the functor $V^G$ from (2.87).
Theorem 4.25. There exists a functor $V^G_C$ such that the following diagram commutes:

$$
\begin{array}{ccc}
GBornCoarse \times \text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s}) & \xrightarrow{V^G_C} & \text{Cat}^{\text{LEX}}_{\infty, s} \\
\downarrow{\times \text{id}} & & \\
GBornCoarse_{tr} \times \text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s})
\end{array}
$$

In particular, $V^G_C$ admits transfers for every $C$ in $\text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s})$.

As a consequence of Theorem 4.25, we will also derive the following:


The proofs of the theorem and the corollary will occupy the rest of this section.

The composition of the functor $\text{Sh}^{\pi_0, G}$ from (2.34) with the inclusions $\text{CAT}^{\text{LEX}}_{\infty, s} \rightarrow \text{CAT}_\infty$ and $\text{op}: \text{CAT}_\infty \rightarrow \text{CAT}_\infty$ gives rise to a cocartesian fibration

$$
(4.8) \quad s^{\pi_0}: \tilde{\text{Sh}}^{\pi_0, G} \rightarrow \text{Coarse} \times \text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s})
$$

which classifies the functor $(X, C) \mapsto \text{Sh}^{\pi_0, G}_C(X)^{\text{op}}$. Taking the opposite category in the target is motivated by the following.

Lemma 4.27. $s^{\pi_0}$ is also a cartesian fibration.

Proof. The opposite of $s^{\pi_0}$, i.e., the map

$$
(4.9) \quad s^{\pi_0, \text{op}}: \tilde{\text{Sh}}^{\pi_0, G, \text{op}} \rightarrow \text{Coarse}^{\text{op}} \times \text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s})^{\text{op}}
$$

is a cartesian fibration classifying $\text{Sh}^{\pi_0, G}$ as a contravariant functor, but such that the fibre over $(X, C)$ is equivalent to $\text{Sh}^{\pi_0, G}_C(X)$. In this picture, the functors $\hat{\phi}^G_*$ for morphisms $f$ in $\text{Coarse}$ and $\hat{\phi}^G_*$ for morphisms $\phi$ in $\text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s})$ have left adjoints. For $f$ this follows from Corollary 2.12, and for $\phi$ we use Corollary 2.10. By Lurie [Lur09, Cor. 5.2.2.5], $s^{\pi_0, \text{op}}$ is also a cocartesian fibration. In particular, its opposite $s^{\pi_0}$ is also both cartesian and cocartesian. \[\square\]

The subfunctor $\text{Sh}^G$ (see (2.36)) of $\text{Sh}^{G, \pi_0, G}$ gives rise to a cocartesian subfibration of $s^{\pi_0}$:

$$
(4.10) \quad s: \tilde{\text{Sh}}^G \rightarrow \text{Coarse} \times \text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s})
$$

For a morphism $f$ in $\text{Coarse}$ the induced morphism $\hat{f}^G_*$ on sheaves has a left adjoint only under additional conditions (e.g., if $f$ is a coarse covering, see Lemma 2.25). Similarly, for a morphism $\phi$ in $\text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, s})$ the induced morphism $\hat{\phi}^G_*$ on sheaves is a morphism in $\text{CAT}^{\text{LEX}}_{\infty, s}$ and not expected to have a left adjoint, except if it is an equivalence. In order to capture this situation, we will employ Barwick’s effective Burnside category formalism. To this end we recall some terminology from [Bar17, Sec. 5].

Definition 4.28. A triple is an $\infty$-category $D$ together with two subcategories $D^\dagger$ and $D^\dagger$ of $D$, both of which contain the maximal Kan complex $iD$ of $D$. ♦

Let $(D, D^\dagger, D^\dagger)$ be a triple.

1. The morphisms in $D^\dagger$ are called ingressive and will be depicted by the symbol $\rightarrow$.
(2) The morphisms in $D_\dagger$ are called egressive and will be depicted by the symbol $\twoheadrightarrow$.

Let $(D, D_\dagger, D_{\dagger})$ be a triple.

**Definition 4.29.** $(D, D_\dagger, D_{\dagger})$ is called adequate if every diagram

$$
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array}
$$

in $D$ can be completed to a pullback square

$$(4.11)$$

$$
\begin{array}{ccc}
Y' & \rightarrow & X' \\
\downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array}
$$

in $D$.

Pullback squares of the form (4.11) are called ambigressive squares. We will often say that a square is ambigressive in $D$, understanding that this refers to a given triple structure on $D$.

Let $G\Coarse_{\dagger}$ be the wide subcategory of $G\Coarse$ of coarse coverings (Definition 2.23), and set $G\Coarse_{\dagger} := G\Coarse$. Then we set

$$
\begin{align*}
D & := G\Coarse \times \Fun(BG, \Cat_{\infty, *}^{\text{LEX}}) \\
D_\dagger & := G\Coarse_{\dagger} \times \Fun(BG, \Cat_{\infty, *}^{\text{LEX}}) \\
D_{\dagger} & := G\Coarse_{\dagger} \times i\Fun(BG, \Cat_{\infty, *}^{\text{LEX}})
\end{align*}
$$

So we have $D_{\dagger} = D$, and a morphism $(f, \phi)$ in $D$ belongs to $D_{\dagger}$ if and only if $f$ is a coarse covering and $\phi$ is an equivalence.

**Lemma 4.30.** The following triples are adequate:

1. $(G\Coarse, G\Coarse_{\dagger}, G\Coarse_{\dagger})$;
2. $(\Fun(BG, \Cat_{\infty, *}^{\text{LEX}}), \Fun(BG, \Cat_{\infty, *}^{\text{LEX}}), i\Fun(BG, \Cat_{\infty, *}^{\text{LEX}}))$;
3. $(D, D_\dagger, D_{\dagger})$.

**Proof.** We use that $G\Coarse$ and $\Fun(BG, \Cat_{\infty, *}^{\text{LEX}})$ admit pullbacks, and that a pullback of a coarse covering or an equivalence is again a coarse covering ([BEKWd, Lem. 2.11]) or an equivalence, respectively.

**Lemma 4.31.**

1. The projection $\widehat{\Sh}^G_D \times D_\dagger \rightarrow D_\dagger$ is a cocartesian fibration.
2. The projection $\widehat{\Sh}^G_D \times D_{\dagger} \rightarrow D_{\dagger}$ is a cartesian fibration.

**Proof.** The map in (1) is equivalent to the cocartesian fibration $s$. For (2) we first observe, using Lemma 4.27, that $\widehat{\Sh}^{\pi_0, G}_D \times D_{\dagger} \rightarrow D_{\dagger}$ is the pullback of a cartesian fibration and hence itself cartesian. We then use that the cartesian lifts of coarse coverings preserve the category $\widehat{\Sh}^{\dagger}$ by the existence of left adjoints asserted in Lemma 2.25, and that we restricted to the maximal Kan complex in the second factor of $D$. 

□
We define $\hat{\mathbf{Sh}}^G \downarrow_1$ to be the subcategory of cocartesian morphisms in
$$\hat{\mathbf{Sh}}^G \times D \downarrow \to D \downarrow .$$
Similarly, we define $\hat{\mathbf{Sh}}^G \downarrow_1$ to be the subcategory of cartesian morphisms in
$$\hat{\mathbf{Sh}}^G \times D \downarrow \to D \downarrow .$$

**Proposition 4.32.** The triple $(\hat{\mathbf{Sh}}^G, \hat{\mathbf{Sh}}^G \downarrow_1, \hat{\mathbf{Sh}}^G \downarrow_1)$ is adequate, and the map of triples

$$s, s\downarrow_1, s\downarrow_1 : (\hat{\mathbf{Sh}}^G, \hat{\mathbf{Sh}}^G \downarrow_1, \hat{\mathbf{Sh}}^G \downarrow_1) \to (D, D \downarrow_1, D \downarrow_1)$$

preserves ambigressive squares.

The proof of the proposition will be prepared by some intermediate results. We fix $C$ in $\mathbf{Fun}(BG, \mathbf{Cat}_{\infty,LEX}^*)$ and consider the specialisation

$$s_C : \hat{\mathbf{Sh}}^G \to \mathbf{GCoarse}$$

of $s$ at $C$ in the second factor of the target. We will similarly write

$$s_C : \hat{\mathbf{Sh}}^G \downarrow_1 \to \mathbf{GCoarse} \quad \text{and} \quad s\downarrow_1 C : \hat{\mathbf{Sh}}^G \downarrow_1 \to \mathbf{GCoarse} \downarrow_1 .$$

Let

$$M' \xrightarrow{\phi'} N' \quad \text{with} \quad \psi' \downarrow \downarrow \psi$$

be a square in $\hat{\mathbf{Sh}}^G$.\quad \text{(4.14)}

**Lemma 4.33.** Assume:

1. Square (4.14) covers an ambigressive square in $\mathbf{GCoarse}$.
2. $\phi$ is cocartesian.
3. $\psi$ is cartesian.

Then $\phi'$ is cocartesian if and only if $\psi'$ is cartesian.

**Proof.** By (1), the square covers a pullback square

$$V \xrightarrow{g} W \quad \text{with} \quad v \downarrow \downarrow w$$

$$X \xrightarrow{f} Y$$
in $G\text{Coarse}$ such that $v$ and $w$ are coarse coverings. By (2) and (3), we get the following situation

\[
\begin{array}{cccc}
L^{\pi_0 \hat{w}^* G} M & \xrightarrow{\text{cocart}} & \hat{g_0^*} L^{\pi_0 \hat{w}^* G} M & \xrightarrow{\text{Corollary 2.28}} \\
\downarrow \cong & & \downarrow \cong & \\
M' & \xrightarrow{\phi'} & N' & \xrightarrow{\text{cart}} \\
\psi & \downarrow \phi & \psi' & \downarrow \hat{f}_s^* M
\end{array}
\]

where the dotted arrows are obtained from the universal properties of the cartesian or cocartesian maps as indicated. The composition of the two dotted arrows is the comparison morphism from Corollary 2.28, which has the indicated direction since the upper right triangle in (4.15) lives in the fibre $\text{Sh}_G^S(W)^{\text{op}}$ (note the superscript $\text{op}$) of $s$ over $(W, C)$.

If $\phi'$ is cocartesian or $\psi'$ is cartesian, then one of the dotted arrows is an equivalence, and hence the other is an equivalence, too. □

The following assertion is a general fact about an inner fibration in $\text{Cat}_\infty$, but for concreteness we formulate it for $s_C$. We consider again a square of the shape (4.14) in $\text{Sh}_C^G$.

**Lemma 4.34.** Assume:

1. $\psi$ is cartesian.
2. $\psi'$ is cartesian.
3. Square (4.14) covers a pullback square in $G\text{Coarse}$.

Then (4.14) is a pullback square.

**Proof.** We contract the notation for mapping spaces $\text{Map}_E(\cdot, \cdot)$ to $E(\cdot, \cdot)$. We further contract $G\text{Coarse}$ to $G$. Then we have the following chain of equivalences of spaces:

\[
\begin{align*}
\text{Sh}_C^G(P, M') & \cong \text{Sh}_C^G(P, M) \\
& \xrightarrow{(1)} \text{Sh}_C^G(P, M) \times_{G\text{C}(s_C(P), s_C(M))} G\text{C}(s_C(P), s_C(M')) \\
& \xrightarrow{(2)} \text{Sh}_C^G(P, M) \times_{G\text{C}(s_C(P), s_C(M))} G\text{C}(s_C(P), s_C(M)) \times_{G\text{C}(s_C(P), s_C(N))} G\text{C}(s_C(P), s_C(N')) \\
& \xrightarrow{(3)} \text{Sh}_C^G(P, M) \times_{G\text{C}(s_C(P), s_C(N))} G\text{C}(s_C(P), s_C(N')) \\
& \xrightarrow{(4)} \text{Sh}_C^G(P, M) \times_{\text{Sh}_C^G(P, N)} \text{Sh}_C^G(P, N) \times_{G\text{C}(s_C(P), s_C(N))} G\text{C}(s_C(P), s_C(N')) \\
& \xrightarrow{(5)} \text{Sh}_C^G(P, M) \times_{\text{Sh}_C^G(P, N)} \text{Sh}_C^G(P, N')
\end{align*}
\]
The equivalence marked by (1) expresses the assumption that $\psi$ is cartesian. The equivalence marked by (2) follows from the isomorphism

$$s_C(M') \cong s_C(M) \times s_C(N')$$

in $G\mathcal{C}$ which holds by assumption (3). For the equivalence marked by (3) we just cancelled the factor $G\mathcal{C}(s_C(P), s_C(M))$, and for the equivalence marked by (4) we introduced the factor $\hat{\text{Sh}}_G(P, N)$. The equivalence marked by (5) expresses the assumption that $\psi'$ is cartesian. The chain of equivalences above is natural in $P$ and shows that $M' \simeq M \times_N N'$, i.e., that (4.14) is a pullback square.

**Lemma 4.35.** If (4.14) is an ambigressive square, then its image in $G\text{Coarse}$ is an ambigressive square.

**Proof.** Applying $s_C$ to (4.14), we obtain the outer square as in the following picture:

![Diagram](image)

We must show that this square is ambigressive in $G\text{Coarse}$.

Let $V'$ be the pullback $X \times_Y W$. Then we get a uniquely determined map $a$ as indicated. Our task is to show that $a$ is an isomorphism.

We choose a cocartesian lift $\alpha$ of $a$ as indicated. Since $\alpha$ is cocartesian, we obtain the maps $\hat{\phi}'$ and $\hat{\psi}$. Since (4.14) is cartesian by assumption, we also get a map $\beta$ as indicated. We conclude that $\beta \circ \alpha \simeq \text{id}_{M'}$ and hence $s_C(\beta) \circ a = \text{id}_{V'}$. Since the square with upper left corner $V'$ is cartesian, we also have $a \circ s_C(\beta) = \text{id}_{V'}$. Hence $a$ is an isomorphism.

Since (4.14) is ambigressive, we know that $s_C(\psi')$ is a coarse covering. By [BEKWd, Lem. 2.11], this implies that $s_C(\psi)$ is also a coarse covering. Hence the outer square in (4.16) is ambigressive. □

**Lemma 4.36.** The triple $(\hat{\text{Sh}}_G, \hat{\text{Sh}}_1, \hat{\text{Sh}}_G^\dagger)$ is adequate.
Proof. Consider the diagram

\[(4.17) \quad \begin{array}{ccc} N' & \downarrow \chi & N \\ \downarrow M & \phi & \downarrow \end{array} \]

in \(\text{Sh}^G\). We must show that it can be extended to an ambigressive square.

The image of \((4.17)\) in \(D = G\text{Coarse} \times \text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, *})\) is a diagram of the form

\[\begin{array}{ccc} (Y', B') & \downarrow (h,v) & (Y, B) \\ \downarrow (id, v) & \downarrow & \downarrow \end{array}\]

Here \(X, Y, Y'\) are in \(G\text{Coarse}\) and \(h\) is a coarse covering. Furthermore, \(C, B, B'\) are in \(\text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty, *})\) and \(v\) is an equivalence. We can decompose the diagram as follows:

\[(4.18) \quad \begin{array}{ccc} (Y', B') & \downarrow (id,v) & (Y', B) \\ \downarrow & \downarrow (h, id) & \downarrow \end{array}\]

\[\begin{array}{ccc} X & \xleftarrow{(id, u)} & (X, C) \\ \xrightarrow{(f, id)} & \downarrow & \xrightarrow{(Y', B')} \end{array}\]

We can extend the diagram

\[\begin{array}{ccc} Y' & \downarrow h & X \xleftarrow{f} \end{array}\]

to a pullback square in \(G\text{Coarse}\)

\[\begin{array}{ccc} Y' & \xleftarrow{f'} & Y' \\ \downarrow g & \downarrow h & \downarrow \end{array}\]

Then \(g\) is also a coarse covering \([\text{BEKWd, Lem. 2.11}]\). Hence we get the diagram

\[\begin{array}{ccc} (Y', B)' & \downarrow (id,v) & (Y, B) \\ \downarrow & \downarrow (g, id) & \downarrow \end{array}\]

\[\begin{array}{ccc} X & \xleftarrow{(id, u)} & (X, C) \\ \xrightarrow{(f, id)} & \downarrow & \xrightarrow{(Y', B)} \end{array}\]
where the square is ambigressive. This gives finally the following four ambigressive squares

\[(4.19) \quad (X', C) \xrightarrow{(id,u)} (X', B') \xrightarrow{(f', id)} (Y', B'), \]

where for the upper left corner we use that \(v\) is an equivalence. Diagram \((4.17)\) can now be extended as

\[\begin{array}{c}
N' \\
\downarrow id \\
N' \\
\downarrow \chi \\
M & \xrightarrow{\nu} & M'' & \xrightarrow{\phi} & N \\
\downarrow \chi & & \downarrow \chi & & \downarrow \chi \\
M & \xrightarrow{\phi} & M'' & \xrightarrow{\phi} & N
\end{array}\]

over \((4.18)\). For the upper right part we use that \(v\) is an equivalence. The morphism \(\nu\) is defined as a cocartesian lift of \((id_X, u)\) with domain \(M\). The morphism \(\phi\) is then obtained from the universal property of the cocartesian lift, and it is also cocartesian since \(\phi\) is cocartesian.

We now chose a cartesian lift \(\gamma\) of the morphism \((g, id)\) to obtain

\[\begin{array}{c}
N' \\
\downarrow id \\
N' \\
\downarrow \chi \\
M' & \xrightarrow{\phi'} & N' \\
\downarrow \chi & & \downarrow \chi \\
M' & \xrightarrow{\phi'} & N'
\end{array}\]

The morphism \(\phi'\) is obtained from the universal property of \(\chi\) being cartesian. By Lemma 4.33, \(\phi'\) is also cocartesian and therefore ingressive as indicated. By Lemma 4.34, the new square is a pullback and hence an ambigressive square. The same argument provides the lower left ambigressive square in the following diagram.
For the upper part we can argue similarly, but, using the fact that $v$ is an equivalence, we can adopt the special choices of the objects as indicated. The outer square is the desired extension of diagram (4.17) to an ambigressive square.

**Proof of Proposition 4.32.** The triple $(\widehat{\mathbf{Sh}}^G, \widehat{\mathbf{Sh}}^G_\dagger, \widehat{\mathbf{Sh}}^G)$ is adequate by Lemma 4.36. It remains to show that $(s, s_\dagger, s_\dagger)$ preserves ambigressive squares. We consider an ambigressive square

\[
\begin{array}{ccc}
M \rightarrow M' & \rightarrow N' \\
\downarrow \phi & \downarrow \phi' & \downarrow \gamma \\
M' \rightarrow M'' & \rightarrow N
\end{array}
\]

in $\widehat{\mathbf{Sh}}^G$. Then the projection to the first factor $G\mathbf{Coarse}$ of its image in $D$ is ambigressive by Lemma 4.35. The image of diagram (4.20) in the second factor $\mathbf{Fun}(BG, \mathbf{Cat}_{\infty, *})$ of $D$ has the form

\[
\begin{array}{ccc}
C' & \rightarrow B' \\
\downarrow \simeq & \downarrow \simeq \\
C & \rightarrow B
\end{array}
\]

and is therefore a pullback square (because of the vertical equivalences). Therefore, it is an ambigressive square in $\mathbf{Fun}(BG, \mathbf{Cat}_{\infty, *})$. \hfill $\square$

We consider the functor $\text{Tw}: \Delta \rightarrow \mathbf{Cat}$ which associates to the poset $[n]$ in $\Delta$ the poset

\[\text{Tw}([n]) := \{(i, j) \in \{0, \ldots, n\}^2 \mid i \leq j\}\]

with the order relation $(i, j) \leq (i', j')$ if and only if $i \leq i' \leq j' \leq j$ (see also the text after Corollary 3.21).

Let $(\mathbf{D}, \mathbf{D}_\dagger, \mathbf{D}_\dagger)$ be an adequate triple. Then we can consider the simplicial set

\[\mathbf{Fun}(\text{Tw}(-), \mathbf{D}): \Delta^{op} \rightarrow \mathbf{Set},\]

where for two $\infty$-categories $\mathbf{A}, \mathbf{B}$ we write $\mathbf{Fun}(\mathbf{A}, \mathbf{B})$ for the set of functors from $\mathbf{A}$ and $\mathbf{B}$, i.e., the set of objects of the functor category $\mathbf{Fun}(\mathbf{A}, \mathbf{B})$. If $D$ in $\mathbf{Fun}(\text{Tw}([n]), \mathbf{D})$ is such a functor, then we write $D(i, j)$ for the value of $D$ at $(i, j)$ in $\text{Tw}([n])$.

Let $(\mathbf{D}, \mathbf{D}_\dagger, \mathbf{D}_\dagger)$ be an adequate triple.
**Definition 4.37.** The effective Burnside $\infty$-category $A^{\text{eff}}(D, D^\dagger, D^\dagger)$ is defined as the simplicial subset of $\text{Fun}(\text{Tw}([-]), D)$ consisting of those functors $D$ such that for all $0 \leq i \leq k \leq l \leq j \leq n$ the square

$$
\begin{array}{ccc}
D(i, j) & \to & D(k, j) \\
\downarrow & & \downarrow \\
D(i, l) & \to & D(k, l)
\end{array}
$$

is an ambigressive square in $D$.

We refer to [Bar17, Sec. 3 and Prop. 5.9] for further details and a proof that $A^{\text{eff}}(D, D^\dagger, D^\dagger)$ is an $\infty$-category. There are canonical inclusions

(4.21) $D^\dagger \to A^{\text{eff}}(D, D^\dagger, D^\dagger)$, $(D^\dagger)_{\text{op}} \to A^{\text{eff}}(D, D^\dagger, D^\dagger)$.

**Remark 4.38.** Unwinding Definition 4.37, one checks that diagrams of the form

$$
\begin{array}{ccc}
D(0, 1) & \to & D(0, 0) \\
\downarrow & & \downarrow \\
D(1, 1)
\end{array}
$$

constitute the 1-simplices in $A^{\text{eff}}(D, D^\dagger, D^\dagger)$.

Applying the effective Burnside category functor to $(s, s^\dagger, s^\dagger)$ in (4.12), we get a map

(4.22) $A^{\text{eff}}(s): A^{\text{eff}}(\hat{\text{Sh}}^G, \hat{\text{Sh}}^G, \hat{\text{Sh}}^G) \to A^{\text{eff}}(D, D^\dagger, D^\dagger)$.

**Proposition 4.39.** $A^{\text{eff}}(s)$ is a cocartesian fibration.

**Proof.** Our goal is to apply [Bar17, Thm. 12.2].

Note that $\hat{\text{Sh}}^G \to D$ is an inner fibration, so [Bar17, Thm. 12.2] shows directly that $A^{\text{eff}}(s)$ is an inner fibration.

We now verify conditions (12.2.1) and (12.2.2) of [Bar17, Thm. 12.2]. If $f: X \to Y$ is in $G\text{Coarse}^\dagger$ and $M$ is a preimage of $X$, there exists a cocartesian lift $\tilde{f}: M \to \tilde{f}_*^G M$ of $f$. By definition, $\tilde{f}$ lies in $\hat{\text{Sh}}^G$. Moreover, $\tilde{f}$ is also cocartesian with respect to the map $\hat{\text{Sh}}^G \to G\text{Coarse}^\dagger$ by [Lur09, Cor. 2.4.2.5] and [Lur09, Prop. 2.4.2.4] (translated from the cartesian to the cocartesian case). This verifies condition (12.2.1) of [Bar17, Thm. 12.2] for morphisms in $G\text{Coarse}^\dagger$. A similar argument applies to morphisms in $\phi: C \to C'$ in $\text{Fun}(BG, \text{Cat}_{\text{LEX}}^\omega)$.

Suppose that we are given a commutative square

(4.23) $\begin{array}{ccc}
M' & \xrightarrow{\phi'} & N' \\
\psi \downarrow & & \downarrow \psi' \\
M & \xrightarrow{\phi} & N
\end{array}$

in $\hat{\text{Sh}}^G$ whose images in $D$ and hence in $G\text{Coarse}$ are ambigressive squares, and such that $\phi'$ is ingressive, $\psi$ is egressive, and $\phi$ is cocartesian. Note that then by definition $\phi'$ is cocartesian and $\psi$ is cartesian, and $\phi$ is ingressive. It follows from Lemma 4.33
that then \( \psi' \) is also cartesian and hence egressive. By Lemma 4.34, square \((4.23)\) is cartesian and hence an ambigressive square. This shows that condition \((12.2.2)\) of \([\text{Bar}17, \text{Thm. 12.2}]\) is satisfied.

\([\text{Bar}17, \text{Thm. 12.2}]\) now asserts that an edge in \( \mathbf{A}^{\mathrm{ef}}(\widehat{\mathbf{Sh}}, \mathbf{Sh}, \mathbf{Sh}) \) is cartesian (w.r.t. \( \mathbf{A}^{\mathrm{ef}}(s) \)) if and only if it is represented by a span

\[
\begin{array}{ccc}
N & \xrightarrow{w} & M \\
\downarrow & & \downarrow f \\
P & \xrightarrow{f} & \mathbf{A}
\end{array}
\]

in which \( w \) is cartesian and \( f \) is cocartesian. Using Lemma 4.31 and the explicit description of the edges in \( \mathbf{A}^{\mathrm{ef}}(\widehat{\mathbf{Sh}}, \mathbf{Sh}, \mathbf{Sh}) \) given in Remark 4.38, we see that every morphism in \( \mathbf{A}^{\mathrm{ef}}(\mathbf{D}, \mathbf{D}, \mathbf{D}) \) given a cocartesian lift. This shows that \( \mathbf{A}^{\mathrm{ef}}(s) \) is a cocartesian fibration \( \square \)

We define the category \( \mathbf{GCoarse}_{\mathrm{tr}} \) of \( G \)-coarse spaces with transfers by

\[(4.24) \quad \mathbf{GCoarse}_{\mathrm{tr}} := \mathbf{A}^{\mathrm{ef}}(\mathbf{GCoarse}, \mathbf{GCoarse}_{\dagger}, \mathbf{GCoarse}^{\dagger}) .\]

An instance of the canonical functor \((4.21)\) applied to \((4.24)\) provides

\[(4.25) \quad \mathbf{GCoarse} \to \mathbf{GCoarse}_{\mathrm{tr}} .\]

Using in addition the canonical inclusion

\[
\operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}) \to \mathbf{A}^{\mathrm{ef}}(\operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}), \operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}), \operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}))
\]

in the second component, we get a functor

\[(4.26) \quad \mathbf{GCoarse}_{\mathrm{tr}} \times \operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}) \to \mathbf{A}^{\mathrm{ef}}(\mathbf{D}, \mathbf{D}, \mathbf{D}) .\]

We define the functor

\[(4.27) \quad \mathbf{Sh}^{G}_{\mathrm{tr}}: \mathbf{GCoarse}_{\mathrm{tr}} \times \operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}) \to \mathbf{CAT}_{\infty} \]

by restricting the cocartesian fibration \( \mathbf{A}^{\mathrm{ef}}(s) \) along the functor from \((4.26)\), applying the straightening functor, and applying \((\cdot)^{\text{op}}\) to the values.

Note that the value of \( \mathbf{Sh}^{G}_{\mathrm{tr}} \) at \((X, C)\) is the object \( \mathbf{Sh}^{G}_{\mathrm{tr}}(X) \) of \( \mathbf{CAT}^{\mathrm{LEX}}_{\infty,s} \). By construction, \( \mathbf{Sh}^{G}_{\mathrm{tr}} \) sends a morphism \( f \) in \( \mathbf{GCoarse} \) to the morphism \( \widehat{f}^{G} \) in \( \mathbf{CAT}^{\mathrm{LEX}}_{\infty,s} \), a coarse covering \( w \) in \( \mathbf{GCoarse}^{\dagger} \) to the morphism \( L_{p_{0}}^{\widehat{w}^{G},*} \) in \( \mathbf{CAT}^{\mathrm{Lex}}_{\infty,s} \), and a morphism \( \phi \) in \( \operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}) \) to the morphism \( \widehat{\phi}^{G} \) in \( \mathbf{CAT}^{\mathrm{Lex}}_{\infty,s} \). This implies that \( \mathbf{Sh}^{G}_{\mathrm{tr}} \) actually takes values in the subcategory \( \mathbf{CAT}^{\mathrm{Lex}}_{\infty,s} \) of \( \mathbf{CAT}_{\infty} \), i.e., we have a functor

\[(4.28) \quad \mathbf{Sh}^{G}_{\mathrm{tr}}: \mathbf{GCoarse}_{\mathrm{tr}} \times \operatorname{Fun}(BG, \mathbf{Cat}^{\mathrm{LEX}}_{\infty,s}) \to \mathbf{CAT}^{\mathrm{Lex}}_{\infty,s} .\]

Let \( \mathbf{GBC} \) be the category with the same objects as \( \mathbf{GBornCoarse} \) and all equivariant maps which are controlled and bornological (Definition 2.33). Consider the wide subcategory \( \mathbf{GBC}^{\dagger} \) of \( \mathbf{GBC} \) of those morphisms which are in addition proper, and the wide subcategory \( \mathbf{GBC}^{\dagger} \) of \( \mathbf{GBC} \) of coverings (Definition 2.41). Using [BEKWd, Lem. 2.20 and 2.21], one verifies that the triple \((\mathbf{GBC}, \mathbf{GBC}^{\dagger}, \mathbf{GBC}^{\dagger})\) is admissible.

**Definition 4.40.** We define

\[
\mathbf{GBornCoarse}_{\mathrm{tr}} := \mathbf{A}^{\mathrm{ef}}(\mathbf{GBC}, \mathbf{GBC}_{\dagger}, \mathbf{GBC}^{\dagger}) .
\]
**Remark 4.41.** One checks using Remark 4.38 that $G\text{BornCoarse}_{\text{tr}}$ precisely coincides with the $\infty$-category defined in [BEKWd, Def. 2.27] and denoted there by the same symbol.

There is an inclusion functor

\[(4.29)\]

\[\iota : G\text{BornCoarse} \to G\text{BornCoarse}_{\text{tr}}\]

which is determined by the requirement that it sends a $G$-bornological coarse space to itself and a morphism $f : X \to X'$ to the span $X \leftarrow \tilde{X} \overset{\ell}{\to} X'$, where $\tilde{X}$ is obtained from $X$ by replacing the bornology on $X$ by $f^{-1}B_{X'}$.

The forgetful functor $G\text{BC} \to G\text{Coarse}$ induces a functor

\[(4.30)\]

\[G\text{BornCoarse}_{\text{tr}} \to G\text{Coarse}_{\text{tr}}\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
G\text{BornCoarse} & \xrightarrow{\iota} & G\text{BornCoarse}_{\text{tr}} \\
\downarrow \text{(4.29)} & & \downarrow \text{(4.30)} \\
G\text{Coarse} & \xrightarrow{\text{tr}} & G\text{Coarse}_{\text{tr}}
\end{array}
\]

If we precompose $\text{Sh}_{\text{tr}}^G$ from (4.28) with the forgetful map (4.30), we obtain a functor (we use the same symbol)

\[\text{Sh}_{\text{tr}}^G : G\text{BornCoarse}_{\text{tr}} \times \text{Fun}(BG, \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*}) \to \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*}\]

such that $\text{Sh}_{\text{tr}}^G \circ (\iota \times \text{id}) \simeq \text{Sh}^G$. In particular, its value at $(X, C)$ is given by

\[\text{Sh}_{\text{tr}}^G(X, C) \simeq \text{Sh}^G_C(X) .\]

It follows from a combination of Lemma 2.49, Lemma 2.51, and Proposition 2.54 that we have a subfunctor

\[\text{Sh}_{\text{tr}}^{G, \text{eqsm}} : G\text{BornCoarse}_{\text{tr}} \times \text{Fun}(BG, \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*}) \to \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*}\]

with values

\[\text{Sh}_{\text{tr}}^{G, \text{eqsm}}(X, C) \simeq \text{Sh}_{C}^{G, \text{eqsm}}(X) .\]

For every pair $(X, C)$ we have a localisation functor $\ell : \text{Sh}_{\text{tr}}^G(X) \to V_{\text{C}}^G(X)$ (Definition 2.77), and we let $\tilde{W}_{X,C}^{\text{eqsm}}$ denote the labelling of $\text{Sh}_{C}^{G, \text{eqsm}}(X)$ generated by the morphisms which are sent by $\ell$ to equivalences. If $f$ is a morphism in $G\text{BornCoarse}$, then $\tilde{f}^G$ sends $\tilde{W}_{X,C}^{\text{eqsm}}$ to $\tilde{W}_{X',C}^{\text{eqsm}}$ by Corollary 2.82. If $\phi$ is a morphism in $\text{Fun}(BG, \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*})$, then $\tilde{\phi}^G$ sends $\tilde{W}_{X,C}^{\text{eqsm}}$ to $\tilde{W}_{X',C}^{\text{eqsm}}$ by Corollary 2.84. Finally, if $w : X \to X'$ is a morphism in $G\text{BornCoarse}$, then $L_{\tau_\nu}w^G$ sends $\tilde{W}_{X,C}^{\text{eqsm}}$ to $\tilde{W}_{X',C}^{\text{eqsm}}$ by Lemma 2.85 (2). This implies that the functor $\text{Sh}_{\text{tr}}^{G, \text{eqsm}}$ refines to a functor

\[\ell\text{Sh}_{\text{tr}}^{G, \text{eqsm}} : G\text{BornCoarse}_{\text{tr}} \times \text{Fun}(BG, \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*}) \to \ell\text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*}\]

given on objects by $(X, C) \mapsto (\text{Sh}_{\text{tr}}^{G, \text{eqsm}}(X), \tilde{W}_{X,C}^{\text{eqsm}})$, see Definition 7.35. Recall the localisation functor $\text{Loc}$ from (7.23).

**Definition 4.42.** We define the functor

\[V_{\text{tr}}^G : G\text{BornCoarse}_{\text{tr}} \times \text{Fun}(BG, \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*}) \xrightarrow{\ell\text{Sh}_{\text{tr}}^{G, \text{eqsm}}} \ell\text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*} \xrightarrow{\text{Loc}} \text{Cat}_{\text{\text{\scriptsize{LEX}}}^\infty,*} .\]
Proof of Theorem 4.25. By construction, $V^G_{tr}$ satisfies
\[ V^G_{tr} \circ (\iota \times \text{id}) \simeq V^G. \]
\[ \square \]

Let $E_{tr} : \text{GBornCoarse}_{tr} \to \text{M}$ be some functor with a target which admits small filtered colimits. Let $\iota$ be the inclusion functor from (4.29). Recall Definition 4.2 of a continuous functor.

**Definition 4.43.** We call $E_{tr}$ continuous if $E_{tr} \circ \iota$ is continuous. ♦

In the following, we show that we can force continuity for $E_{tr}$ in such a way that we obtain a functor $E^c_{tr}$ satisfying $E^c_{tr} \circ \iota \simeq (E_{tr} \circ \iota)^c$ (see Definition 4.4).

We consider the inclusion of the full $\infty$-subcategory
\[ i_{tr} : \text{GBornCoarse}_{tr}^{mb} \to \text{GBornCoarse}_{tr} \]
of $G$-bornological coarse spaces with the minimal bornology. We consider the following diagram
\[ \text{GBornCoarse}_{tr}^{mb} \xrightarrow{E_{tr} \circ i_{tr}} \text{M}, \]
\[ G \text{BornCoarse}_{tr} \]

where $E^c_{tr}$ is defined by left Kan extension. Similarly as in Definition 4.4, we call $E^c_{tr}$ the functor obtained from $E_{tr}$ by forcing continuity.

We let $E := E_{tr} \circ \iota$ and consider the following diagram
\[ \text{GBornCoarse}_{tr}^{mb} \xrightarrow{E_{tr} \circ i_{tr}} \text{M}, \]
\[ G \text{BornCoarse}_{tr} \]
\[ \text{GBornCoarse}_{tr}^{mb} \xrightarrow{i_{mb}} G \text{BornCoarse}_{tr} \]
\[ \text{GBornCoarse}_{tr} \xrightarrow{E \circ \iota} \text{M}, \]
\[ G \text{BornCoarse}_{tr} \]

where $i_{mb}$ is defined as the restriction of $\iota$ and $E^c$ is defined by left Kan extension. The universal property of left Kan extensions provides a transformation
\[ (4.31) \quad E^c \to E^c_{tr} \circ \iota. \]

**Lemma 4.44.** Transformation (4.31) is an equivalence.

**Proof.** Let $X$ be in $\text{GBornCoarse}$. We must show that
\[
\colim_{(F \to X) \in (\text{GBornCoarse}_{tr}^{mb})/X} E_{tr}(\iota(F)) \to \colim_{(F \to X) \in (\text{GBornCoarse}_{tr}^{mb})/X} E_{tr}(F)
\]
is an equivalence, where the morphism is induced by the functor
\[ \iota_X : (\text{GBornCoarse}_{tr}^{mb})/X \to (\text{GBornCoarse}_{tr}^{mb})/X \]
between index categories induced by $\iota$. 
We claim that $\iota_X$ is cofinal. We first show that the index categories are filtered. We give the argument for $(\text{GBornCoarse}_{tr}^{mb})/X$. The case of $(\text{GBornCoarse}_{tr}^{mb})/X$ is similar and simpler.

Let $F: K \to (\text{GBornCoarse}_{tr}^{mb})/X$ be a morphism from a finite simplicial set. Then for every $k$ in $K$ we have a span

$$F(k) \leftarrow F'(k) \xrightarrow{\kappa_k} X,$$

whose left leg is a covering, whose right leg $\kappa_k$ is a morphism in $\text{GBornCoarse}$, and such that $F(k)$ carries the minimal bornology. It is easy to see using Definition 2.41 that $F'(k)$ is then also equipped with the minimal bornology. We consider the subset $\tilde{F} := \bigcup_{k \in K} \kappa_k(F'(k))$ of $X$ with the induced $G$-bornological coarse structure. Then $\tilde{F}$ is a $G$-invariant locally finite subset of $X$ and therefore belongs to $\text{GBornCoarse}_{tr}^{mb}$. We extend the map $F$ to a map $F_{\triangledown}: K_{\triangledown} \to (\text{GBornCoarse}_{tr}^{mb})/X$

by sending the cone point to $\tilde{F} \xleftarrow{\text{id}_{\tilde{F}}} \xrightarrow{\text{incl}} X$.

We now see that $\iota_X$ is cofinal since the object $\tilde{F} \xleftarrow{\text{id}_{\tilde{F}}} \xrightarrow{\text{incl}} X$ belongs to the image of $\iota_X$ (since the left arrow is the identity). □

Let $C$ be in $\text{Fun}(BG, \text{Cat}^{\text{LEX}}_\infty, \ast)$, and let $V_{tr,C}^G$ denote the specialisation of the functor $V_{tr}^G$ from Definition 4.42 at $C$. We let $V_{tr,C}^{G,c}: \text{GBornCoarse}_{tr} \to \text{Cat}^{\text{Lex}}_\infty, \ast$ be obtained from $V_{tr,C}^G$ by forcing continuity in the sense of Definition 4.4.

Proof of Corollary 4.26. The equivalence

$$V_{tr}^{G,c} \circ \iota \simeq V^{G,c}$$

follows immediately from Theorem 4.25 and Lemma 4.44. □

5. Calculations

In this section we calculate the values of the functors $V_{C,G}^G$ and $V_{C,C}^{G,c}$ (see Definition 4.19) on certain $G$-bornological coarse spaces. The result is an essential ingredient in the proof of Theorem 6.23.

The group $G$, acting on itself on the left, is an object of $G\text{Orb}$. The right action of $G$ on itself identifies $\text{Aut}_{G\text{Orb}}(G)$ with $G$. We thus obtain the inclusion (1.2)

$$j: BG \to G\text{Orb}$$

sending the unique object $*_{BG}$ of $BG$ to the object $G$ of $G\text{Orb}$, and the automorphism $g$ in $\text{Aut}_{BG}(*_{BG})$ to the right-multiplication by $g^{-1}$ in $\text{Aut}_{G\text{Orb}}(G)$.

Since $\text{Cat}^{\text{Lex}}_\infty, \ast$ is cocomplete by Proposition 7.9, we have an adjunction

$$\text{Ind}^G: \text{Fun}(BG, \text{Cat}^{\text{Lex}}_\infty, \ast) \simeq \text{Fun}(G\text{Orb}, \text{Cat}^{\text{Lex}}_\infty, \ast) : \text{Res}^G,$$

where $\text{Res}^G := j^*$ is the restriction functor along $j$, and $\text{Ind}^G := j_!$ is the left Kan extension functor along $j$. 

**Remark 5.1.** For a subgroup $H$ of $G$ we consider the object $G/H$ in $G\text{Orb}$. We then have an equivalence of categories $BH \xrightarrow{\sim} BG/(G/H)$ which sends the unique object $\ast_{BH}$ of $BH$ to the projection $G \rightarrow G/H$ and the element $h$ in $H = \text{Aut}_{BH}(\ast_{BH})$ to the automorphism of $(G \rightarrow G/H)$ in $BG/(G/H)$ given by right-multiplication on $G$ with $h^{-1}$. If $C$ is in $\text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}})$, then the pointwise formula for the left Kan extension provides the first equivalence in

$$\text{Ind}^{G}(C)(G/H) \simeq \colim_{BG/(G/H)} C \simeq \colim_{BH} \text{Res}^{G}_{H} C,$$

where the second equivalence uses the equivalence of index categories discussed above.

By restriction from $G$-sets to $G$-orbits, the functor $G\text{Set} \rightarrow G\text{BornCoarse}$ from (2.45) gives rise to the functor

$$i: G\text{Orb} \rightarrow G\text{BornCoarse}, \quad S \mapsto S_{\text{min,max}}.$$

Let $(-)^{\omega}: \text{Cat}_{\infty,*}^{\text{LEX}} \rightarrow \text{Cat}_{\infty,*}^{\text{Lex}}$ be the functor taking the subcategory of cocompact objects (Definition 7.7). By postcomposition, it induces a functor

$$\text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{LEX}}) \rightarrow \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}}), \quad C \mapsto C^{\omega}.$$

Let $C$ be in $\text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}})$ and recall the definition of $V_{C,G}$ from Definition 4.19 (2).

**Proposition 5.2.** There is an equivalence

$$\text{Ind}^{G}(C^{\omega}) \simeq V_{C,G} \circ i$$

of functors $G\text{Orb} \rightarrow \text{Cat}_{\infty,*}^{\text{Lex}}$.

**Proof.** We consider the functors

$$\text{Sh}_{C,\text{eq}}, \text{Sh}_{C,\text{eq}}^{\text{sm}}, V_{C,\text{eq}}: G\text{Orb} \rightarrow \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}})$$

defined as the compositions (compare with the first three morphisms in (4.4))

$$G\text{Orb} \cong G\text{Orb} \times G\text{BornCoarse} \times \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}}) \rightarrow \text{Fun}(BG \times BG, \text{BornCoarse} \times \text{Cat}_{\infty,*}^{\text{Lex}})$$

$$\xrightarrow{\text{diag}_{BG}} \text{Fun}(BG, \text{BornCoarse} \times \text{Cat}_{\infty,*}^{\text{Lex}})$$

$$\xrightarrow{\text{Sh}_{C,\text{sm}}, V_{C}} \text{Fun}(BG, \text{Cat}_{\infty,*}^{\text{Lex}}).$$

We first show that there is an equivalence

$$V_{C,G} \circ i \simeq \colim_{BG} \text{Sh}_{C,\text{sm}}^{\text{sm}} \circ i.$$

Since the functor $i: G\text{Orb} \rightarrow G\text{BornCoarse}$ equips the transitive $G$-sets with the minimal coarse structure, the localisation morphism $\ell: \text{Sh}_{C}^{\text{sm}} \rightarrow V_{C}$ induces an equivalence

$$\text{Sh}_{C}^{\text{sm}} \circ \text{res}_{\{1\}} \circ i \simeq V_{C} \circ \text{res}_{\{1\}}^{G} \circ i$$

(see Example 2.80 and recall that we omit to write $\text{Res}_{\{1\}}^{G}$ in front of $C$, but we do not omit $\text{res}_{\{1\}}^{G}$ at the space variables).
Let \( S \) be in \( G\text{-}\text{Orb} \). Using (5.6), the left vertical equivalence in the square (2.71), and the equivalence (4.1), we identify the transformation
\[
V_C^c(\text{res}_G^G S_{\min,max}) \to V_C^c(\text{res}_G^G S_{\min,max})
\]
with the canonical map
\[
\text{colim}_{F \subseteq S \text{ finite}} \prod_F C^\omega \to \prod_S C^\omega.
\]
Since the latter is obviously an equivalence, we obtain the first equivalence in
\[
(5.7) \quad V_C^c \circ \text{res}_G^{G} \circ i \simeq V_C \circ \text{res}_G^{G} \circ i \simeq \text{Sh}_{C, eq}^{\text{eq}} \circ \text{res}_G^{G} \circ i.
\]
We now take the \( G \)-actions on \( S \) and \( C \) into account which by functoriality induce \( G \)-actions on the two sides of this equivalence. We therefore have an equivalence
\[
(5.8) \quad V_C^c \circ \text{eq} \simeq \text{Sh}_{C, eq}^{\text{eq}}.
\]
Applying \( \text{colim}_{BG} \) and Definition 4.19 of \( V_{C,G}^c \) for the first equivalence, we get the chain of equivalences
\[
(5.9) \quad V_{C,G}^c \circ i \simeq \text{colim}_{BG} \circ V_{C,eq}^c \simeq \text{colim}_{BG} \circ \text{Sh}_{C,eq}^{\text{eq}}
\]
showing (5.5).

The functor corresponding to \( \text{Sh}_{C,eq}^{\text{eq}} \) by the exponential law, is given by the composition
\[
\text{Sh}_{C,eq}^{\text{eq}} : G\text{Orb} \times BG \xrightarrow{(\alpha, \text{Copr})} \text{BornCoarse} \times \text{Cat}_{\infty}^{\text{Lex}} \xrightarrow{\text{Sh}_{\text{eq}}^{\text{eq}}} \text{Cat}_{\infty}^{\text{Lex}},
\]
where \( \text{pr} : G\text{Orb} \times BG \to BG \) is the projection and \( \alpha : G\text{Orb} \times BG \to \text{BornCoarse} \) is the functor sending objects \((S, \ast_{BG})\) to \( \text{res}_G^{G} S_{\min,max} \) and morphisms \((S \xrightarrow{\phi} S', \ast_{BG} \xrightarrow{g} \ast_{BG})\) to
\[
\text{res}_G^{G} S_{\min,max} \xrightarrow{\text{res}_G^{G} S_{\min,max}} \text{res}_G^{G} S'_{\min,max} \xrightarrow{g} \text{res}_G^{G} S'_{\min,max}.
\]
Consider the functor \( p := (j, \text{id}) : BG \to G\text{Orb} \times BG \) and the projection \( \pi : G\text{Orb} \times BG \to G\text{Orb} \). We then have the following diagram (the left triangle commutes, and the two fillers \( \tau \) and \( \sigma \), which are not necessarily equivalences, will be explained below):

\[
\begin{array}{ccc}
BG & \xrightarrow{j} & G\text{Orb} \\
\downarrow{p} & & \downarrow{\pi} \\
G\text{Orb} \times BG & \xrightarrow{\text{Sh}_{C,eq}^{\text{eq}}} & \text{Cat}_{\infty}^{\text{Lex}} \\
\downarrow{\text{colim}_{BG} \circ \text{Sh}_{C,eq}^{\text{eq}}} & & \downarrow{\sigma} \\
\end{array}
\]

In the following we use the subscript \( ! \) in order to denote left Kan extension functors. There is an obvious canonical natural transformation \( \sigma \) exhibiting \( \text{colim}_{BG} \circ \text{Sh}_{C,eq}^{\text{eq}} \) as a left Kan extension of \( \text{Sh}_{C,eq}^{\text{eq}} \) along \( \pi \), i.e., we have an equivalence
\[
(5.10) \quad \text{colim}_{BG} \circ \text{Sh}_{C,eq}^{\text{eq}} \simeq \pi_{!} \text{Sh}_{C,eq}^{\text{eq}}.
\]
The functor $\alpha \circ \eta : p: BG \to \text{BornCoarse}$ sends $\ast_{BG}$ to $\text{res}_{\{1\}}^G G_{\text{min,max}}$ and a morphism $g$ in $G = \text{Aut}_{BG}(\ast_{BG})$ to the conjugation by $g$ on $\text{res}_{\{1\}}^G G_{\text{min,max}}$. Since conjugation fixes the identity element, the inclusion of the identity element $i_{\ast} : \ast \cong \{e\} \to \text{res}_{\{1\}}^G G_{\text{min,max}}$ induces a natural transformation

$$\tau : \mathbf{C}^{\omega} \cong \text{Sh}_{\mathbf{C}}^{\text{eqsm}}(\ast) \overset{i_\ast}{\longrightarrow} \text{Sh}_{\mathbf{C}}^{\text{eqsm}} \circ p$$

of functors $BG \to \text{Cat}^{\text{Lex}}_{\infty,*}$. We claim that it exhibits $\text{Sh}_{\mathbf{C}}^{\text{eqsm}}$ as the left Kan extension of $\mathbf{C}^{\omega}$ along $p$, i.e., that $\tau$ induces an equivalence

$$(5.11) \quad p ! \mathbf{C}^{\omega} \cong \text{Sh}_{\mathbf{C}}^{\text{eqsm}}.$$

Since $\pi \circ \eta = j$ the claim implies the desired equivalence (5.4) by

$$\text{Ind}_G^G(\mathbf{C}^{\omega}) \cong j ! \mathbf{C}^{\omega} \cong \pi_! p ! \mathbf{C}^{\omega} \cong \pi_! \text{Sh}_{\mathbf{C}}^{\text{eqsm}} \cong \text{colim}_{BG} \text{Sh}_{\mathbf{C}}^{\text{eqsm}} \cong \text{V}^c_{\mathbf{C},G} \circ i.$$

It remains to show the claim. We need to check for $S$ in $\text{GOrb}$ that the induced morphism

$$\tau(S) : \text{colim}_{p/(S,*)} \mathbf{C}^{\omega} \to \text{Sh}_{\mathbf{C}}^{\text{eqsm}}(S, \ast_{BG}) \cong \text{Sh}_{\mathbf{C}}^{\text{eqsm}}(\text{res}_{\{1\}}^G S_{\text{min,max}})$$

is an equivalence, where $p/(S,*)$ is a shorthand for $BG \times \text{GOrb} \times BG(\text{GOrb} \times BG)/(S,*), \text{with $p$ being implicitly used in the pullback construction. The set of objects of the category } p/(S,*) \text{ can be identified with the set } S \times G \text{ such that } (s,g) \text{ corresponds to the pair } \ast_{BG}, (G \overset{c_{\text{eqsm}}}{\longrightarrow} S, \ast_{BG} \overset{g}{\longrightarrow} \ast_{BG}).$ A morphism $(s,g) \to (s',g')$ exists (and is then unique) if and only if $g'g^{-1}s' = s$ (see also the proof of Lemma 2.56). Consequently, the functor $S \to p/(S,*)$ sending $s$ in $S$ to the object $(s,e)$ is an inverse to the projection functor $p/(S,*) \to S$. This explicit inverse induces the left vertical equivalence in the commutative diagram

$$\text{colim}_{p/(S,*)} \mathbf{C}^{\omega} \overset{\tau(S)}{\longrightarrow} \text{Sh}_{\mathbf{C}}^{\text{eqsm}}(\text{res}_{\{1\}}^G S_{\text{min,max}}) \cong \bigcup_{s \in S} \Sigma_{i_\ast} \tau_{s,*}$$

where $i_\ast : \{s\} \to \text{res}_{\{1\}}^G S_{\text{min,max}}$ denotes the inclusion maps. The diagonal arrow is an equivalence by Lemma 2.71, so $\tau(S)$ is an equivalence for all $S$ in $\text{GOrb}$. This finishes the proof of the claim.

Let $\mathbf{C}$ be in $\text{Fun}(BG, \text{Cat}^{\text{Lex}}_{\infty,*})$. Recall the $G$-bornological coarse space $G_{\text{can,min}}$ from Example 2.39. Recall also the idempotent completion functor $\text{Idem}$ from (7.14).

**Proposition 5.3.** There are equivalences

$$\text{Idem} V^G_C(G_{\text{can,min}} \otimes (-)_{\text{min,max}}) \cong \text{Idem} V^G_C(G_{\text{can,min}} \otimes (-)_{\text{min,max}}) \cong \text{Sh}_{\mathbf{C}}^{G}((-)_{\text{min}})^{\omega}$$

(5.12)

of functors $G\text{Set} \to \text{Cat}^{\text{Lex}}_{\infty,*}$. 


Proof. Let $X$ be in $G\text{Set}$. Denote by $\pi: G_{can} \otimes X_{min} \to X_{min}$ the projection and by $\iota: G_{min} \otimes X_{min} \to G_{can} \otimes X_{min}$ be the morphism in $G\text{Coarse}$ induced by the identity of the underlying $G$-sets. Then we get the bold part of the following commutative diagram:

\[
\begin{array}{ccc}
\text{Sh}^G_c(G_{min} \otimes X_{min}) & \xrightarrow{\iota_*} & \text{Sh}^G_c(G_{can} \otimes X_{min}) \\
\sigma(2.73) & \downarrow & \pi_*^G \\
\text{Sh}^G_c(G_{min} \otimes X_{min}) & \rightarrow & \text{Sh}^G_c(X_{min}) \\
\end{array}
\]

The left square is a case of (2.83), and the upper triangle is an instance of the upper right triangle in (2.65). The arrow $\iota_*^G \otimes X_{min}$ is an equivalence by Example 2.80.

In order to get the dotted arrow $p$ and the corresponding triangle, we use the universal property of $\iota_*^G \otimes X_{min}$ and the fact that $\pi_*^G$ sends the morphisms $M \to V^G M$ for all $M$ in $\text{Sh}^G_c(G_{can} \otimes X_{min})$ and $V$ in $G_{can} \otimes X_{min}$ to identity morphisms.

The whole diagram (5.13) is natural in $X$ in $G\text{Set}$.

Note that $\text{Sh}^G_c(X_{min})$ is an object of $\text{Cat}^{\text{LEX}}_{\infty,*}$ by Remark 2.70 and therefore has a well-defined notion of cocompact objects. Note that if $G$ is not finite, then $\pi$ does not induce a morphism in $G\text{BornCoarse}$ from $G_{can,min} \otimes X_{min,max}$ to $X_{min,max}$. Since this projection would not be proper. We nevertheless can show that $\pi_*^G$ restricts to a functor

\[
\pi_*^G: \text{Sh}^G_c(G_{can,min} \otimes X_{min,max}) \to \text{Sh}^G_c(X_{min})^\omega.
\]

Let $M$ be in $\text{Sh}^G_c(G_{can,min} \otimes X_{min,max})$. By Proposition 2.73 we have $M \in \text{PSH}^G_c(G \times X)^\omega$. Since the functor $\text{PSH}^G_c(G \times X) \to \text{PSH}^G_c(X)$ induced by $\pi$ is a morphism in $\text{Cat}^{\text{LEX}}_{\infty,*}$, it preserves cocompact objects, and we have $\pi_*^G(M) \in \text{PSH}^G_c(X)^\omega$. Since the inclusion $\text{Sh}^G_c(X_{min}) \to \text{PSH}^G_c(X)$ clearly detects cocompactness, using Lemma 2.25 we conclude that $\pi_*^G(M) \in \text{Sh}^G_c(X_{min})^\omega$.

We consider the morphism $\iota: G_{min,min} \otimes X_{min,max} \to G_{can,min} \otimes X_{min,max}$ in $G\text{BornCoarse}$ and note that $\iota_*^G$ preserves equivariantly small objects by Lemma 2.49. By restricting the diagram in (5.13) to equivariantly small objects, using the equivalence in (2.72) for the upper corner, and the second assertion of Corollary 2.60, we get the following commutative diagram

\[
\begin{array}{ccc}
\text{Sh}^G_c(G_{min,min} \otimes X_{min,max}) & \xrightarrow{\iota_*^G} & \text{Sh}^G_c(G_{can,min} \otimes X_{min,max}) \\
\sigma(2.74) & \downarrow & \pi_*^G \\
\text{Sh}^G_c(G_{min,min} \otimes X_{min,max}) & \rightarrow & \text{Sh}^G_c(X_{min})^\omega \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{V}_c^G(G_{min,min} \otimes X_{min,max}) & \xrightarrow{\iota_*^G} & \text{V}_c^G(G_{can,min} \otimes X_{min,max}) \\
\rightarrow & \pi_*^G \\
\text{V}_c^G(G_{min,min} \otimes X_{min,max}) & \rightarrow & \text{V}_c^G(G_{can,min} \otimes X_{min,max})
\end{array}
\]
which is natural in $X$ in $G\text{Set}$. In the next step we construct a factorisation of $\iota_*$ over the canonical morphism $i$ as indicated by the dotted arrow in the following diagram:

\[
\begin{aligned}
\text{Sh}^G_{\text{eqsm}}(G_{\text{min,min}} \otimes X_{\text{min,max}}) &\xrightarrow{\simeq} \text{V}^G_C(G_{\text{min,min}} \otimes X_{\text{min,max}}) \\
\downarrow &\quad \downarrow \\
\text{Sh}^G_{\text{eqsm,c}}(G_{\text{min,min}} \otimes X_{\text{min,max}}) &\xrightarrow{i_*} \text{V}^G_C(G_{\text{can,min}} \otimes X_{\text{min,max}})
\end{aligned}
\]

We first observe that the dashed arrows exist and the diagram commutes for obvious reasons. It now suffices to show that the dashed vertical arrow is an equivalence. In order to see this note that every invariant locally finite subset of $G_{\text{min,min}} \otimes X_{\text{min,max}}$ is isomorphic in $G\text{BornCoarse}$ to the image of $G_{\text{min,min}} \otimes F_{\text{min,max}}$ for some finite subset $F$ of $X$ (equipped with the trivial $G$-action) under the map $(g,f) \mapsto (g,gf)$.

By the first assertion of Lemma 2.72 we obtain the vertical equivalences in the commutative diagram

\[
\begin{aligned}
\colim_{F \subseteq X \text{ finite}} \text{Sh}^G_{\text{eqsm}}(G_{\text{min,min}} \otimes F_{\text{min,max}}) &\xrightarrow{\simeq} \text{Sh}^G_{\text{eqsm}}(G_{\text{min,min}} \otimes X_{\text{min,max}}) \\
\downarrow &\quad \downarrow \\
\colim_{F \subseteq X \text{ finite}} \text{Sh}^G_{\text{eqsm}}(\text{res}_1^G F_{\text{min,max}}) &\xrightarrow{\simeq} \text{Sh}^G_{\text{eqsm}}(\text{res}_1^G X_{\text{min,max}})
\end{aligned}
\]

By Lemma 2.71, the lower horizontal arrow is evidently an equivalence. Hence the upper horizontal morphism is the desired equivalence

\[
\text{Sh}^G_{\text{eqsm,c}}(G_{\text{min,min}} \otimes X_{\text{min,max}}) \xrightarrow{\simeq} \text{Sh}^G_{\text{eqsm}}(G_{\text{min,min}} \otimes X_{\text{min,max}}).
\]

The morphism $i$ in (5.16) is fully faithful by Lemma 4.22. We show that $p$ in (5.15) is fully faithful, too. Indeed, for $M, N$ in $\text{Sh}^G_{\text{eqsm}}(G_{\text{can,min}} \otimes X_{\text{min,max}})$ we have the following chain of equivalences:

\[
\begin{aligned}
\text{Map}_{\text{Sh}^G_{\text{can}}(X_{\text{min}})}(p(M), p(N)) &\xrightarrow{(2.5)} \text{Map}_{\text{Sh}^G_{\text{can}}(X_{\text{min}})}(\hat{\pi}^G M, \hat{\pi}^G N) \\
&\xrightarrow{(2.13)} \text{Map}_{\text{Sh}^G_{\text{can}}(G \times X)}(V^* G M, N) \\
&\xrightarrow{(2.81)} \text{Map}_{\text{Sh}^G_{\text{can}}(G \times X)}(\lim_{V \in G_{\text{can}} \otimes X_{\text{min}}} V^* G M, N) \\
&\xrightarrow{(\hat{\pi}^G)} \text{Map}_{\text{Sh}^G_{\text{can}}(G \times X)}(\hat{\pi}^G M, \hat{\pi}^G N) \\
&\xrightarrow{(\pi^G)} \text{Map}_{\text{Sh}^G_{\text{can}}(G \times X)}(\pi^G M, \pi^G N) \\
&\xrightarrow{\text{Map}_{\text{Sh}^G_{\text{can}}(X_{\text{min}})}(\ell M, \ell N)}
\end{aligned}
\]
The equivalence marked by ! holds because $N$ is cocompact in $\mathbf{PSh}_{G}^{G}(G \times X)$ by Proposition 2.73. For the equivalence marked by !! we observe that for every $B$ in $\mathcal{P}(G \times X)$ the family $(V[B])_{V \in G_{\overline{can}} \otimes X_{\min}}$ is a $U$-covering family of $G \times \pi(B) = \pi^{-1}(\pi(B))$ for every entourage $U$. Using that $M$ is a sheaf it follows that the natural transformation $V[-] \to \pi^{-1} \circ \pi(-)$ of endofunctors of $\mathcal{P}(G \times X)$ induces an equivalence

$$\hat{\pi}^{*,G} \hat{\pi}^{G} M \cong \lim_{V \in G_{\overline{can}} \otimes X_{\min}} V^{*,G} M.$$ 

Finally, the equivalence marked by !!! follows from the definition of $p$.

We now apply the functor $\text{Idem}$ to the bold and dotted part of (5.15), contract its middle line, and use (2.75) in order to see that the left column consists of idempotent complete categories. Furthermore, we apply $\text{Idem}$ to the right triangle of (5.16). The combination of the resulting two commutative diagrams yields the commutative diagram (5.17)

which is natural in $X$ in $G\text{Set}$. By Lemma 7.15 the functors $\text{Idem}(p)$ and $\text{Idem}(i)$ are again fully faithful inclusions of idempotent complete subcategories. By the second assertion of Corollary 2.60 the functor $\text{coind}_{G_{(1)}}$ generates $\text{Sh}_{C}^{G}(X_{\min})^{\omega}$ under finite limits and retracts. By the commutativity of (5.17) we can then conclude that the functors $\text{Idem}(p)$ and $\text{Idem}(p) \circ \text{Idem}(i)$ are essentially surjective. Hence we have the asserted natural equivalences

$$\text{Idem} V_{C}^{G}(G_{\overline{can}} \otimes X_{\min,\max}) \cong \text{Idem} V_{C}^{G}(G_{\overline{can}} \otimes X_{\min,\max})$$

Corollary 5.4. There are equivalences

$$\text{Idem} V_{C}^{G}(G_{\overline{can}} \otimes (-)_{\min,\max}) \cong \text{Idem} V_{C}^{G,G}(G_{\overline{can}} \otimes (-)_{\min,\max})$$

(5.18)

of functors $G\text{Orb} \to \text{Cat}_{\infty,\ast}^{\text{lex}}$.

Proof. Let $\iota: BG^{op} \to BG$ be the inversion functor. We use the notation introduced in the proof of Proposition 5.2. Since $\iota$ is an equivalence, we have the first equivalence
in the chain
\[(5.19) \quad \Sh^G((-)_\text{min})^\omega \simeq (\lim_{BG} \iota^* \Sh_{C,eq}(-))^\omega \simeq \colim_{BG} \Sh_{C,eq}(-)^\omega.\]

We furthermore have equivalences
\[(5.20) \quad \Sh_{C,eq}(-)^\omega \simeq \colim_{BG} \Sh_{C,eq}(-) \simeq \V_{C,G} \circ \iota(-).\]

Applying \(\colim_{BG}\), we get the second equivalence in
\[(5.21) \quad \Sh^G((-)_\text{min})^\omega \simeq \colim_{BG} \Sh_{C,eq}(-)^\omega \simeq \V_{C,G} \circ \iota(-).\]

Combining the equivalence (5.21) with (5.12), we get (5.18). \(\square\)

6. Equivariant coarse homology theories

6.1. Basic definitions. Consider a cocomplete stable \(\infty\)-category \(\mathcal{M}\), and let \(E: G\text{-BornCoarse} \to \mathcal{M}\) be a functor.

Recall Definition 3.4 of a flasque \(G\)-bornological coarse space.

**Definition 6.1.** We say that \(E\) vanishes on flasques if \(E\) sends flasque \(G\)-bornological coarse spaces to zero objects.

Since \(\mathcal{M}\) is stable, by Remark 3.8 this condition on \(E\) is actually equivalent to the condition that \(E\) is flasqueness preserving.

**Definition 6.2** ([BEKWa, Def. 3.10]). \(E\) is called an equivariant coarse homology theory if it is

1. coarsely invariant (Definition 3.1),
2. excisive (Definition 3.27) and
3. \(u\)-continuous (Definition 3.16), and
4. vanishes on flasques (Definition 6.1).

If \(E\) is a coarse homology theory, then it may additionally be

1. continuous (Definition 4.2),
2. strongly additive (Definition 3.37) (where we must assume that \(\mathcal{M}\) has set-indexed products) or
3. strong (Definition 6.5),
4. and it may admit transfers (Definition 4.23).

In the following, we recall the notion of strongness. In [BEKWa, Sec. 4.1] we have constructed a universal equivariant coarse homology theory

\[\Yo^*: G\text{-BornCoarse} \to G\text{Sp}\mathcal{X}.\]

It has the universal property that precomposition by \(\Yo^*\) induces an equivalence between the \(\infty\)-category of \(\mathcal{M}\)-valued coarse homology theories (considered as a subcategory of \(\text{Fun}(G\text{-BornCoarse}, \mathcal{M})\)), and the \(\infty\)-category \(\text{Fun}^{\text{colim}}(G\text{Sp}\mathcal{X}, \mathcal{M})\) of colimit preserving functors from \(G\text{Sp}\mathcal{X}\) to \(\mathcal{M}\) for any cocomplete stable \(\infty\)-category \(\mathcal{M}\).

Let \(X\) be in \(G\text{-BornCoarse}\).

**Definition 6.3** ([BEKWa, Def. 4.17]). We call \(X\) weakly flasque if it admits an endomorphism \(f: X \to X\) such that

1. \(\Yo^*(f) \simeq \text{id}_{\Yo^*(X)}\).
2. \(f\) implements pre-flasqueness of \(X\) (see Definition 3.5).
We say that $f$ implements weak flasqueness of $X$.

**Remark 6.4.** For a flasque space (Definition 3.4), we require $f$ to be close to the identity. Weak flasqueness replaces this by assumption (1). Since $Y^0$ is coarsely invariant, a flasque $G$-bornological coarse space is weakly flasque. ♦

Let $E: G\text{BornCoarse} \rightarrow M$ be an equivariant coarse homology theory.

**Definition 6.5 ([BEKWa, Def. 4.18]).** We call $E$ strong if it annihilates weakly flasque bornological coarse spaces. ♦

**Remark 6.6.** The condition of $E$ being strong is important if one wants to construct equivariant homology theories from equivariant coarse homology theories by precomposing with the cone functor. Strongness of $E$ implies homotopy invariance of the composition. We refer to [BEKWa, Sec. 11.3] for more details. ♦

### 6.2. Homological functors.

We consider a functor $H_g: \text{Cat}^{\text{Lex}}_{\infty, \ast} \rightarrow M$. We say that $H_g$ inverts Morita equivalences if it sends the morphism $C \rightarrow \text{Idem}(C)$ (the unit of adjunction (7.14)) to an equivalence for every $C$ in $\text{Cat}^{\text{Lex}}_{\infty, \ast}$.

**Definition 6.7.** The functor $H_g$ is called homological if it has the following properties:

1. $M$ is stable and cocomplete.
2. $H_g$ preserves filtered colimits.
3. $H_g$ sends excisive squares in $\text{Cat}^{\text{Lex}}_{\infty, \ast}$ (Definition 7.49) to pushout squares.

The functor $H_g$ will be called a finitary localising invariant if it in addition inverts Morita equivalences.

Since the functor $\text{Idem}: \text{Cat}^{\text{Lex}}_{\infty, \ast} \rightarrow \text{Cat}^{\text{lex, perf}}_{\infty, \ast}$ preserves filtered colimits (since it is a left adjoint), excisive squares in $\text{Cat}^{\text{Lex}}_{\infty, \ast}$ by Lemma 7.51, and inverts Morita equivalences (by definition), we get:

**Corollary 6.8.** If $H_g$ is a homological functor, then $H_g \circ \text{Idem}$ is a finitary localising invariant.

Following [BGT13, Def. 8.1], a functor $\text{Cat}^{\text{Lex}}_{\infty} \rightarrow M$ is called a stable finitary localising invariant if $M$ is stable and cocomplete, and the functor inverts Morita equivalences, preserves filtered colimits, and sends Verdier sequences to fibre sequences.

We have an adjunction (Lemma 7.41)

\[
\text{Sp}: \text{Cat}^{\text{Lex}}_{\infty, \ast} \rightleftarrows \text{Cat}^{\text{ex}}_{\infty, \ast} : \text{incl},
\]

where $\text{Sp}$ is the stabilisation functor. If $H_g$ is a finitary localising invariant, then $H_g \circ \text{incl}$ is clearly a stable finitary localising invariant. The following lemma justifies our terminology and shows that finitary localising invariants correspond to stable finitary localising invariants by precomposition with the stabilisation functor $\text{Sp}$.

**Lemma 6.9.**

---

\(^5\)We added the adjective *stable* in order to distinguish this notion from the one introduced in Definition 6.7. We further added the word *finitary* in order to highlight that the functor preserves filtered colimits, as one might want to drop this assumption in certain applications.
(1) If $H_g$ is a finitary localising invariant, then the natural transformation
$$H_g \to H_g \circ \text{incl} \circ \tilde{Sp}$$
(induced by the unit of adjunction (6.1)) is an equivalence.

(2) If $L$ is a stable finitary localising invariant, then $L \circ \tilde{Sp}$ is a finitary localising invariant and the transformation
$$L \circ \tilde{Sp} \circ \text{incl} \to L$$
(induced by the counit of adjunction (6.1)) is an equivalence.

Proof. Let $C$ be in $\text{Cat}_{\infty, s}^{\text{Lex}}$. Then we have the following excisive square in $\text{Cat}_{\infty, s}^{\text{Lex}}$:

\[
\begin{array}{ccc}
0 & \longrightarrow & C \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{Sp}(C)
\end{array}
\]

Indeed, the horizontal morphisms are fully faithful, and the induced morphism on stable cofibres (Lemma 7.48) is the identity of $\tilde{Sp}(C)$. The functor $H_g$ sends this square to a pushout square in $M$. Since $H_g(0) \simeq 0$, we conclude that $H_g(C) \to H_g(\tilde{Sp}(C))$ is an equivalence.

For the second assertion, we observe that $L \circ \tilde{Sp}$ is a finitary localising invariant since $\tilde{Sp}$ commutes with filtered colimits and idempotent completion (see [BKW, Cor. 2.32], preserves fully faithful functors by Lemma 7.43 and sends stable cofibres to Verdier quotients Lemma 7.47. Moreover, the counit $\tilde{Sp} \circ \text{incl} \to \text{id}$ is an equivalence. □

Example 6.10. We let

\[
U_{\text{loc}} : \text{Cat}_{\infty, s}^{\text{Lex}} \to \mathcal{M}_{\text{loc}}
\]

denote the universal (stable finitary) localising invariant of Blumberg–Gepner–Tabuada [BGT13, Thm. 8.7]. The target $\mathcal{M}_{\text{loc}}$ is a presentable stable $\infty$-category. The composition

\[
\text{UK} : \text{Cat}_{\infty, s}^{\text{Lex}} \xrightarrow{\tilde{Sp}} \text{Cat}_{\infty}^{\text{Lex}} \xrightarrow{U_{\text{loc}}} \mathcal{M}_{\text{loc}}
\]

is a finitary localising invariant by Lemma 6.9. ♦

Let $H_g : \text{Cat}_{\infty, s}^{\text{Lex}} \to M$ be a functor.

Lemma 6.11. If $H_g$ is homological, then it preserves coproducts.

Proof. For $C, D$ in $\text{Cat}_{\infty, s}^{\text{Lex}}$ we consider the pushout square

\[
\begin{array}{ccc}
0 & \longrightarrow & C \\
\downarrow & & \downarrow \\
D & \longrightarrow & C \sqcup D
\end{array}
\]

This square is excisive, since the horizontal morphisms are fully faithful and the induced morphism of stable cofibres is equivalent to the identity on $\tilde{Sp}(C)$. Since
Hg is homological, the square
\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Hg}(C) \\
\downarrow & & \downarrow \\
\text{Hg}(D) & \longrightarrow & \text{Hg}(C \sqcup D)
\end{array}
\]
is a pushout. 

\section*{Remark 6.12.} As a consequence of Lemma 6.11, a homological functor is additive. More precisely, let \( F, G : C \to D \) be two morphisms in \( \text{Cat}^{\text{Lex},*}_{\infty} \) between the same objects. Then we have an equivalence
\[
(6.4) \quad \text{Hg}(F + G) \simeq \text{Hg}(F) + \text{Hg}(G)
\]
of morphisms from \( \text{Hg}(C) \) to \( \text{Hg}(D) \). 

\section*{Lemma 6.13.} If \( \text{Hg} \) is homological, then it annihilates flasques.

\textbf{Proof.} Let \((C, S)\) be in \( \text{Fl}(\text{Cat}^{\text{Lex}}_{\infty,*}) \) (Definition 3.10 (2)). Since \( \text{Hg} \) is homological and therefore additive, the relation \( S \simeq \text{id}_C + S \) implies \( \text{Hg}(S) \simeq \text{id}_{\text{Hg}(C)} + \text{Hg}(S) \). Since \( \text{M} \) is stable and therefore additive, this in turn implies that \( \text{Hg}(C) \simeq 0 \) (Remark 3.8). 

We consider two functors
\[
\text{Hg} : \text{Cat}^{\text{Lex},*}_{\infty} \to \text{M} \quad , \quad \text{V} : G\text{BornCoarse} \to \text{Cat}^{\text{Lex}}_{\infty,*}.
\]
We use the notation \( \text{HgV} \) for the composition
\[
\text{Hg} \circ \text{V} : G\text{BornCoarse} \to \text{M}.
\]

\section*{Lemma 6.14.} Assume:

1. \( \text{V} \) is
   \begin{enumerate}
   \item coarsely invariant (Definition 3.1),
   \item \( l \)-excisive (Definition 3.29) and
   \item \( u \)-continuous (Definition 3.16), and
   \item preserves flasques (Definition 3.7).
   \end{enumerate}
2. \( \text{Hg} \) is homological (Definition 6.7).

Then the functor \( \text{HgV} : G\text{BornCoarse} \to \text{M} \) is an equivariant coarse homology theory (Definition 6.2).

\textbf{Proof.} First note that the target category \( \text{M} \) is stable and cocomplete since it is the target of a homological functor. We show that the functor \( \text{HgV} \) has the properties listed in Definition 6.2.

\( \text{HgV} \) is coarsely invariant since \( \text{V} \) is so.

\( \text{HgV} \) is excisive since \( \text{V} \) (being \( l \)-excisive) sends complementary pairs to excisive squares in \( \text{Cat}^{\text{Lex}}_{\infty,*} \), and \( \text{Hg} \) (being homological) sends these squares to pushout squares. At this point, we also employ that \( \text{Hg} \) (being homological) preserves filtered colimits in order to justify the equivalence \( \text{HgV}(\mathcal{Y}) \simeq \text{Hg}(\text{V}(\mathcal{Y})) \) for every big family \( \mathcal{Y} \) on an object of \( G\text{BornCoarse} \) (see (3.12) for notation).

\( \text{HgV} \) is \( u \)-continuous since \( \text{V} \) is \( u \)-continuous by assumption, and \( \text{Hg} \) preserves filtered colimits.

\( \text{HgV} \) vanishes on flasques since \( \text{V} \) preserves flasques by assumption, and \( \text{Hg} \) annihilates flasques by Lemma 6.13. 
\]
We now discuss the additional properties a coarse homology could have.

**Lemma 6.15.** We retain the assumptions of Lemma 6.14.

1. If \( V \) is continuous, then so is \( Hg \, V \) (Definition 4.2).
2. If \( V \) is hyperexcisive, then so is \( Hg \, V \) (Definition 4.16).
3. If \( V \) admits transfers, then so does \( Hg \, V \) (Definition 4.23).
4. Assume:
   1. \( V \) is strongly additive (Definition 3.37).
   2. \( M \) admits set-indexed products.
   3. \( Hg \) preserves set-indexed products.

Then \( Hg \, V \) is strongly additive.

**Proof.** (1) and (2) follow from the fact that \( Hg \) preserves filtered colimits. (3) and (4) are obvious. □

We finally discuss the condition of being strong (Definition 6.5). Recall Definition 3.11 of a functorially pre-flasqueness preserving functor. Since in the following the auxiliary \( \infty \)-category \( P \) satisfies \( P \simeq \ast \), we just say that \( V \) is pre-flasqueness preserving.

**Lemma 6.16.** We retain the assumptions of Lemma 6.14. If \( V \) is pre-flasqueness preserving, then \( Hg \, V \) is strong.

**Proof.** We assume that \( X \) is in \( G \text{BornCoarse} \) and that \( f : X \to X \) implements weak flasqueness (Definition 6.3). Since \( V \) is pre-flasqueness preserving we have an endofunctor \( S : V(X) \to V(X) \) such that

\[
\text{id}_{V(X)} + V(f) \circ S \simeq S.
\]

We apply \( Hg \) and use (6.4) in order to conclude that

\[
\text{id}_{Hg\, V(X)} + Hg\, V(f) \circ Hg(S) \simeq Hg(S).
\]

Since by assumption on \((X, f)\) we have an equivalence \( Yo^\ast (f) \simeq \text{id}_{Yo^\ast (X)} \), and since \( Hg \, V \) is a coarse homology theory, we have \( Hg\, V(f) \simeq \text{id}_{Hg\, V(X)} \). Hence (6.5) yields an equivalence

\[
\text{id}_{Hg\, V(X)} + Hg(S) \simeq Hg(S),
\]

which implies that \( Hg\, V(X) \simeq 0 \) since \( M \) is stable and hence additive (see Remark 3.8). □

Let \( Hg : \text{Cat}^{\text{Lex}}_{\infty,*} \to M \) be a functor, and let \( C \) be in \( \text{Fun}(BG, \text{Cat}^{\text{LEX}}_{\infty,*}) \). Recall the functor \( V^G_{\infty,*} \) from Definition 4.19 (1).

**Corollary 6.17.** If \( Hg \) is a homological functor, then \( Hg \, V^G_{\infty,*} \) is an equivariant coarse homology theory which in addition is

1. strong,
2. continuous and
3. strongly additive (provided \( M \) admits set-indexed products and \( Hg \) preserves set-indexed products), and
4. admits transfers.

**Proof.** The first assertion follows from Lemma 6.14 together with Corollary 4.20. For (1) we use Corollary 4.20 and Lemma 6.16. For (2) and (3) we use Corollary 4.20 and Lemma 6.15. For (4) we use Corollary 4.26 and Lemma 6.15. □

Recall the functor \( V^G_{\infty,*} \) from Definition 4.19 (2).
Corollary 6.18. If \( H_g \) is a homological functor, then \( H_g V^G_G, C \) is an equivariant coarse homology theory which in addition is

1. hyperexcisive and
2. strong.

**Proof.** This follows from Corollary 4.21 together with Lemmas 6.14 to 6.16. \( \square \)

6.3. **CP-functors.** Let \( E: \text{GBornCoarse} \to M \) be a functor. For every \( X \) in \( \text{GBornCoarse} \) we can define a new functor

\[
E_X: \text{GBornCoarse} \to M, \quad Y \mapsto E(X \otimes Y)
\]
called the twist of \( E \) by \( X \). If \( E \) is a coarse homology theory, then so is \( E_X \), see e.g. [BEKWa, Sec. 10.4]. Below, the twist \( E_{G, \text{can}, \text{min}} \) is of particular importance (see Example 2.39 for the definition of \( G_{\text{can}, \text{min}} \)).

Let \( \text{GOrb} \) be the orbit category of \( G \). By restricting the functor \( i: G\text{Set} \to \text{GOrb} \) from (2.45), we obtain the functor (compare with (5.3))

\[
i: \text{GOrb} \to \text{GBornCoarse}, \quad S \mapsto S_{\text{min}, \text{max}}.
\]

Let \( M: \text{GOrb} \to M \) be a functor.

**Definition 6.19** ([BEKWc, Def. 1.8]). We call \( M \) a CP-functor if it satisfies the following conditions:

1. \( M \) is stable, complete, cocomplete, and compactly generated.
2. There exists an equivariant coarse homology theory \( E: \text{GBornCoarse} \to M \) satisfying:
   (a) \( M \) is equivalent to \( E_{G, \text{can}, \text{min}} \circ i \) (see (6.6) and (6.7) for notation).
   (b) \( E \) is
      (i) strongly additive (Definition 3.37),
      (ii) continuous (Definition 4.2) and
      (iii) admits transfers (Definition 4.23).

**Example 6.20.** Assume that \( E: \text{GBornCoarse} \to M \) is an equivariant coarse homology theory with a stable, complete, cocomplete, and compactly generated target category. In addition, assume that \( E \) admits transfers, is strongly additive and continuous. Then the functor

\[
E_{G_{\text{can}, \text{min}}} \circ i: \text{GOrb} \to M
\]
is a CP-functor.

Let \( H_g \) be a homological functor, and let \( C \) be in \( \text{Fun}(BG, \text{Cat}^{\text{Lex}}_{\infty, \text{*,}}) \). In view of Example 6.20, Corollary 6.17 immediately implies:

**Corollary 6.21.** Assume that \( H_g \) has a complete and compactly generated target category and is product-preserving. Then

\[
(H_g V^{G, C}_{G})_{G, \text{can}, \text{min}} \circ i
\]
is a CP-functor.

Let \( C \) be in \( \text{Fun}(BG, \text{Cat}^{\text{Lex}}_{\infty, \text{*,}}) \). We define \( \text{Ind}^{G}(C): \text{GOrb} \to \text{Cat}^{\text{Lex}}_{\infty, \text{*,}} \) using the induction functor \( \text{Ind}^{G} \) from (5.1). We consider a functor \( H_g: \text{Cat}^{\text{Lex}}_{\infty, \text{*,}} \to M \).

**Definition 6.22.** We define the composition

\[
H_g C_G := H_g \circ \text{Ind}^{G}(C): \text{GOrb} \to M.
\]
Theorem 6.23. Assume:
(1) $M$ is stable, complete and cocomplete, and compactly generated.
(2) $Hg$ is a finitary localising invariant and preserves products.
(3) $C$ belongs to $\text{Fun}(BG, \text{Cat}_{\text{Lex}, \text{perf}}^{\infty})$.

Then $Hg \cdot C_G$ is a CP-functor.

Proof. We set $\hat{C} := \text{Pro}_{\omega}(C)$ in $\text{Fun}(BG, \text{Cat}_{\text{LEX}}^{\infty})$ (see (7.12)). Then (3) implies that $C \simeq \hat{C}^\omega$. Combining Proposition 5.2 and Corollary 5.4, we have an equivalence

$$Hg \cdot C_G \simeq Hg \circ V_C \circ i \simeq (Hg \circ \text{Idem}(V_C^{G,c}))_{G, \text{can}, \text{min}} \circ i$$

(note that the operations of twisting by $G_{\text{can}, \text{min}}$ and postcomposing with $Hg$ obviously commute). Since $Hg$ is localising by (2), we have an equivalence

$$(Hg \circ \text{Idem}(V_C^{G,c}))_{G, \text{can}, \text{min}} \circ i \simeq (Hg \circ V_C^{G,c})_{G, \text{can}, \text{min}} \circ i.$$

This last functor is a CP-functor by Corollary 6.21. \hfill $\square$

We now consider the notion of hereditary CP-functors. If $\phi: G \to Q$ is a surjective homomorphism of groups, then we get a functor

$$\text{Res}_\phi: Q\text{ Orb} \to G\text{ Orb}$$

which sends the $Q$-set $S$ to $S$ considered as a $G$-set via $\phi$. Since $\phi$ is surjective, $S$ is still transitive as a $G$-set.

Let $M: G\text{ Orb} \to M$ be a functor.

Definition 6.24 ([BKW, Def. 2.5]). We call $M$ a hereditary CP-functor if $M \circ \text{Res}_\phi$ is a CP-functor for every surjective group homomorphism $\phi$.

Recall Definition 6.22 of the functor $Hg \cdot C_G$.

Theorem 6.25. We retain the assumptions of Theorem 6.23. Then $Hg \cdot C_G$ is a hereditary CP-functor.

Proof. Let $\phi: G \to Q$ be a surjective homomorphism. Let $j^G: BG \to G\text{ Orb}$ and $j^Q: BQ \to Q\text{ Orb}$ denote the fully faithful inclusions (see (1.2)). The homomorphism $\phi$ induces a map $G \to \text{Res}_\phi(Q)$ in $G\text{ Orb}$ which defines a natural transformation $j^G \to \text{Res}_\phi \circ j^Q \circ B\phi$ of functors $BG \to G\text{ Orb}$. The resulting natural transformation

$$C \xrightarrow{\simeq} j^G, j^G_C \to B\phi^* j^Q, j^G_C \to \text{Res}_\phi^* j^G_C$$

induces an adjoint transformation

$$j^Q_B \phi^*_C \to \text{Res}_\phi^* j^G_C.$$

We claim that this transformation is an equivalence. Using the pointwise formulas for left Kan extensions, we must show for every $S$ in $Q\text{ Orb}$ that the map

$$\text{colim}_{(j^Q_B \phi^*_C)S} C \to \text{colim}_{j^Q_{\text{Res}_\phi(S)}} C$$

induced by the functor

$$(j^Q \circ B\phi)_S \to j^G_{\text{Res}_\phi(S)}, \quad (Q \to S) \mapsto (G \xrightarrow{\phi} \text{Res}_\phi(Q) \to \text{Res}_\phi(S))$$

is an equivalence. This is clear since the functor on indexing categories is an equivalence. We conclude that

$$Hg \cdot C_G \circ \text{Res}_\phi \simeq Hg(B\phi^*_C)_Q.$$

The latter is a CP-functor by Theorem 6.23.

6.4. Algebraic K-theory. The prototypical example of a finitary localising invariant is algebraic K-theory. Recall the universal localising invariant $U_{\text{loc}}: \text{Cat}_{\infty}^\text{ex} \to \mathcal{M}_{\text{loc}}$ from (6.2).

**Definition 6.26.** The (non-connective) algebraic K-theory functor for stable $\infty$-categories is defined by

$$K_{\text{st}} := \text{map}_{\mathcal{M}_{\text{loc}}}(U_{\text{loc}}(\text{Sp}^{\omega}), U_{\text{loc}}(-)): \text{Cat}_{\infty}^\text{ex} \to \text{Sp}.$$  

The (non-connective) algebraic K-theory functor for left-exact $\infty$-categories is defined by

$$K := K_{\text{st}} \circ \tilde{\text{Sp}}: \text{Cat}_{\infty, \ast}^\text{Lex} \to \text{Sp}.$$  

**Corollary 6.27.**

1. The algebraic K-theory functor for stable $\infty$-categories is a stable finitary localising invariant.
2. The algebraic K-theory functor for left-exact $\infty$-categories is a finitary localising invariant (Definition 6.7).

**Proof.** We note that $U_{\text{loc}}(\text{Sp}^{\omega})$ is compact in $\mathcal{M}_{\text{loc}}$. This implies that $K_{\text{st}}$ preserves filtered colimits since $U_{\text{loc}}$ does so. Since $\mathcal{M}_{\text{loc}}$ is stable, any corepresentable functor preserves finite colimits. This implies the statement (1). The statement (2) then follows from (1) and Lemma 6.9 (2). □

**Remark 6.28.** By Lemma 6.9 (2), we have $K_{\text{st}}(C) \simeq K(C)$ for every stable $\infty$-category $C$. If $K^{\text{BGT}}$ denotes the non-connective algebraic K-theory functor from [BGT13, Def. 9.6], [BGT13, Thm. 9.8] yields an identification $K(C) \simeq K^{\text{BGT}}(C^{\text{op}})$. ♦

**Proposition 6.29.** The algebraic K-theory functor for left-exact $\infty$-categories (Definition 6.26) preserves set-indexed products.

**Proof.** This result will be deduced from the corresponding statement about $K_{\text{st}}$. Let $(C_i)_{i \in I}$ be a family in $\text{Cat}_{\infty, \ast}^\text{Lex}$. By [BKW, Lem. 2.39] the canonical map

$$U_{\text{loc}}(\tilde{\text{Sp}}(\prod_{i \in I} C_i)) \to U_{\text{loc}}(\prod_{i \in I} \tilde{\text{Sp}}(C_i))$$

is an equivalence. Note that it is important here to apply $U_{\text{loc}}$ since the map between the arguments of $U_{\text{loc}}$ is not an equivalence.

Then the canonical comparison map factors as

$$K(\prod_{i \in I} C_i) \simeq K^\text{st}(\text{Sp}(\prod_{i \in I} C_i)) \simeq K^\text{st}(\prod_{i \in I} \text{Sp}(C_i)) \simeq K(\prod_{i \in I} \text{Sp}(C_i)) \overset{!}{\to} \prod_{i \in I} K(\text{Sp}(C_i)) \simeq \prod_{i \in I} K(C_i),$$

where the morphism marked by $!$ is an equivalence by [KW19, Theorem 1.3]. □

Let $C$ be in $\text{Fun}(BG, \text{Cat}_{\infty, \ast}^\text{Lex, perf})$. Applying Definition 6.22 for $Hg = K$, we get the functor $K_C$.

---

6 The result in the reference is stated for right-exact $\infty$-categories. We obtain the corresponding result for left-exact $\infty$-categories by considering opposites.
Corollary 6.30. The functor \( \mathcal{K}_G : \mathsf{GOrb} \to \mathsf{Sp} \) is a hereditary CP-functor.

Proof. We apply Theorem 6.25 with \( H_g = K \). The target \( \mathsf{Sp} \) of \( K \) has the required properties. Furthermore, \( K \) is homological by Corollary 6.27 and preserves products by Proposition 6.29. □

Let \( C \) be in \( \mathbf{Fun}(BG, \mathbf{Cat}_{\mathsf{Lex}, \ast}^{\mathsf{perf}}) \).

Definition 6.31. We define the coarse algebraic \( K \)-homology functor with coefficients in \( C \) by
\[
\mathcal{K}^G_{C} := \mathcal{K}^G_{C, :} : \mathsf{GBornCoarse} \to \mathsf{Sp}.
\]

Corollary 6.17 implies:

Corollary 6.32. \( \mathcal{K}^G_{C} \) is an equivariant coarse homology theory which is continuous, strong and strongly additive, and admits transfers.

Definition 6.33. We define the functor \( \mathcal{K}^G_{C,G} := \mathcal{K}^G_{C,G, :} : \mathsf{GBornCoarse} \to \mathsf{Sp} \).

Corollary 6.18 implies:

Corollary 6.34. \( \mathcal{K}^G_{C,G} \) is a strong and hyperexcisive equivariant coarse homology theory.

6.5. Split injectivity results. One of the main goals of the present paper is to produce new examples to which the results about split injectivity of assembly maps from \( \mathbf{BEKWc} \) can be applied. For the sake of completeness, we list these results, all shown in \( \mathbf{BEKWc} \).

In view of Corollary 6.30, all these results apply to the functor \( \mathcal{K}_G \) in place of \( M : \mathsf{GOrb} \to \mathsf{M} \) for any \( C \) in \( \mathbf{Fun}(BG, \mathbf{Cat}_{\mathsf{Lex}, \ast}^{\mathsf{perf}}) \).

Apart from the assumption on the functor \( M \) being a CP-functor, the theorems in \( \mathbf{BEKWc} \) contain the geometric assumption of finite decomposition complexity. In the following, we recall the relevant definitions from \( \mathbf{BEKW19}, \text{Sec. 3.1} \).

Let \( U \) be an entourage on a set \( X \) and consider two subsets \( Y, Z \). Recall the definition of the thickening from (2.7).

Definition 6.35. \( Y \) and \( Z \) are \( U \)-disjoint if \( U[Y] \cap Z = \emptyset \) and \( Y \cap U[Z] = \emptyset \). ★

Let \( X \) be a \( G \)-set.

Definition 6.36. An equivariant family of subsets of \( X \) is a family of subsets \( (Y_i)_{i \in I} \) indexed by a \( G \)-set \( I \) such that \( gY_i = Y_{gi} \) for every \( g \in G \) and \( i \) in \( I \). ★

Let now \( X \) be in \( \mathsf{GCoarse} \), and let \( (Y_i)_{i \in I} \) be an equivariant family of subsets.

Definition 6.37. We define the \( G \)-coarse space \( \bigsqcup_{i \in I} Y_i \) as follows:

1. The underlying \( G \)-set of \( \bigsqcup_{i \in I} Y_i \) is \( \bigsqcup_{i \in I} Y_i \).
2. The coarse structure on \( \bigsqcup_{i \in I} Y_i \) is the maximal one such that the family of its subsets \( (Y_i)_{i \in I} \) is coarsely disjoint and the canonical map \( \bigsqcup_{i \in I} Y_i \to X \) is controlled. ★

By definition, we have a canonical morphism \( \bigsqcup_{i \in I} Y_i \to X \) in \( \mathsf{GCoarse} \). If \( (Y_i)_{i \in I} \) and \( (Y'_i)_{i \in I} \) are two equivariant families with \( Y_i \subseteq Y'_i \), then we have a morphism \( \bigsqcup_{i \in I} Y_i \to \bigsqcup_{i \in I} Y'_i \) in \( \mathsf{GCoarse} \).

Recall the notion of a coarse equivalence (Remark 3.2).
Definition 6.38. We call the family \((Y_i)_{i \in I}\) nice if for every \(U\) in \(C^G_X\) the canonical morphism
\[
\bigsqcup_{i \in I} Y_i \to \bigsqcup_{i \in I} U[Y_i]
\]
is a coarse equivalence. ♦

In the following, we will consider classes \(\mathcal{C}\) of \(G\)-coarse spaces.

Example 6.39. Let \(X\) be in \(G\Coarse\). A subset \(Y\) in \(P_X\) is bounded if \(Y \times Y \in C^G_X\). A \(G\)-coarse space \(X\) is semi-bounded if every coarse component (Definition 2.19) of \(X\) is bounded. We consider the class \(SB\) of all semi-bounded \(G\)-coarse spaces. ♦

Definition 6.40. We say that \(X\) is decomposable over \(\mathcal{C}\) if for every \(U\) in \(C^G_X\) there exist nice equivariant families \((Y_i)_{i \in I}\) and \((Z_j)_{j \in J}\) of subsets of \(X\) such that the following conditions are satisfied:
1. \((Y_i)_{i \in I}\) and \((Z_j)_{j \in J}\) are pairwise \(U\)-disjoint.
2. \(X = \bigsqcup_{i \in I} Y_i \cup \bigsqcup_{j \in J} Z_j\).
3. \(\bigsqcup_{i \in I} Y_i\) and \(\bigsqcup_{j \in J} Z_j\) belong to the class \(\mathcal{C}\). ♦

Let \(\mathcal{C}\) be a class of \(G\)-coarse spaces.

Definition 6.41. The class \(\mathcal{C}\) is closed under decomposition if every \(G\)-coarse space which is decomposable over \(\mathcal{C}\) belongs to \(\mathcal{C}\). ♦

Definition 6.42. We let \(GFDC\) be the smallest class of \(G\)-coarse spaces which is closed under decomposition and contains the class \(SB\) of all semi-bounded \(G\)-coarse spaces. ♦

Let \(X\) be a \(G\)-coarse space.

Definition 6.43. \(X\) has \(G\)-finite decomposition complexity (\(G\)-FDC) if it belongs to the class \(GFDC\). ♦

Let \(\mathcal{F}\) be a set of subgroups of \(G\).

Definition 6.44. \(\mathcal{F}\) is called a family of subgroups if it is closed under
1. taking subgroups and
2. conjugation in \(G\). ♦

Let \(\mathcal{F}\) be a family of subgroups.

Definition 6.45.
1. We let \(G_{\mathcal{F}}\text{Set}\) denote the full subcategory of \(G\text{Set}\) on \(G\)-sets with stabilisers in \(\mathcal{F}\).
2. By \(G_{\mathcal{F}}\text{Orb}\) we denote the full subcategory of \(G_{\mathcal{F}}\text{Set}\) of transitive \(G\)-sets with stabilisers in \(\mathcal{F}\). ♦

Let \(X\) be in \(G\Coarse\).

Definition 6.46. \(X\) has \(G_{\mathcal{F}}\text{-FDC}\) if the \(G\)-coarse space \(S_{min} \otimes X\) has \(G\text{-FDC}\) for all \(S\) in \(G_{\mathcal{F}}\text{Set}\). ♦

Example 6.47. Examples of families of subgroups are:
(1) \{1\} - the family consisting of the trivial subgroup.
(2) \text{Fin} - the family of all finite subgroups.
(3) \text{Vcyc} - the family of virtually cyclic subgroups.
(4) \text{FDC} - the family of subgroups \( V \) of \( G \) such that \( V_{\text{can}} \) has \( V_{\text{Fin}} \)-\text{FDC}.
(5) \text{CP} denotes the family of subgroups of \( G \) generated by those subgroups \( V \) such that \( E_{\text{Fin}}V \) (see Definition 6.48 below) is a compact object of \( \text{PSh}(\text{VOrb}) \).
(6) \text{FDC}^{\text{cp}} denotes the intersection of \text{FDC} and \text{CP}.

In the following, for an \( \infty \)-category \( \mathcal{C} \) we use the standard notation \( \text{PSh}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc}) \). We have an adjunction

\[ \text{Ind}_{F} : \text{PSh}(G_{F}\text{Orb}) \rightleftarrows \text{PSh}(G_{\text{Orb}}) : \text{Res}_{F}, \]

where \( \text{Res}_{F} \) is the restriction along the inclusion \( G_{F}\text{Orb} \rightarrow G_{\text{Orb}} \). We let \( *_{F} \) denote the final object of \( \text{PSh}(G_{F}\text{Orb}) \).

Definition 6.48. We define the classifying space of the family \( \mathcal{F} \) by

\[ E_{F}G := \text{Ind}_{F}(*)_{F}. \]

By definition, \( E_{F}G \) is an object of \( \text{PSh}(G_{\text{Orb}}) \). The object \( E_{\text{Fin}}G \) is also called the classifying space for proper actions.

By Elmendorf’s theorem, \( \text{PSh}(G_{\text{Orb}}) \) is equivalent to the \( \infty \)-category of \( G \)-topological spaces (see [BEKWc, Rem. 1.12] for further explanations).

Definition 6.49. \( E_{F}G \) admits a finite-dimensional model if there exists a finite-dimensional \( G \)-CW-complex modelling the homotopy type of \( E_{F}G \) in \( G \)-topological spaces.

Let \( \mathcal{M} \) be a cocomplete \( \infty \)-category, and let \( M : G_{\text{Orb}} \rightarrow \mathcal{M} \) be a functor. Let \( \mathcal{F} \) and \( \mathcal{F}' \) be families of subgroups such that \( \mathcal{F}' \subseteq \mathcal{F} \).

Definition 6.50. The relative assembly map \( \text{Ass}_{\mathcal{F},M}^{\mathcal{F}'} \) is the morphism

\[ \text{Ass}_{\mathcal{F},M}^{\mathcal{F}} : \colim_{G_{F}\text{Orb}} M \rightarrow \colim_{G_{F}'\text{Orb}} M \]

in \( \mathcal{M} \) canonically induced by the inclusion \( G_{F}\text{Orb} \rightarrow G_{F}'\text{Orb} \).

Theorem 6.51 ([BEKWc, Thm. 1.11]). Assume:

(1) \( M \) is a CP-functor.
(2) One of the following conditions holds:
   (a) \( \mathcal{F} \) is a subfamily of \( \text{FDC}^{\text{cp}} \) such that \( \text{Fin} \subseteq \mathcal{F} \).
   (b) \( \mathcal{F} \) is a subfamily of \( \text{FDC} \) such that \( \text{Fin} \subseteq \mathcal{F} \) and \( G \) admits a finite-dimensional model for \( E_{\text{Fin}}G \).

Then the relative assembly map \( \text{Ass}_{\text{Fin},M}^{\mathcal{F}} \) admits a left inverse.

Theorem 6.52 ([BEKWc, Cor. 1.13]). If \( M \) is a CP-functor, then the relative assembly map \( \text{Ass}_{\text{Vcyc},M}^{\text{Fin}} \) admits a left inverse.

Theorem 6.53 ([BEKWc, Cor. 1.14]). Assume:

(1) \( M \) is a CP-functor;
(2) \( G \) admits a finite-dimensional model for \( E_{\text{Fin}}G \);
(3) \( G_{\text{can}} \) has \( G_{\text{Fin}} \)-\text{FDC}.

Then the assembly map \( \text{Ass}_{\text{Fin},M}^{\text{All}} \) admits a left inverse.
Note that the finite decomposition complexity assumption is in general not easy to check. The following two theorems are consequences of the theorems on groups with finite decomposition complexity.

**Theorem 6.54** ([BEKWc, Thm. 2.9]). Assume:

(1) $M$ is a hereditary $CP$-functor.

(2) $G$ fits into an exact sequence $1 \to S \to G \xrightarrow{\phi} Q \to 1$ such that
   (a) $S$ is virtually solvable and has finite Hirsch length.
   (b) $Q$ admits a finite-dimensional model for $E_{\text{Fin}}Q$.

Then $\text{Ass}_{\text{Fin},M}^{\phi^{-1}(\text{Fin}(Q))}$ admits a left inverse.

Here $\text{Fin}(Q)$ denotes the family of finite subgroups of $Q$.

**Theorem 6.55** ([BEKWc, Cor. 2.11]). Assume:

(1) $M$ is a hereditary $CP$-functor.

(2) $G$ admits a finite-dimensional $CW$-model for the classifying space $E_{\text{Fin}}G$.

(3) $G$ is a finitely generated subgroup of a linear group over a commutative ring with unit or of a virtually connected Lie group.

Then $\text{Ass}_{\text{Fin},M}^{\text{All}}$ admits a left inverse.

### 7. $\infty$-CATEGORY BACKGROUND

#### 7.1. Left-exact $\infty$-categories

We let $\text{Cat}_\infty$ denote the large $\infty$-category of small $\infty$-categories. It is an object of the very large $\infty$-category $\text{CAT}_\infty$ of large $\infty$-categories. We have a chain of inclusions of subcategories

\begin{equation}
\text{Cat}_{\text{Lex},\text{perf}}^{\infty,\text{*,s}} \subseteq \text{Cat}_{\text{Lex}}^{\infty,\text{*,s}} \subseteq \text{Cat}_{\text{Lex}}^{\infty,\text{*,s}} \subseteq \text{Cat}_\infty.
\end{equation}

For ease of reference, they are described in the following collection of examples.

**Example 7.1.** $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ is the subcategory of small $\infty$-categories admitting finite limits and finite limit preserving functors. A typical object of $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ is the opposite of the $\infty$-category $\text{Spc}_\infty$ of compact spaces.

**Example 7.2.** $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ is the full subcategory of $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ of pointed objects called the $\infty$-category of left-exact $\infty$-categories. A typical object of $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ is the opposite $\text{Spc}_\infty^{\text{CP,op}}$ of the $\infty$-category $\text{Spc}_\infty^{\text{CP}}$ of compact pointed spaces.

**Example 7.3.** The $\infty$-category $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ of small stable $\infty$-categories is a full subcategory of $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$. Both the $\infty$-category $\text{Sp}_{\text{CP}}$ of compact spectra and its opposite $\text{Sp}_{\text{CP,op}}$ are stable. See Example 1.4 for more examples.

**Example 7.4.** $\text{Cat}_{\text{Lex},\text{per}f}^{\infty,\text{*,s}}$ is the full subcategory of $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ of idempotent complete objects. The $\infty$-categories $\text{Spc}_{\text{CP,op}}$ and $\text{Sp}_{\text{CP,op}}$ are idempotent complete.

We will also consider the chain

\begin{equation}
\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}} \subseteq \text{CAT}_{\text{Lex}}^{\infty,\text{*,s}} \subseteq \text{CAT}_{\text{Lex}}^{\infty,\text{*,s}} \subseteq \text{CAT}_\infty
\end{equation}

of subcategories described in the following list of examples.

**Example 7.5.** $\text{CAT}_{\text{Lex}}^{\infty,\text{*,s}}$ is the subcategory of large pointed $\infty$-categories admitting finite limits and finite limit preserving functors. It is the large analogue of $\text{Cat}_{\text{Lex}}^{\infty,\text{*,s}}$ in Example 7.2.
Example 7.6. \( \mathbf{CAT}_{\infty,*,}^{\text{LEX}} \) is the subcategory of \( \mathbf{CAT}_{\infty,*,}^{\text{Lex}} \) of \( \infty \)-categories which admit all small limits and limit preserving functors.

If \( C \) is in \( \mathbf{CAT}_{\infty,*,}^{\text{LEX}} \), then we can consider the notion of a cocompact object \( C \) in \( C \):

Definition 7.7. \( C \) is cocompact if the natural morphism
\[
(7.2) \quad \text{colim}_{I} \text{Map}_{C}(T,C) \xrightarrow{\sim} \text{Map}_{C}(\text{lim}_{I} T, C)
\]
is an equivalence for every cofiltered diagram \( T: I \to C \).

The \( \infty \)-category \( C_{\omega} \) of cocompact objects of \( C \) is an object of \( \mathbf{CAT}_{\infty,*,}^{\text{LEX}} \).

Example 7.8. \( \mathbf{Cat}_{\infty,*,}^{\text{LEX}} \) is the subcategory of \( \mathbf{CAT}_{\infty,*,}^{\text{LEX}} \) whose opposites are pointed \( \omega \)-presentable. The morphisms in \( \mathbf{Cat}_{\infty,*,}^{\text{LEX}} \) are right adjoint functors which preserve cocompact objects (equivalently, their left adjoints preserve cofiltered limits [Lur09, Prop. 5.5.7.2]). In other words, by definition we have an equivalence
\[
(7.3) \quad (-)^{\text{op}}: \mathbf{Pr}_{\omega,*,}^{\text{L}} \xrightarrow{\sim} \mathbf{Cat}_{\infty,*,}^{\text{LEX}}
\]
There are two ways to consider objects of \( \mathbf{Pr}_{\omega,*,}^{\text{L}} \) as \( \infty \)-categories. The first is just by forgetting that they are presentable. In this case we would use the notation \( \mathbf{Pr}_{\omega,*,}^{\text{L}} \). The other way is by considering the opposite \( \infty \)-category. In order to make this distinction clear we will use the notation \( \mathbf{Cat}_{\infty,*,}^{\text{LEX}} \) in this case. If \( C \) is in \( \mathbf{Cat}_{\infty,*,}^{\text{LEX}} \), then its full subcategory \( C_{\omega} \) of cocompact objects belongs to \( \mathbf{Cat}_{\infty,*,}^{\text{LEX}} \). Note that \( C_{\omega} \simeq ((C^{\text{op}})^{\text{op}})^{\text{op}} \).

Proposition 7.9. The \( \infty \)-category \( \mathbf{Cat}_{\infty,*,}^{\text{LEX}} \) has small limits as well as small filtered colimits. Furthermore, the inclusion \( \mathbf{Cat}_{\infty,*,}^{\text{LEX}} \to \mathbf{Cat}_{\infty} \) preserves small limits and small filtered colimits.

Proof. By [Lur09, Prop. 5.5.7.6 and 5.5.7.10], \( \mathbf{Cat}_{\infty}^{\text{LEX}} \) has small limits, and the functor \( \mathbf{Cat}_{\infty}^{\text{LEX}} \to \mathbf{Cat}_{\infty} \) preserves small limits. Similarly, by [Lur09, Prop. 5.5.7.11], \( \mathbf{Cat}_{\infty}^{\text{LEX}} \) has small filtered colimits and \( \mathbf{Cat}_{\infty}^{\text{LEX}} \to \mathbf{Cat}_{\infty} \) preserves them.

By [Lur09, Prop. 5.2.7.8] the functor \( C \mapsto C_{*/} \) (where \(* \) is a final object of \( C \)) is the left adjoint of a localisation
\[
(7.4) \quad (-)_{*/}: \mathbf{Cat}_{\infty}^{\text{Lex}} \xrightarrow{\sim} \mathbf{Cat}_{\infty,*,}^{\text{Lex}} : \text{incl}
\]
in the sense of [Lur09, Def. 5.2.7.2 and 5.2.7.6].

Since the formation of the coslice functor \( C \mapsto C_{*/} \) preserves small filtered colimits, we see that \( \mathbf{Cat}_{\infty,*,}^{\text{Lex}} \) has small filtered colimits and that the full inclusion functor from \( \mathbf{Cat}_{\infty,*,}^{\text{Lex}} \to \mathbf{Cat}_{\infty}^{\text{Lex}} \) preserves filtered colimits.

The inclusion functor also commutes with small limits since it is a right adjoint. Thus in order to prove the existence of limits in \( \mathbf{Cat}_{\infty,*,}^{\text{Lex}} \), by [Lur09, Prop. 4.4.2.6], it is sufficient to prove the existence of pullbacks and of products. This amounts to checking that the property of being pointed is preserved by taking pullbacks or small products. The case of products is trivial, and the case of pullbacks is a rather degenerate particular case of [Lur09, Lem. 5.4.5.5] (to prove the existence of pullbacks, one may also argue that the functor \( C \mapsto C_{*/} \) commutes with pullbacks, or, for a third argument, provide a pedestrian proof, using directly [Lur09, Lem. 5.4.5.5]). □
Lemma 7.10. The ∞-category $\text{Cat}_{\infty,*,\text{LEX}}$ has small limits. Furthermore, the inclusion $\text{Cat}_{\infty,*,\text{LEX}} \to \text{CAT}_{\infty}$ preserves small limits.

Proof. The ∞-category $\text{Pr}_{\omega,*}$ has small limits by a pointed version (see the proof of Proposition 7.9 for the derivation of the pointed from the unpointed version) of [Lur09, Prop. 5.5.7.6]. Hence $\text{Cat}_{\infty,*,\text{LEX}}$ also has small limits. Since $\text{Pr}_{\omega,*}$, the inclusion $\text{Pr}_{\omega,*} \to \text{Cat}_{\infty}$ preserves small limits, the inclusion $\text{Cat}_{\infty,*,\text{LEX}} \to \text{CAT}_{\infty}$ preserves limits. □

In the following we discuss limits in $\text{Cat}_{\infty}$ and $\text{Cat}_{\infty,*,\text{LEX}}$. There are analogous results for the large cases $\text{CAT}_{\infty}$ and $\text{CAT}_{\infty,*,\text{LEX}}$.

Let $C$ be any of the above categories, and let $I$ be a small category. Then we consider a diagram $C$ in $\text{Fun}(I,C)$ and assume that

$$\eta: \lim_I C \to C$$

presents an object $\lim_1 C$ as the limit of the diagram in $C$.

For every object $i$ in $I$ we have an evaluation functor

$$ev_i: \text{Fun}(I,C) \to C.$$

Applied to $C$ we get the underlying object $ev_i(C) \simeq C(i)$ in $C$. If we apply $ev_i$ to $\eta$ from (7.5) and use the canonical equivalence $ev_i((-)) \simeq \text{id}$, then we get the evaluation morphism

$$e_i: \lim_I C \to C(i)$$

in $C$.

If $\phi: C \to D$ is a morphism in $\text{Fun}(I,\text{Cat}_\infty)$, then the following diagram commutes for every $i$ in $I$:

$$\begin{array}{ccc}
\lim_I C & \xrightarrow{e_i} & C(i) \\
\downarrow \phi & & \downarrow \phi(i) \\
\lim_I D & \xrightarrow{e_i} & D(i)
\end{array}$$

Lemma 7.11. The collection of functors $(e_i)_{i \in I}$ detects equivalences.

Proof. If $D$ is in $\text{Cat}_\infty$ with objects $D, D'$, then we can present the mapping space by the pullback in $\text{Cat}_\infty$

$$\begin{array}{ccc}
\text{Map}_D(D, D') & \longrightarrow & \text{Fun}(\Delta^1, D) \\
\downarrow & & \downarrow \\
\Delta^0 \times \Delta^0 & \xrightarrow{(D, D')} & D \times D
\end{array}$$

where the right vertical map is given by the evaluations at the two boundaries of $\Delta^1$. Thus for a pair of objects $X, Y$ of $\lim_1 C$ we have a pullback in $\text{Cat}_\infty$

$$\begin{array}{ccc}
\text{Map}_{\lim_1 C}(X, Y) & \longrightarrow & \text{Fun}(\Delta^1, \lim_1 C) \\
\downarrow & & \downarrow \\
\Delta^0 \times \Delta^0 & \xrightarrow{(X, Y)} & \lim_1 C \times \lim_1 C
\end{array}$$
We now define $M(X,Y)$ in $\text{Fun}(I, \text{Cat}_\infty)$ by the pullback square

$$
\begin{array}{ccc}
M(X,Y) & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{C}) \\
\downarrow & & \downarrow \\
\Delta^0 \times \Delta^0 & \longrightarrow & \mathcal{C} \times \mathcal{C}
\end{array}
$$

where the lower horizontal map corresponds to $(\_ \lim \_)$.

We apply $\lim_I$ to this diagram. Since the functor $\lim_I$ preserves pullbacks, $\lim_I \Delta^0 \simeq \Delta^0$, and $\lim_I \text{Fun}(\Delta^1, \mathcal{C}) \simeq \text{Fun}(\Delta^1, \lim_I \mathcal{C})$ we get the pullback diagram (7.10). In other words, we have an equivalence

$$
\text{Map}_{\lim_I} \mathcal{C}(X,Y) \simeq \lim_I M(X,Y).
$$

On the other hand, for $i \in I$ the functor $ev_i : \text{Fun}(I, \text{Cat}_\infty) \rightarrow \text{Cat}_\infty$ preserves pullback diagrams. In view of the definition of the functor $e_i$ and using the equivalences $ev_i(\Delta^0) \simeq \Delta^0$ and $\text{Fun}(\Delta^1, \mathcal{C}(i)) \simeq ev_i(\text{Fun}(\Delta^1, \mathcal{C}))$ we get the pullback diagram

$$
\begin{array}{ccc}
ev_i(M(X,Y)) & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{C}(i)) \\
\downarrow & & \downarrow \\
\Delta^0 \times \Delta^0 & \longrightarrow & \mathcal{C}(i)
\end{array}
$$

i.e., the equivalence

$$
ev_i(M(X,Y)) \simeq \text{Map}_{\mathcal{C}(i)}(e_i(X), e_i(Y)).
$$

Let now $f : Y \rightarrow Y'$ be a morphism in $\lim_I \mathcal{C}$ such that $e_i(f)$ is an equivalence for every $i \in I$. Then for every $X$ in $\lim_I \mathcal{C}$ the induced morphism

$$
\text{Map}_{\mathcal{C}(i)}(e_i(X), e_i(Y)) \rightarrow \text{Map}_{\mathcal{C}(i)}(e_i(X), e_i(Y'))
$$

is an equivalence. Hence the induced map $ev_i(M(X,Y)) \rightarrow ev_i(M(X,Y'))$ is an equivalence for all $i$ in $I$. We conclude that the morphism $M(X,Y) \rightarrow M(X,Y')$ induces an equivalence after applying $\lim_I$. In view of the equivalence (7.10), we thus have shown that the induced morphism $\text{Map}_{\lim_I} \mathcal{C}(X,Y) \rightarrow \text{Map}_{\lim_I} \mathcal{C}(X,Y')$ is an equivalence. Since $X$ is arbitrary, we conclude that $f : Y \rightarrow Y'$ is an equivalence.

Let $\mathcal{C} \rightarrow \mathcal{D}$ be a morphism in $\text{Fun}(I, \text{Cat}_\infty)$.

**Lemma 7.12.** Assume that for every $i$ in $I$ the morphism $\mathcal{C}(i) \rightarrow \mathcal{D}(i)$ is fully faithful. Then we have the following assertions:

1. The morphism $\lim_I \mathcal{C} \rightarrow \lim_I \mathcal{D}$ is fully faithful.
2. The essential image of the morphism in (1) consists of those objects $D$ of $\lim_I \mathcal{D}$ whose evaluation $e_i(D)$ belongs to the essential image of $\mathcal{C}(i) \rightarrow \mathcal{D}(i)$ for all $i$ in $I$.

**Proof.** (1) is well-known, but also follows from the discussion of mapping spaces in the proof of Lemma 7.11. In order to see (2), one easily checks that the indicated subcategory of $\lim_I \mathcal{D}$ has the required universal property.

We consider a diagram $\mathcal{C}$ in $\text{Fun}(I, \text{Cat}_{\infty, \text{Lex}})$.

**Lemma 7.13.** The collection of functors $(e_i)_{i \in I}$ detects finite limits.
Proof. Let $J$ be a finite category and $X$ be in $(\lim J)$. Let furthermore $Y$ be an object of $\lim J$ and $\iota: Y \to \lim J X$ be a morphism. We want to show that $\iota$ is an equivalence provided the induced functor

$$e_i(Y) \xrightarrow{e_i(\iota)} e_i(\lim X) \xrightarrow{1} \lim e_i(X)$$

is an equivalence for all $i$ in $I$. Since $e_i$ is left-exact, the marked morphism is an equivalence. Hence Lemma 7.13 follows from the fact that the collection $(e_i)_{i \in I}$ detects equivalences (Lemma 7.11).

We consider a diagram $C$ in $Fun(I, \mathbf{CAT}^{\text{LEX}}_{\infty, \ast})$.

**Lemma 7.14.** The collection of functors $(e_i)_{i \in I}$ detects limits.

Proof. The argument is the same as for Lemma 7.13. We just drop all finiteness assumptions on $J$ and use that in this case the evaluations (being morphisms in $\mathbf{CAT}^{\text{LEX}}_{\infty, \ast}$) preserve small limits.

In the following, we consider colimits of diagrams of left-exact $\infty$-categories. We will show that $\mathbf{Cat}^{\text{Lex, perf}}_{\infty, \ast}$ and $\mathbf{Cat}^{\text{Lex}}_{\infty, \ast}$ are cocomplete (the cases of limits and filtered colimits have already been settled in Proposition 7.9). We further study instances where colimits preserve fully faithfulness of transformations.

We have a chain of inclusions

$$\mathbf{Cat}^{\text{Rex, perf}}_{\infty, \ast} \subseteq \mathbf{Cat}^{\text{Rex}}_{\infty, \ast} \subseteq \mathbf{Cat}_{\infty}$$

with the following description:

1. $\mathbf{Cat}^{\text{Rex}}_{\infty, \ast}$ is the subcategory of small pointed $\infty$-categories admitting finite colimits, and finite colimit-preserving functors [Lur09, Not. 5.5.7.7].
2. $\mathbf{Cat}^{\text{Rex, perf}}_{\infty, \ast}$ is the full subcategory of $\mathbf{Cat}^{\text{Rex}}_{\infty, \ast}$ of idempotent complete objects [Lur09, Sec. 4.4.5].

As above, we let $\mathbf{Pr}^{\text{L}}_{\omega, \ast}$ denote the very large $\infty$-category of $\omega$-presentable pointed $\infty$-categories and left adjoint functors which preserve compact objects. We have the $\omega$-Ind-completion functor

$$\text{Ind}_{\omega}: \mathbf{Cat}^{\text{Rex}}_{\infty, \ast} \to \mathbf{Pr}^{\text{L}}_{\omega, \ast}.$$
We then have an $\omega$-Pro-completion $\text{Pro}_\omega : \mathbf{Cat}^{\text{Lex},*}_{\infty,*} \to \mathbf{Cat}^{\text{LEX}}_{\infty,*}$ functor defined such that
\[
\begin{array}{c}
\mathbf{Cat}^{\text{Lex}}_{\infty,*} \xrightarrow{\text{Pro}_\omega} \mathbf{Cat}^{\text{LEX}}_{\infty,*} \\
\xrightarrow{\text{op}} \mathbf{Cat}^{\text{Lex}}_{\infty,*} \xrightarrow{\text{op}} \mathbf{Cat}^{\text{LEX}}_{\infty,*}
\end{array}
\]
commutes. It induces an equivalence
\[
\text{Pro}_\omega : \mathbf{Cat}^{\text{Lex},\text{perf}}_{\infty,*} \xrightarrow{\simeq} \mathbf{Cat}^{\text{LEX}}_{\infty,*}.
\]
The inverse of the functor (7.12) is the functor
\[
(\_)^\omega : \mathbf{Cat}^{\text{LEX}}_{\infty,*} \to \mathbf{Cat}^{\text{Lex,perf}}_{\infty,*}
\]
taking the full subcategory of cocompact objects (Definition 7.7). Finally, we have an adjunction
\[
\text{Idem} := (\_)^\omega \circ \text{Pro}_\omega : \mathbf{Cat}^{\text{Lex}}_{\infty,*} \rightleftarrows \mathbf{Cat}^{\text{Lex,perf}}_{\infty,*} : \text{incl}.
\]

**Lemma 7.15.** The functor $\text{Idem}$ preserves fully faithfulness.

**Proof.** The operations $\text{Ind}_\omega$, $\text{op}$ and $(-)^\omega$ going into the definition of $\text{Pro}_\omega$ preserve fully faithfulness. □

**Lemma 7.16.** The $\infty$-category $\mathbf{Cat}^{\text{Lex,perf}}_{\infty,*}$ admits small colimits.

**Proof.** We let $\mathbf{Pr}^R_\omega$ denote the $\infty$-category of presentable $\infty$-categories and right adjoint functors which preserve filtered colimits. By [Lur09, Prop. 5.5.7.6], this $\infty$-category admits small limits which are preserved by the inclusion $\mathbf{Pr}^R_\omega \to \mathbf{Cat}_\infty$. We have an equivalence
\[
\text{ad} : \mathbf{Pr}^L_\omega \rightleftarrows \mathbf{Pr}^R_\omega,
\]
which is the identity on objects and replaces morphisms by their right adjoints. Consequently, $\mathbf{Pr}^L_\omega$ admits all small colimits. This implies that $\mathbf{Pr}^L_\omega$ also admits small colimits. In view of equivalence (7.3), $\mathbf{Cat}^{\text{Lex,perf}}_{\infty,*}$ admits small colimits. Finally, $\mathbf{Cat}^{\text{Lex,perf}}_{\infty,*}$ also admits small colimits by the equivalence (7.12). □

If $I$ is a groupoid, then we have an equivalence $\iota : I^\text{op} \to I$.

Let $C : I \to \mathbf{Cat}^{\text{Lex}}_{\infty,*}$ be a diagram. The colimit in the following lemma is interpreted in $\mathbf{Cat}^{\text{Lex,perf}}_{\infty,*}$.

**Lemma 7.17.** Assume that $I$ is a groupoid.

1. There is an equivalence
\[
\text{colim}_i C^\omega \simeq \left(\text{lim}^\omega i^* C\right)^\omega.
\]
2. For every object $i$ in $I$ the diagram
\[
\begin{array}{c}
\text{colim}_i C^\omega \xrightarrow{(7.15)} (\text{lim}^\omega i^* C)^\omega \xrightarrow{\text{incl}} \text{lim}^\omega i^* C
\end{array}
\]

commutes, where the arrow $e_{i,*}$ is the right adjoint of the canonical morphism $e_i: \lim_{\mathbf{I}^p} \iota^* C \to C(i)$, and the two unmarked horizontal arrows are the inclusions of the full subcategories of cocompact objects.

(3) The essential images of the functors $\text{can}(i)$ for all $i$ in $\mathbf{I}$ generate $\text{colim}_I C^\omega$ under finite limits and retracts.

Proof. By assumption, we have an equivalence $\iota: \mathbf{I}^p \to \mathbf{I}$ and an equivalence of diagrams $\text{ad}(C^{op}) \simeq \iota^* C^{op}$ in $\text{Fun}(\mathbf{I}^p, \mathbf{Cat}_\infty)$. This gives, using the description of colimits in $\mathbf{Cat}_{\text{Lex}, \text{perf}}^\omega$ provided by the proof of Lemma 7.16 in the first step,

$$\text{colim}_I C^\omega \simeq \left( \left( \lim_{\mathbf{I}^p} \iota^* C^{op} \right)^{op} \right)^\omega$$

$$\simeq \left( \left( \lim_{\mathbf{I}^p} \iota^* C \right)^{op} \right)^\omega$$

$$\simeq \lim_{\mathbf{I}^p} \iota^* C \omega.$$ 

Since $\iota^* C^{op}$ defines diagrams in both $\mathbf{Pr}_L^\omega$ and $\mathbf{Pr}_R^\omega$, and since both inclusions $\mathbf{Pr}_L^\omega \to \mathbf{Cat}_\infty$ and $\mathbf{Pr}_R^\omega \to \mathbf{Cat}_\infty$ preserve limits, for every $i$ in $\mathbf{I}$ the canonical transformation $e_i: \lim_{\mathbf{I}^p} C \to C(i)$ admits both adjoints. By the proof of Lemma 7.16, its right adjoint corresponds to the canonical transformation

$$e_{i,*}: C(i) \to \text{colim}_I C,$$

which evidently restricts to the subcategories of cocompact objects yielding $\text{can}(i)$.

We are left with showing that the essential images of the transformations $\text{can}(i)$ for all $i$ in $\mathbf{I}$ generate the target under finite limits and retracts. Since $\text{colim}_I C$ has a set of cocompact generators, it suffices to check that a morphism $f: x \to y$ in $\text{colim}_I C$ is an equivalence whenever

$$f^*: \text{Map}_{\text{colim}_I C}(x, e_{i,*}(c)) \to \text{Map}_{\text{colim}_I C}(y, e_{i,*}(c))$$

is an equivalence for all $i$ in $\mathbf{I}$ and $c$ in $C(i)$ (The proof of [AR94, Thm. 1.11] carries over to the setting of $\infty$-categories.). Since

$$\text{Map}_{\text{colim}_I C}(x, e_{i,*}(c)) \simeq \text{Map}_{C(i)}(e_i(x), c),$$

such a morphism has the property that $e_i(f)$ is an equivalence for all $i$ in $\mathbf{I}$. So it is indeed an equivalence by Lemma 7.11. \hfill \Box

Proposition 7.18. The $\infty$-category $\mathbf{Cat}_{\text{Lex}, \text{perf}}^\omega$ admits small colimits.

Proof. Let $C: \mathbf{I} \to \mathbf{Cat}_{\text{Lex}, \text{perf}}^\omega$ be a diagram. Then for every $i$ in $\mathbf{I}$ we have a canonical morphism

$$\iota_i: C(i) \to \text{Idem}(C(i)) \xrightarrow{e_{i,*}} \text{colim}_I \text{Idem}(C),$$

where the colimit is interpreted in $\mathbf{Cat}_{\text{Lex}, \text{perf}}^\omega$. We let $D$ be the full left-exact subcategory of $\text{colim}_I \text{Idem}(C)$ generated by the images of the functors $\iota_i$ for all $i$.
in \( \mathbf{I} \). For every \( \mathbf{T} \) in \( \mathbf{Cat}_{\infty,*}^{\text{Lex}} \) we then have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Map}_{\mathbf{Fun}(1, \mathbf{Cat}_{\infty,*}^{\text{Lex}})}(\text{Idem}(\mathbf{C}), \text{Idem}(\mathbf{T})) & \cong & \text{Map}_{\mathbf{Cat}_{\infty,*}^{\text{Lex}}}(\text{colim}_1 \text{Idem}(\mathbf{C}), \text{Idem}(\mathbf{T})) \\
\uparrow \cong & & \uparrow \cong \\
\text{Map}_{\mathbf{Fun}(1, \mathbf{Cat}_{\infty,*}^{\text{Lex}})}(\mathbf{C}, \text{Idem}(\mathbf{T})) & & \text{Map}_{\mathbf{Cat}_{\infty,*}^{\text{Lex}}}(\mathbf{D}, \text{Idem}(\mathbf{T})) \\
\downarrow \cong & & \downarrow \cong \\
\text{Map}_{\mathbf{Fun}(1, \mathbf{Cat}_{\infty,*}^{\text{Lex}})}(\mathbf{C}, \mathbf{T}) & \cong & \text{Map}_{\mathbf{Cat}_{\infty,*}^{\text{Lex}}}(\mathbf{D}, \mathbf{T})
\end{array}
\]

The morphism ! is induced by the morphism \( \mathbf{C} \to \text{Idem}(\mathbf{C}) \) and is an equivalence by the universal property of the latter. The morphism !! is induced by the inclusion \( \mathbf{D} \to \text{colim}_1 \text{Idem}(\mathbf{C}) \) and is an equivalence by a similar reason since the induced morphism \( \text{Idem}(\mathbf{D}) \to \text{colim}_1 \text{Idem}(\mathbf{C}) \) is an equivalence by construction of \( \mathbf{D} \). The morphisms marked by !!! and !!!! are induced by the fully faithful functor \( \mathbf{T} \to \text{Idem}(\mathbf{T}) \). They are inclusions of collections of components. The \( \infty \)-category \( \mathbf{D} \) is constructed exactly such that the dotted arrow exists and is a bijection on \( \pi_0 \). Since it is natural in \( \mathbf{T} \) we can conclude that the colimit of the diagram \( \mathbf{C} \) in \( \mathbf{Cat}_{\infty,*}^{\text{Lex}} \) exists and is represented by \( \mathbf{D} \).

Let \( \mathbf{I} \) be a small \( \infty \)-category, and let \( \mathbf{C} \to \mathbf{D} \) be a morphism in \( \mathbf{Fun}(\mathbf{I}, \mathbf{Cat}_{\infty,*}^{\text{Lex}}) \).

**Lemma 7.19.** Assume:

1. \( \mathbf{I} \) is a groupoid.
2. The functor \( \mathbf{C}(i) \to \mathbf{D}(i) \) is fully faithful for all \( i \) in \( \mathbf{I} \).

Then the functor \( \text{colim}_1 \mathbf{C} \to \text{colim}_1 \mathbf{D} \) is fully faithful.

**Proof.** We first consider the analogous assertion for diagrams in \( \mathbf{Cat}_{\infty,*}^{\text{Lex, perf}} \). In this case, we can apply the idea of the proof of Lemma 7.17. We use the formula

\[
\text{colim}_1 \mathbf{C} \cong \left( (\text{ad}^{-1} \lim_{\mathbf{I}^{\text{op}}} \text{ad}((\text{Pro}_{\omega}(\mathbf{C})^{\text{op}}))^{\text{op}} \right)^{\omega}.
\]

for the colimit given in the proof of Lemma 7.16. As the operations \( \text{Pro}_{\omega} \), \( (-)^{\omega} \) and \( (-)^{\text{op}} \) preserve fully faithfulness (Lemma 7.15) we must show the following assertion.

Assume that \( f: \mathbf{P} \to \mathbf{Q} \) is a morphism in \( \mathbf{Fun}(\mathbf{I}, \mathbf{Pr}^{L}_{\omega}) \) such that \( f(i) \) is fully faithful for every \( i \) in \( \mathbf{I} \). Then the functor \( (\text{ad}^{-1} \lim_{\mathbf{I}^{\text{op}}} \text{ad})(f) \) is fully faithful.

We have a morphism \( \text{ad}(f): \text{ad}(\mathbf{Q}) \to \text{ad}(\mathbf{P}) \) in \( \mathbf{Fun}(\mathbf{I}^{\text{op}}, \mathbf{Cat}_{\infty}) \). The functor \( \text{ad} \) (which replaces functors by their right adjoints) does not preserve fully faithfulness. To overcome this problem we use that the fully faithful left adjoints \( f(i) \) of the functors \( \text{ad}(f(i)) \) for all \( i \) in \( \mathbf{I} \) assemble to a natural transformation in the opposite direction. Since \( \mathbf{I} \) is a groupoid, we have an equivalence \( \iota: \mathbf{I}^{\text{op}} \to \mathbf{I} \). We get the diagram in \( \mathbf{Fun}(\mathbf{I}^{\text{op}}, \mathbf{Cat}_{\infty}) \):

\[
\begin{array}{ccc}
\text{ad}(\mathbf{P}) & \cong & \text{ad}(\mathbf{Q}) \\
\downarrow \cong & & \downarrow \cong \\
\iota^{*}\mathbf{P} & \cong & \iota^{*}\mathbf{Q}
\end{array}
\]
where \( \tilde{f} \) is defined by commutativity of the lower square. By construction, \( \tilde{f}(i) \) is the left adjoint \( f(i) \) of \( \text{ad}(f(i)) \). We now apply \( \lim_{\mathbb{P}} \) to the upper line and get an adjunction

\[
\begin{array}{c}
\lim_{\mathbb{P}} \text{ad}(f) \\
\approx \\
\lim_{\mathbb{P}} \text{ad}(P) \\
\approx \\
\lim_{\mathbb{P}} \text{ad}(Q)
\end{array}
\]

In particular, we have an equivalence

\[
\lim_{\mathbb{P}} \tilde{f} \simeq (\text{ad}^{-1} \lim_{\mathbb{P}} \text{ad})(f).
\]

Since a limit of a diagram of fully faithful functors in \( \text{Cat}_{\infty} \) is again fully faithful we can conclude that \( \lim_{\mathbb{P}} \tilde{f} \) and hence \( (\text{ad}^{-1} \lim_{\mathbb{P}} \text{ad})(f) \) are fully faithful. This finishes the case of diagrams with values in \( \text{Cat}_{\infty,*}^{\text{Lex, perf}} \).

Assume now that \( f: C \to D \) is a morphism of \( \mathcal{I} \)-indexed diagrams with values in \( \text{Cat}_{\infty,*}^{\text{Lex}} \) which is objectwise fully faithful. Then \( \text{Idem}(f): \text{Idem}(C) \to \text{Idem}(D) \) is such a diagram in \( \text{Cat}_{\infty,*}^{\text{Lex, perf}} \) with the same property by Lemma 7.15. We just have shown that the lower line in the square in \( \text{Cat}_{\infty} \)

\[
\begin{array}{c}
\underset{\text{colim}_{\mathcal{I}} C}{\text{colim}} \\
\approx \\
\underset{\text{colim}_{\mathcal{I}} f}{\text{colim}} \\
\approx \\
\text{colim}_{\text{Idem}(C)} \text{colim}_{\text{Idem}(f)} \text{colim}_{\text{Idem}(D)}
\end{array}
\]

is fully faithful. In the proof of Proposition 7.18, we have presented the colimits (in \( \text{Cat}_{\infty,*}^{\text{Lex}} \)) in the upper line as full subcategories of the colimits (in \( \text{Cat}_{\infty,*}^{\text{Lex, perf}} \)) in the lower line. Consequently, the vertical arrows are fully faithful. This implies that also the upper horizontal arrow is fully faithful. \( \square \)

In general, if \( C \) is a pointed \( \infty \)-category admitting finite products and coproducts, then for every finite set \( F \) and family \((C_f)_{f \in F}\) in \( C \) we have a natural morphism

\[
q: \prod_{f \in F} C_f \to \prod_{f' \in F} C_{f'}.
\]

This morphism is classified by the collection of morphisms

\[
(q_{f'}: \prod_{f \in F} C_f \to C_{f'})_{f' \in F},
\]

where \( q_{f'} \) itself is classified by the collection of morphisms

\[
(q_{f,f'}: C_f \to C_{f'})_{f \in F}
\]

such that \( q_{f,f'} \) is zero for \( f \neq f' \) and \( \text{id}_{C_f} \) for \( f = f' \).

**Definition 7.20** ([Lurie, Def. 6.1.6.13]). The \( \infty \)-category \( C \) is called semi-additive, if it is pointed, admits finite products and coproducts, and if the morphism (7.16) is an equivalence for every finite set \( F \) and family \((C_f)_{f \in F}\) of objects in \( C \).

**Lemma 7.21.** The \( \infty \)-category \( \text{Cat}_{\infty,*}^{\text{Lex}} \) is semi-additive.
Proposition 7.9 and therefore admits products. \( \mathbf{Cat}_{\infty, *}^{\text{Lex}} \) also admits coproducts by Lemma 7.16. Finally, \( \mathbf{Cat}_{\infty, *}^{\text{Lex}} \) is pointed by the one-point category \( * \).

We now fix a finite set \( F \) and a family \( (C_f)_{f \in F} \) in \( \mathbf{Cat}_{\infty, *}^{\text{Lex}} \). For \( f \) in \( F \) let \( \iota_f : C_f \to \prod_{f \in F} C_{f'} \) be the canonical inclusion and \( \pi_f : \prod_{f \in F} C_{f'} \to C_f \) be the canonical projection. For every \( f'' \) in \( F \) we have a morphism

\[
p_{f''} := \iota_{f''} \circ \pi_{f''} : \prod_{f \in F} C_{f'} \to \prod_{f \in F} C_f .
\]

Their product\(^7\) in the left-exact \( \infty \)-category \( \prod_{f \in F} C_f \) is a morphism

\[
p := \times_{f'' \in F} p_{f''} : \prod_{f \in F} C_{f'} \to \prod_{f \in F} C_f .
\]

We claim that \( p \) and the morphism \( q \) from (7.16) are mutually inverse equivalences. In order to show that \( p \circ q \simeq \text{id}_{\prod_{f \in F} C_f} \), it suffices to provide equivalences \( p \circ q \circ \iota_f \simeq \iota_f \) for all \( f \) in \( F \). They are given by the following chains of equivalences:

\[
p \circ q \circ \iota_f \simeq (\times_{f'' \in F} f_{f''} \circ \pi_{f''}) \circ q \circ \iota_f
\]

\[
\simeq \times_{f'' \in F} (f_{f''} \circ \pi_{f''} \circ q \circ \iota_f)
\]

\[
\simeq \times_{f'' \in F} f_{f''} \circ q_{f''} \circ \iota_f
\]

\[
\simeq \times_{f'' \in F} (f_{f''} \circ q_{f''} \circ \iota_f)
\]

\[
\simeq \iota_f,
\]

where we use the notation from (7.17) and (7.18). Similarly, in order to show that \( q \circ p \simeq \text{id}_{\prod_{f' \in F} C_{f'}} \), it suffices to provide an equivalence \( \pi_{f'} \circ q \circ p \simeq \pi_{f'} \) for every \( f' \) in \( F \). They are given by the following chains of equivalences:

\[
\pi_{f'} \circ q \circ p \simeq \pi_{f'} \circ q \circ (\times_{f'' \in F} f_{f''} \circ \pi_{f''})
\]

\[
\simeq q_{f'} \circ (\times_{f'' \in F} f_{f''} \circ \pi_{f''})
\]

\[
\simeq \times_{f'' \in F} q_{f'} \circ f_{f''} \circ \pi_{f''}
\]

\[
\simeq \pi_{f'},
\]

where at the marked equivalence we use that \( q_{f'} \) preserves finite products. \( \square \)

In the remainder of this subsection, we consider a situation where a limit and a colimit can be interchanged.

Remark 7.22. We consider a small category \( I \), functors \( T, C, D \) in \( \mathbf{Fun}(I, \mathbf{Cat}_\infty) \), and a transformation \( f : C \to D \). We consider the subspace \( \text{Map}'_{\mathbf{Fun}(I, \mathbf{Cat}_\infty)}(T, D) \) consisting of the components of \( \text{Map}'_{\mathbf{Fun}(I, \mathbf{Cat}_\infty)}(T, D) \) of those transformations \( u : T \to D \) such that for every \( i \) in \( I \) the functor \( u(i) : T(i) \to D(i) \) takes values in the essential image of \( f(i) : C(i) \to D(i) \). If \( f \) is objectwise fully faithful, then the canonical map induces an equivalence

\[
\text{Map}'_{\mathbf{Fun}(I, \mathbf{Cat}_\infty)}(T, C) \xrightarrow{\simeq} \text{Map}'_{\mathbf{Fun}(I, \mathbf{Cat}_\infty)}(T, D) .
\]

\(^7\)This product of functors can formally be understood as a right Kan extension along the functor of discrete categories \( F \to * \). It exists since \( F \) is finite and \( \prod_{f \in F} C_f \) being left-exact admits finite products.
Let $I$ and $J$ be small categories and let $C : I \times J \to \mathsf{Cat}_\infty$ be a functor.

**Lemma 7.23.** Assume:

1. $I$ is filtered.
2. $J$ has only finitely many objects.
3. For every morphism $i \to i'$ in $I$ and every object $j$ in $J$ the functor $C(i, j) \to C(i', j)$ is fully faithful.

Then the natural functor

$$\mathsf{colim}_I \mathsf{lim}_J C \to \mathsf{lim}_J \mathsf{colim}_I C$$

is an equivalence.

**Proof.** By assumption, the transformation $C(i, -) \to C(i', -)$ of diagrams $J \to \mathsf{Cat}_\infty$ is objectwise fully faithful for every morphism $i \to i'$ in $I$. Since limits of diagrams of fully faithful functors are fully faithful, the induced functor $\mathsf{lim}_J C(i, -) \to \mathsf{lim}_J C(i', -)$ is fully faithful. Since $I$ is filtered, it follows that the functor

$$\mathsf{lim}_J C(i, -) \to \mathsf{colim}_I \mathsf{lim}_J C$$

is fully faithful for every $i$ in $I$.

Similarly, the canonical functor $C(i, j) \to \mathsf{colim}_I C(-, j)$ is fully faithful for all $i$ in $I$ and $j$ in $J$. We conclude that the induced map

$$\mathsf{lim}_J C(i, -) \to \mathsf{lim}_J \mathsf{colim}_I C$$

is fully faithful for every $i$ in $I$.

Since $I$ is filtered and since we have a commutative diagram

$$\begin{array}{ccc}
\mathsf{lim}_J C(i, -) & \longrightarrow & \mathsf{lim}_I \mathsf{colim}_J C \\
\downarrow & & \downarrow \\
\mathsf{colim}_I \mathsf{lim}_J C & \longrightarrow & \mathsf{lim}_J \mathsf{colim}_I C
\end{array}$$

for every $i$ in $I$, it follows that $\mathsf{colim}_I \mathsf{lim}_J C \to \mathsf{lim}_J \mathsf{colim}_I C$ is fully faithful.

We are left with showing essential surjectivity of the functor (7.19). Since $\mathsf{lim}_J$ is right adjoint to the functor $-\to \mathsf{colim}_I$ taking constant $J$-diagrams, an object $A$ in $\mathsf{lim}_I \mathsf{colim}_J C$ corresponds to a natural transformation

$$\Delta^0 \to \mathsf{colim}_I C$$

of diagrams $J \to \mathsf{Cat}_\infty$. Since $\Delta^0$ is compact and since $J$ has only finitely many objects, there exists some $i_0$ in $I$ such that $\Delta^0(j) = \Delta^0 \to \mathsf{colim}_I C(-, j)$ factors for every $j$ in $J$ through the canonical map $C(i_0, j) \to \mathsf{colim}_I C(-, j)$. Applying Remark 7.22 yields a transformation $\Delta^0 \to C(i_0, -)$ which fits into a commutative triangle

$$\begin{array}{ccc}
\Delta^0 & \longrightarrow & \mathsf{colim}_I C \\
\downarrow & & \downarrow \\
C(i_0, -) & \longrightarrow &
\end{array}$$

Hence, we obtain an object in $\mathsf{lim}_J C(i_0, -)$ whose image under the canonical map $\mathsf{lim}_J C(i_0, -) \to \mathsf{colim}_I \mathsf{lim}_J C$ provides the required preimage of $A$. $\square$
Let $\mathbf{M}$ be an $\infty$-category admitting small filtered colimits and small products.

**Definition 7.24.** We say that filtered colimits distribute over products in $\mathbf{M}$ if for any family of small filtered categories $(\mathbf{F}_i)_{i \in I}$ and family of functors $(E_i: \mathbf{F}_i \to \mathbf{M})_{i \in I}$ the canonical morphism

$$\text{colim}_{(F_i)_{i \in I}} \prod_{i \in I} E_i(F_i) \to \prod_{i \in I} \text{colim}_{F_i \in \mathbf{F}_i} E_i(F_i)$$

is an equivalence.

**Example 7.25.** Examples of $\infty$-categories satisfying the condition of **Definition 7.24** are $\text{Spc}$, $\text{Cat}_\infty$, and hence also $\text{Cat}_{\text{Lex}}^\text{Lex}$ by **Proposition 7.9**.

7.2. The calculus of fractions formula. In this section we explicitly work with the model of $\text{Cat}_\infty$ given by quasi-categories. Objects of $\text{Cat}_\infty$ are in particular simplicial sets. There is a parallel version of the theory below for $\text{CAT}_\infty$ involving large simplicial sets.

**Definition 7.26.** A pair $(\mathbf{C}, W)$ is an object of $\text{Cat}_\infty$ together with a subcategory $W$ containing the maximal Kan complex of $\mathbf{C}$.

If $(\mathbf{C}, W)$ is a pair, then we can consider the Dwyer-Kan localisation

$$\ell: \mathbf{C} \to \mathbf{C}[W^{-1}]$$

It is characterised by the universal property that for every $D$ in $\text{Cat}_\infty$ the functor $\ell$ induces an equivalence

$$\text{Fun}(\mathbf{C}[W^{-1}], D) \to \text{Fun}^W(\mathbf{C}, D),$$

where the right-hand side is the subcategory of functors which send the one-simplices of $W$ to equivalences.

We consider a pair $(\mathbf{C}, W)$, and we let $A$ be an object of $\mathbf{C}$.

**Definition 7.27.** A putative right calculus of fractions at $A$ is a pair $(W(A), \pi)$ of a simplicial set $W(A)$ and a functor $\pi: W(A) \to \mathbf{C}$ such that the following conditions are satisfied:

a) The simplicial set $W(A)$ has a final object $e_A$ such that $\pi(e_A) = A$.

b) The image by $\pi$ of any map in $W(A)$ with target $e_A$ belongs to $W$.

Let $(W(A), \pi)$ be a putative right calculus of fractions at $A$.

**Definition 7.28.** $(W(A), \pi)$ is a right calculus of fractions at $A$ if the functor

$$\text{colim}_{(A' \to e_A) \in (W(A)/e_A)^{op}} \text{Map}_\mathbf{C}(\pi(A'), -): \mathbf{C} \to \text{Spc}$$

sends morphisms in $W$ to equivalences.

There is also the notion of a (putative) left calculus of fractions: it corresponds to a (putative) right calculus of fractions in $\mathbf{C}^{\text{op}}$.

Let $(\mathbf{C}, W)$ be a pair, and let $A$ be an object of $\mathbf{C}$. The following is taken from [Cis19, Ex. 7.2.3].

---

8A final object in a simplicial set is a point that becomes a final object in the $\infty$-category obtained by fibrant-replacement in the Joyal model structure.

9Here we use the reformulation of [Cis19, Def. 7.2.5] in terms of [Cis19, Eq. (7.2.6.2)].
Definition 7.29. The maximal putative right calculus of fractions at $A$ is given by the pair $(W(A), \pi)$, where $W(A)$ is the full subcategory of $C/A$ whose objects are the morphisms $A' \to A$ in $W$, and $\pi : W(A) \to C/A \to C$ is the canonical functor.

Let $(C, W)$ be a pair.

Definition 7.30. We say that $(C, W)$ satisfies a right calculus of fractions formula if for every object $A$ the maximal putative right calculus of fractions at $A$ is a right calculus of fractions at $A$.

By [Cis19, Thm. 7.2.7], we have the following characterisation of a right calculus of fractions. Let $(C, W)$ be a pair, and let $A$ be an object of $C$. Let furthermore $(W(A), \pi)$ be a putative right calculus of fractions at $A$.

Proposition 7.31. $(W(A), \pi)$ is a right calculus of fractions at $A$ if and only if the canonical map

\[
\text{colim}_{(A' \to e_A) \in (W(A)/e_A)^{op}} \text{Map}_C(\pi(A'), B) \to \text{Map}_C(W^{-1}(\ell(A), \ell(B)))
\]

is an equivalence of spaces for every object $B$ of $C$.

Proof. By [Cis19, Thm. 7.2.7] the property of being a right calculus implies that the morphism (7.21) is an equivalence for every object $B$ of $C$. For the converse, if (7.21) is an equivalence for every object $B$ of $C$, then the condition for a right calculus of fractions (Definition 7.28) is satisfied since the functor $\text{Map}_C(W^{-1}(\ell(A), \ell(-))$ sends the elements of $W$ to equivalences.

We recall [Cis19, Thm. 7.2.16]:

Proposition 7.33. If $W$ is preserved by pullbacks, then the pair $(C, W)$ satisfies a right calculus of fractions.

Let $(C, W)$ be a pair.

Definition 7.32. We say that $W$ is preserved by pullbacks if every diagram

\[
\begin{array}{ccc}
B' & \to & B \\
\downarrow^f & & \downarrow^e \\
A & \to & B
\end{array}
\]

in $C$ with $f$ in $W$ can be extended to a cartesian diagram

\[
\begin{array}{ccc}
D & \to & B' \\
\downarrow^g & & \downarrow^f \\
A & \to & B
\end{array}
\]

in $C$ with $g$ in $W$.

We recall [Cis19, Thm. 7.2.16]:

Proposition 7.34. Assume:

1. $W$ is preserved by pullbacks.
2. $W$ has the two-out-of-three property.
3. $C$ admits finite limits.
Then:

1. \( C[W^{-1}] \) has finite limits.
2. \( \ell: C \rightarrow C[W^{-1}] \) is left-exact.
3. For any left-exact functor \( \phi: C \rightarrow D \) which sends the maps of \( W \) to equivalences, the induced functor \( C[W^{-1}] \rightarrow D \) is left-exact.

Proof. Since \( C \) has finite limits, it may be considered as a category with weak equivalences and fibrations in the sense of [Cis19, Def. 7.4.12], where the weak equivalences are the elements of \( W \), and the fibrations are all maps in \( C \). Hence we can apply [Cis19, Prop. 7.5.6].

7.3. Labellings and localisation. We state the version for \( \text{Cat}_\infty \). There is again a parallel version for \( \text{CAT}_\infty \).

Recall Definition 7.26 of a pair \((C, W)\). A functor \((C, W) \rightarrow (C', W')\) between pairs is a functor \( \phi: C \rightarrow C' \) such that \( \phi(W) \subseteq W' \).

As explained in [Bar16, Sec. 1], one can define a large \( \infty \)-category \( \text{Pair}_\infty \) of pairs as a subcategory of the arrow category \( \text{Fun}(\Delta^1, \text{Cat}_\infty) \).

Definition 7.35. A labelled left-exact \( \infty \)-category \((C, W)\) is a pair such that \( C \) is in \( \text{Cat}_{\infty, *}^{\text{Lex}} \).

A functor between labelled left-exact \( \infty \)-categories \((C, W) \rightarrow (C', W')\) is a morphism \( \phi: C \rightarrow C' \) in \( \text{Cat}_{\infty, *}^{\text{Lex}} \) such that \( \phi(W) \subseteq W' \).

We have a functor \( \text{Pair}_{\infty} \rightarrow \text{Cat}_{\infty} \) which takes the underlying \( \infty \)-category. The \( \infty \)-category \( \ell \text{Cat}_{\infty, *}^{\text{Lex}} \) of labelled left-exact \( \infty \)-categories is defined as the pullback in \( \text{CAT}_\infty \)

\[
\ell \text{Cat}_{\infty, *}^{\text{Lex}} := \text{Cat}_{\infty, *}^{\text{Lex}} \times_{\text{Cat}_{\infty}} \text{Pair}_\infty.
\]

Remark 7.36. Let \( C \) be a left-exact \( \infty \)-category. In order to give a labelled left-exact \( \infty \)-category \((C, W)\) it suffices to give the wide subcategory \( \text{Ho}(W) \) of \( \text{Ho}(C) \). So we must only prescribe a set of one-morphisms \( W^1 \) which generates \( \text{Ho}(W) \) as a category. We then say that \( W \) is generated by this set. A left-exact functor \( C \rightarrow C' \) induces a functor between labelled left-exact \( \infty \)-categories given in this way if the induced functor \( \text{Ho}(C) \rightarrow \text{Ho}(C') \) sends \( W^1 \) to \( W'^1 \).

We consider a labelled left-exact \( \infty \)-category \((C, W)\). We then let \( \bar{W} \) denote the smallest subcategory of \( C \) with the following properties:

1. \( W \subseteq \bar{W} \).
2. \( \bar{W} \) is preserved by pullbacks (Definition 7.32).
3. \( \bar{W} \) has the two-out-of-three property.

Definition 7.37. We define the \( \infty \)-category

\[
W^{-1}C := C[W^{-1}]
\]

and denote by \( \ell: C \rightarrow W^{-1}C \) the localisation functor.

By construction and Proposition 7.34, we know that \( W^{-1}C \) has finite limits, and that the localisation functor \( \ell \) is left-exact.

Proposition 7.38. For any \( D \) in \( \text{Cat}_{\infty, *}^{\text{Lex}} \), the functor

\[
\ell^*: \text{Fun}_{\text{Cat}_{\infty, *}^{\text{Lex}}}^W(W^{-1}C, D) \rightarrow \text{Fun}_{\text{Cat}_{\infty, *}^{\text{Lex}}}^W(C, D)
\]
is an equivalence of $\infty$-categories, where $\text{Fun}_{\operatorname{Cat}_{\infty,*,\text{lex}}}^W(C, D)$ is the full subcategory of $\text{Fun}_{\operatorname{Cat}_{\infty,*,\text{lex}}}(C, D)$ of functors which send maps of $W$ to equivalences.

Proof. See [Cis19, Prop. 7.5.11].

In the following, we consider the functorial dependence of the localisation on the data $(C, W)$. To this end we consider the functor

$$M : \text{Cat}_{\infty,*,\text{lex}} \to \ell \text{Cat}_{\infty,*,\text{lex}}, \quad M(C) := (C, i(C)),$$

where $i(C)$ is the maximal Kan complex in $C$. Then Proposition 7.38 gives an equivalence

$$\text{Map}_{\text{Cat}_{\infty,*,\text{lex}}}(W^{-1}C, D) \cong \text{Map}_{\ell \text{Cat}_{\infty,*,\text{lex}}}(((C, W), M(D))$$

functorially in $D$. This means that the functor $M$ fits into an adjunction

$$(7.23) \quad \text{Loc} : \ell \text{Cat}_{\infty,*,\text{lex}} \rightleftarrows \text{Cat}_{\infty,*,\text{lex}} : M$$

where Loc is the localisation functor such that $\text{Loc}(C, W) \cong W^{-1}C$.

Remark 7.39. Note the difference in notation. The symbol $C[W^{-1}]$ denotes the Dwyer-Kan localisation. There is no reason that this localisation is left-exact. In contrast, $W^{-1}C$ denotes the localisation in the realm of left-exact $\infty$-categories. The corresponding universal property is stated in Proposition 7.38. We have canonical equivalences of $\infty$-categories:

$$C[W^{-1}] \cong W^{-1}C \cong W^{-1}C.$$

In particular, if $W$ is stable under pullbacks and has the two-out-of-three property, then $C[W^{-1}] \cong W^{-1}C$.

We consider a pair $(C, W)$ and the localisation $\ell : C \to C[W^{-1}]$. Since the functor $C \to W^{-1}C$ sends the morphisms in $W$ to equivalences, we get an essentially unique (dotted) factorisation

$$(7.24) \quad W^{-1}C \overset{\cong}{\longrightarrow} C[W^{-1}]
\begin{array}{ccc}
\uparrow & \quad & \downarrow \\
C & \xrightarrow{\ell} & C[W^{-1}]
\end{array}$$

Lemma 7.40. Assume that the functor $\ell : C \to C[W^{-1}]$ preserves finite limits. Then $C[W^{-1}]$ is left-exact and the marked morphism in $(7.24)$ is an equivalence in $\text{Cat}_{\infty,*,\text{lex}}$. Furthermore, the localisation functor $C \to C[W^{-1}]$ is essentially surjective on morphisms.

Proof. Let $\tilde{W}$ be the labelling generated by the morphisms which are sent to equivalence by $\ell$. Then $\tilde{W}$ satisfies the two-out-of-three property and $W \subseteq \tilde{W}$. By our assumption on $\ell$, we also know that $\tilde{W}$ is preserved by pullbacks. Hence $\tilde{W} \subseteq W$ and we get the vertical dashed arrow in $(7.24)$. Since clearly $C[W^{-1}] \cong C[\tilde{W}^{-1}]$ we have found an inverse equivalence to the dotted arrow.
The last assertion on the essential surjectivity at the level of morphisms follows right away from the right calculus of fractions, because any map \( A \to B \) in \( \mathcal{C}[W^{-1}] \) may be written up to equivalence as a composition \( fs^{-1} \), where \( f : A' \to B \) is a map in \( \mathcal{C} \), while \( s : A' \to A \) is a map in \( \overline{W} \).

7.4. Stabilisation and cofibres. We have a functor
\[
\hat{\mathbf{Sp}} : \text{Cat}^{\text{Lex}}_{\infty,*} \to \text{Cat}^{\text{Lex}}_{\infty,*}, \quad \hat{\mathbf{Sp}}(\mathcal{C}) := \text{colim}(\mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \to \ldots)
\]
and a natural transformation \( \hat{\Omega}^\infty : \text{id} \to \hat{\mathbf{Sp}} \). Recall that \( \text{Cat}^{\infty}_{\text{ex}} \) denotes the full subcategory of \( \text{Cat}^{\text{Lex}}_{\infty,*} \) of stable \( \infty \)-categories.

Lemma 7.41. The functor \( \hat{\mathbf{Sp}} \) has an essentially unique factorisation
\[
\text{Cat}^{\infty}_{\text{ex}} \xrightarrow{\hat{\mathbf{Sp}}} \text{Cat}^{\infty}_{\text{ex}} \xleftarrow{\hat{\mathbf{Sp}}} \text{Cat}^{\text{Lex}}_{\infty,*}
\]
which fits into an adjunction
\[
\overline{\mathbf{Sp}} : \text{Cat}^{\text{Lex}}_{\infty,*} \hookrightarrow \text{Cat}^{\infty}_{\text{ex}} : \text{incl}
\]

Proof. If \( \mathcal{C} \) is in \( \text{Cat}^{\text{Lex}}_{\infty,*} \), then the endofunctor \( \Omega : \hat{\mathbf{Sp}}(\mathcal{C}) \to \hat{\mathbf{Sp}}(\mathcal{C}) \) is an equivalence. Consequently, the \( \infty \)-category \( \hat{\mathbf{Sp}}(\mathcal{C}) \) is stable. This implies the existence of the factorisation. The second assertion follows from the claim that pullback along \( \Omega^\infty : \mathcal{C} \to \hat{\mathbf{Sp}}(\mathcal{C}) \) (induced by \( \hat{\Omega}^\infty \)) induces an equivalence
\[
\text{Map}_{\text{Cat}^{\infty}_{\text{ex}}}(\hat{\mathbf{Sp}}(\mathcal{C}), \mathcal{D}) \cong \text{Map}_{\text{Cat}^{\text{Lex}}_{\infty,*}}(\mathcal{C}, \mathcal{D})
\]
for every \( \mathcal{D} \) in \( \text{Cat}^{\infty}_{\text{ex}} \). In order to see the claim note that by the universal property of the colimit, we have an equivalence
\[
\text{Map}_{\text{Cat}^{\infty}_{\text{ex}}}(\hat{\mathbf{Sp}}(\mathcal{C}), \mathcal{D}) \cong \text{Map}_{\text{Cat}^{\text{Lex}}_{\infty,*}}(\hat{\mathbf{Sp}}(\mathcal{C}), \mathcal{D}) \cong \text{Map}_{\text{Fun}(\mathcal{N}, \text{Cat}^{\text{Lex}}_{\infty,*})}(\tilde{\mathcal{C}}, \mathcal{D})
\]
where \( \mathcal{D} \) is the constant diagram with value \( \mathcal{D} \), and
\[
\tilde{\mathcal{C}} := \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \to \ldots
\]
We thus have
\[
\text{Map}_{\text{Fun}(\mathcal{N}, \text{Cat}^{\text{Lex}}_{\infty,*})}(\tilde{\mathcal{C}}, \mathcal{D}) \cong \lim_N \text{Map}_{\text{Cat}^{\text{Lex}}_{\infty,*}}(\tilde{\mathcal{C}}, \mathcal{D}) \cong \text{Map}_{\text{Cat}^{\text{Lex}}_{\infty,*}}(\mathcal{C}, \mathcal{D})
\]
where the second equivalence is a consequence of the fact that the argument of the limit is essentially constant since \( \mathcal{D} \) is stable.

Definition 7.42. We call \( \hat{\mathbf{Sp}} : \text{Cat}^{\text{Lex}}_{\infty,*} \to \text{Cat}^{\infty}_{\text{ex}} \) the stabilisation functor.

Lemma 7.43. The stabilisation functor \( \hat{\mathbf{Sp}} \) preserves fully faithfulness.

Proof. The functor \( \hat{\mathbf{Sp}} \) from (7.25) is given by a filtered colimit in \( \text{Cat}^{\text{Lex}}_{\infty,*} \). Note that filtered colimits in \( \text{Cat}^{\text{Lex}}_{\infty,*} \) can be calculated in \( \text{Cat}^\infty \) by Proposition 7.9. Furthermore a filtered colimit of fully faithful functors in \( \text{Cat}^\infty \) is fully faithful. This implies the assertion.
Let $C$ be in $\mathbf{Cat}^{\aleph_0}_{\ast, s}$, and let $f : C \to C'$ be a morphism in $C$. Using the existence of finite limits and the zero object in $C$ the fibre of $f$ can be defined by

$$\text{Fib}(f) := 0 \times_{C'} C.$$  

If $\phi : D \to C$ is a morphism in $\mathbf{Cat}^{\aleph_0}_{\ast, s}$, then we define the labelled left-exact $\infty$-category $(C, W_\phi)$ such that $W_\phi$ is generated by the set of morphisms $f$ in $C$ with the property that $\text{Fib}(f)$ belongs to the essential image of $\phi$. Recall from Definition 7.37 the left-exact localisation of a labelled left-exact $\infty$-category.

**Lemma 7.44.** We have an equivalence

$$W_{\mathbf{Sp}(\phi)}^{-1} \mathbf{Sp}(C) \overset{\simeq}{\to} \mathbf{Sp}(W_{\phi}^{-1}C).$$

**Proof.** We apply the functor $\mathbf{Sp}$ to the sequence

$$D \xrightarrow{\phi} C \xrightarrow{\ell_\phi} W_{\phi}^{-1}C$$

and get

$$(7.27)$$

$$\begin{array}{ccc}
\mathbf{Sp}(D) & \xrightarrow{\mathbf{Sp}(\phi)} & \mathbf{Sp}(C) \\
\downarrow \mathbf{Sp}(\phi) & & \downarrow \mathbf{Sp}(\phi) \\
\mathbf{Sp}(W_{\phi}^{-1}C) & \xrightarrow{\mathbf{Sp}(\ell_\phi)} & \mathbf{Sp}(W_{\phi}^{-1}C) \\
\downarrow \mathbf{Sp}(\ell_\phi) & & \downarrow \mathbf{Sp}(\ell_\phi) \\
W_{\mathbf{Sp}(\phi)}^{-1} \mathbf{Sp}(C) & \xrightarrow{\kappa} & W_{\mathbf{Sp}(\phi)}^{-1} \mathbf{Sp}(C)
\end{array}$$

We get the arrow marked with $!$ from the universal property of the localisation $\ell_\phi$. Assume that $f$ is a morphism in $C$ with $\text{Fib}(f)$ in the essential image of $\phi$. Since the functor $\iota : C \to \mathbf{Sp}(C)$ is left-exact and hence preserves fibres, the commutative left square implies that $\text{Fib}(\iota(f))$ belongs to the essential image of $\mathbf{Sp}(\phi)$, i.e., we have $\iota(f) \in W^{-1}_{\mathbf{Sp}(\phi)}$.

We now observe that $W^{-1}_{\mathbf{Sp}(\phi)} \mathbf{Sp}(C) \in \mathbf{Cat}^{\aleph_0}_{\ast}$. To this end, note that the suspension functor $\Omega$ on $\mathbf{Sp}(C)$ preserves $W_{\mathbf{Sp}(\phi)}$ and therefore induces an autoequivalence of $(\mathbf{Sp}(C), W_{\mathbf{Sp}(\phi)})$ in $\mathcal{L}_{\mathbf{Cat}, \ast}^{\aleph_0}$. Consequently, it induces an equivalence on $W_{\mathbf{Sp}(\phi)}^{-1} \mathbf{Sp}(C)$. We now obtain the dotted arrow from $!$ and the universal property of the functor $\kappa$ formulated in Lemma 7.41.

Let $\tilde{A} \to \tilde{B}$ be a morphism in $\mathbf{Sp}(C)$ belonging to $W_{\mathbf{Sp}(\phi)}$. The morphism is represented by a morphism $f : A \to \Omega^n B$ in $C$, where $\tilde{A} \simeq f_{m+n}(A)$ and $\tilde{B} \simeq f_n(B)$ for some integers $n, m$, and $f_n : C \to \mathbf{Sp}(C)$ places $A$ at the $n$-stage of the system $\tilde{C}$ from (7.26). The condition that the morphism belongs to $W_{\mathbf{Sp}(\phi)}$ is equivalent to the condition that, after replacing $A$ by $\Omega^l A$, $n$ by $n + l$ and $m = m + l$ for sufficiently large $l$ in $\mathbb{N}$, the fibre of $A \to \Omega^m B$ belongs to the essential image of $\phi$, i.e., that $(A \to \Omega^m B) \in W_\phi$. But then $\mathbf{Sp}(\ell_\phi)(A \to B)$ is an equivalence. We get the dashed arrow by the universal property of the morphism $\ell_{\mathbf{Sp}(\phi)}$. It is easy to see that the dotted and the dashed arrows are inverse to each other. \qed
Definition 7.45. We define the stable cofibre of \( \phi \) to be the stable \( \infty \)-category
\[
\text{Cofib}^s(\phi) := \widetilde{\text{Sp}}(W_{\phi}^{-1}C).
\]

We have a natural left-exact functor
\[
(7.28) \quad \ell^s_\phi : C \rightarrow W_{\phi}^{-1}C \rightarrow \widetilde{\text{Sp}}(W_{\phi}^{-1}C) \simeq \text{Cofib}^s(\phi).
\]

Remark 7.46. We consider \( \text{Cofib}^s(\phi) \) as the legitimate cofibre of \( \phi \). The morphism (7.28) has the appropriate universal property if one tests with stable \( \infty \)-categories.

We consider the bold part of the following diagram in \( \text{Cat}_{\infty, *}^{\text{Lex}} \):
\[
\begin{array}{ccc}
D & \xrightarrow{\phi} & C \\
\downarrow & & \downarrow \ell^s_\phi \\
0 & \xrightarrow{\sigma} & \text{Cofib}^s(\phi) \\
\end{array}
\]

If \( E \) is stable, then \( \sigma \) sends the morphisms in \( W_\phi \) to equivalences, and we get an essentially unique dotted arrow. In case that \( D \) and \( C \) are in \( \text{Cat}_{\infty}^{\text{Lex}} \), this shows that \( C/D \simeq \text{Cofib}^s(\phi) \) in \( \text{Cat}_{\infty}^{\text{Lex}} \).

Lemma 7.47. If \( \phi \) is fully faithful, then the sequence (7.29) is a Verdier sequence.

Proof. \( \widetilde{\text{Sp}}(\phi) \) is fully faithful by Lemma 7.43. It remains to show that
\[
(7.30) \quad \widetilde{\text{Sp}}(C)/\widetilde{\text{Sp}}(D) \simeq \text{Cofib}^s(\phi).
\]

By Remark 7.46, we have the first equivalence in
\[
\widetilde{\text{Sp}}(C)/\widetilde{\text{Sp}}(D) \simeq \text{Cofib}^s(\widetilde{\text{Sp}}(\phi)) \overset{\text{Definition 7.45}}{\simeq} \widetilde{\text{Sp}}(W_{\widetilde{\text{Sp}}(\phi)}^{-1}\widetilde{\text{Sp}}(C)),
\]
where the quotient is interpreted in \( \text{Cat}_{\infty}^{\text{Lex}} \). By Lemma 7.44, the category \( W_{\widetilde{\text{Sp}}(\phi)}^{-1}\widetilde{\text{Sp}}(C) \) is already stable. This gives the first equivalence in
\[
\widetilde{\text{Sp}}(W_{\widetilde{\text{Sp}}(\phi)}^{-1}\widetilde{\text{Sp}}(C)) \simeq W_{\widetilde{\text{Sp}}(\phi)}^{-1}\widetilde{\text{Sp}}(C) \overset{\text{Lemma 7.44}}{\simeq} \text{Cofib}^s(\phi). \quad \square
\]

7.5. Excisive squares in \( \text{Cat}_{\infty, *}^{\text{Lex}} \). We consider a commutative square
\[
(7.31) \quad \begin{array}{ccc}
D & \xrightarrow{\phi} & C \\
\psi' \downarrow & & \downarrow \psi \\
D' & \xrightarrow{\phi'} & C'
\end{array}
\]
in \( \text{Cat}_{\infty, *}^{\text{Lex}} \).
Lemma 7.48. We get an induced functor
\[ \tilde{\psi} : \text{Cofib}^\ast(\phi) \to \text{Cofib}^\ast(\phi') . \]

Proof. We first show that \( \psi(W_\phi) \subseteq W_{\phi'} \). Assume that \( f \) belongs to \( W_\phi \). Then there exists \( D \) in \( D \) such that \( \text{Fib}(f) \simeq \phi(D) \). Since \( \psi \) preserves pullbacks we get the first equivalence in the following chain
\[ \text{Fib}(\psi(f)) \simeq \psi(\text{Fib}(f)) \simeq \psi(\phi(D)) \simeq \phi'(\psi'(D)) . \]
Hence \( \psi(f) \) belongs to \( W_{\phi'} \). We get an induced morphism between left-exact localisations
\[ W^{-1}_\phi C \to W^{-1}_{\phi'} C' . \]
We finally apply the functor \( \tilde{\text{Sp}} \) in order to get the desired functor between the stable cofibres. □

Definition 7.49. A square (7.31) in \( \text{Cat}^{\text{Lex}}_{\infty,*} \) is called excisive if:

1. The functors \( \phi : D \to C \) and \( \phi' : D' \to C' \) are fully faithful.
2. The induced functor on stable cofibres \( \psi : \text{Cofib}^\ast(\phi) \to \text{Cofib}^\ast(\phi') \) is an equivalence. ♦

We consider an \( I \)-indexed diagram of squares of the shape (7.31).

Lemma 7.50. Assume:

1. One of the following holds:
   
   a. \( I \) is filtered.
   
   b. \( I \) is a groupoid.

2. The evaluation of the diagram at every object of \( I \) is an excisive square in \( \text{Cat}^{\text{Lex}}_{\infty,*} \).

Then
\[ \colim_I D \xrightarrow{\colim_I \phi} \colim_I C \]
\[ \colim_I \psi' \downarrow \quad \quad \quad \colim_I \psi \]
\[ \colim_I D \xrightarrow{\colim_I \phi'} \colim_I C' \]

is an excisive square in \( \text{Cat}^{\text{Lex}}_{\infty,*} \).

Proof. If \( I \) is filtered, then the functors \( \colim_I \phi \) and \( \colim_I \phi' \) are fully faithful since a filtered colimit of fully faithful functors in \( \text{Cat}^{\text{Lex}}_{\infty,*} \) (which can be calculated in \( \text{Cat}_{\infty} \), see Proposition 7.9) is again fully faithful. In the other case, i.e., when \( I \) is a groupoid, we use Lemma 7.19 in order to conclude fully faithfulness.

By (7.30), we have an equivalence
\[ \text{Cofib}^\ast(\colim_I \phi) \simeq \tilde{\text{Sp}}(\colim_I D)/\tilde{\text{Sp}}(\colim_I C) \]
in \( \text{Cat}^{\text{ex}}_{\infty} \). Since \( \tilde{\text{Sp}} \) preserves colimits and colimits preserve quotients, we have
\[ \colim_I \text{Cofib}^\ast(\phi) \simeq \text{Cofib}^\ast(\colim_I \phi) . \]
This equivalence implies that the diagram (7.32) induces an equivalence
\[ \text{Cofib}^\ast(\colim_I \phi) \xrightarrow{\simeq} \text{Cofib}^\ast(\colim_I \phi') . \] □
Lemma 7.51. If square (7.31) is excisive, then

\[(7.33) \quad \text{Idem}(C) \xrightarrow{\text{Idem}(\phi)} \text{Idem}(D) \xrightarrow{\text{Idem}(\psi)} \\text{Idem}(C') \xrightarrow{\text{Idem}(\psi')} \text{Idem}(D')\]

is excisive.

Proof. By Lemma 7.15, the horizontal maps in (7.33) are fully faithful. It thus remains to show that the induced morphisms between the stable cofibres of the horizontal morphisms is an equivalence. By (7.30), we have equivalences

\[
\text{Cofib}^s(\phi) \simeq \overline{\text{Sp}(D)}/\overline{\text{Sp}(C)} \quad \text{Cofib}^s(\phi') \simeq \overline{\text{Sp}(D')}/\overline{\text{Sp}(C')} \]

in $\text{Cat}^{ex}_\infty$. We have a commutative square

\[
\begin{array}{ccc}
\text{Cat}^{ex, perf}_\infty & \longrightarrow & \text{Cat}^{ex}_\infty \\
\downarrow & & \downarrow \\
\text{Cat}^{lex, perf}_\infty & \longrightarrow & \text{Cat}^{lex}_\infty
\end{array}
\]

of inclusions of full subcategories. Their left adjoints are stabilisation and idempotent completion functors. We further observe that stabilisation preserves idempotent completeness and idempotent completion preserves stability. Consequently, we get a commutative square

\[
\begin{array}{ccc}
\text{Cat}^{lex}_\infty & \longrightarrow & \text{Cat}^{lex, perf}_\infty \\
\downarrow & & \downarrow \\
\text{Cat}^{ex}_\infty & \longrightarrow & \text{Cat}^{ex, perf}_\infty
\end{array}
\]

expressing the fact that Idem commutes with stabilisation. Furthermore the adjunction (7.14) restricts to an adjunction

\[
\text{Idem} : \text{Cat}^{ex}_\infty \rightleftarrows \text{Cat}^{ex, perf}_\infty : \text{incl}.
\]

The functor $\text{Idem} : \text{Cat}^{ex}_\infty \rightleftarrows \text{Cat}^{ex, perf}_\infty$ preserves quotients since, being a left adjoint, it preserves colimits. All these observation together imply an equivalence

\[
\text{Idem}(\text{Cofib}^s(\phi)) \simeq \text{Cofib}^s(\text{Idem}(\phi)),
\]

and similarly for $\phi'$ in place of $\phi$. By assumption, we have an equivalence $\text{Cofib}^s(\phi) \simeq \text{Cofib}^s(\phi')$ induced by (7.31). Consequently, (7.33) induces an equivalence

\[
\text{Cofib}^s(\text{Idem}(\phi)) \simeq \text{Cofib}^s(\text{Idem}(\phi')).
\]

\[\square\]

7.6. The universal property of the bounded derived category. Let $A$ be a small additive (ordinary) category. By $\text{Ch}^b(A)_\infty$ we denote the localisation of (the nerve of) the category of bounded chain complexes $\text{Ch}^b(A)$ by chain homotopy equivalences. Then $\text{Ch}^b(A)_\infty$ is a stable $\infty$-category, equipped with a canonical finite coproduct-preserving functor

\[(7.34) \quad z_A : A \rightarrow \text{Ch}^b(A)_\infty\]
sending an object of $\mathcal{A}$ to the corresponding chain complex concentrated in degree zero. The purpose of this subsection is to prove that the functor $z_\mathcal{A}$ above is the universal finite coproduct-preserving functor from $\mathcal{A}$ to a stable $\infty$-category. As a consequence, we show that the functor $\text{Ch}^b(-)_\infty$ commutes with colimits indexed by small groupoids.

Let $\mathcal{C}$ be an $\infty$-category.

**Definition 7.52.** $\mathcal{C}$ is additive if it is semi-additive (see Definition 7.20) and its homotopy category $\text{ho}(\mathcal{C})$ is additive. ♦

Given a locally small additive $\infty$-category $\mathcal{C}$, we let $\mathcal{P}_\Sigma(\mathcal{C})$ be the $\infty$-category of additive presheaves on $\mathcal{C}$, i.e., the $\infty$-category of finite product-preserving functors from $\mathcal{C}^{\text{op}}$ to the $\infty$-category of spaces $\text{Spc}$. Given two additive $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\text{Fun}_\Sigma(\mathcal{C}, \mathcal{D})$ the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ spanned by finite coproduct-preserving functors.

Given a right-exact $\infty$-category $\mathcal{C}$, we denote by $\text{SW}(\mathcal{C}) := \tilde{\text{Sp}}(\mathcal{C}^{\text{op}})^{\text{op}}$ (see Definition 7.42) the Spanier-Whitehead stabilisation of $\mathcal{C}$; see also [Lurb, Constr. C.1.1.1].

Let $\mathcal{C}$ be an $\infty$-category.

**Definition 7.53** ([Lurb, Def. C.1.2.1]). $\mathcal{C}$ is called prestable if it satisfies the following conditions:

1. $\mathcal{C}$ is right-exact.
2. The suspension functor $\Sigma: \mathcal{C} \to \mathcal{C}$ is fully faithful.
3. $\mathcal{C}$ is closed under extensions: For every morphism $f: Y \to \Sigma Z$ in $\mathcal{C}$ there exists a pullback square

   \[
   \begin{array}{ccc}
   X & \xrightarrow{f'} & Y \\
   \downarrow & & \downarrow f \\
   0 & \to & \Sigma Z
   \end{array}
   \]

   Furthermore, this square is also a pushout. ♦

Let $\mathcal{A}$ be a small additive category.

**Proposition 7.54.**

1. $\mathcal{P}_\Sigma(\mathcal{A})$ is prestable.
2. $\mathcal{P}_\Sigma(\mathcal{A})$ is additive.
3. The canonical functor $\mathcal{P}_\Sigma(\mathcal{A}) \to \text{SW}(\mathcal{P}_\Sigma(\mathcal{A}))$ is fully faithful.
4. The essential image of the functor from (3) is closed under extensions.

**Proof.** Assertion (1) follows from [Lurb, Prop. C.1.5.7]. The remaining assertions are then given by [Lurb, Ex. C.1.5.6] and [Lurb, Prop. C.1.2.2]. □

Let $\mathcal{P}_{\Sigma,f}(\mathcal{A})$ be the smallest prestable subcategory of $\mathcal{P}_\Sigma(\mathcal{A})$ containing the representable presheaves.

**Proposition 7.55.** There is a canonical equivalence of $\infty$-categories

\[\text{Ch}^b(\mathcal{A})_\infty \xrightarrow{\simeq} \text{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{A}))\]
fitting into the commutative diagram

\[(7.35)\]

\[ \begin{array}{ccc}
A \& \xrightarrow{\mathcal{Z}_A} \mathbb{R} \& \xrightarrow{\zeta} \mathcal{P}_{\Sigma, f} (A) \\
\xrightarrow{\text{Yoneda}} \& \xrightarrow{\approx} \& \\
\mathcal{P}_{\Sigma, f} (A) \& \rightarrow \& \mathbf{S} \mathbb{W} (\mathcal{P}_{\Sigma, f} (A))
\end{array} \]

**Proof.** Let \( \mathcal{A} := \text{Fun}_{\Sigma}(\mathcal{A}^{\text{op}}, \text{Ab}) \) be the abelian category of additive presheaves on \( A \) with values in abelian groups. This is an abelian category with a set of compact projective generators provided by the image of the Yoneda embedding \( y_A : A \rightarrow A \).

Note that any projective in \( A \) is a retract of a sum of representables.

Let \( D^- (A) := \mathbb{N} \mathbb{D} (\mathcal{C}^- (\mathcal{A}_{\text{proj}})) \) be the bounded below derived category of \( A \), where \( \mathcal{A}_{\text{proj}} \) denotes the full subcategory of \( A \) spanned by the projective objects.

Note that \( y_A \) induces a fully faithful functor \( \mathcal{C}^- (A) \rightarrow \mathcal{C}^- (\mathcal{A}_{\text{proj}}) \) of dg-categories.

In view of the equivalence \( \mathcal{C}^- (A) \rightarrow \mathcal{C}^- (\mathcal{A}_{\text{proj}}) \) (see [BC, Rem. 2.9]), we get a fully faithful functor

\[(7.36)\]

\[ \mathcal{C}^- (A) \rightarrow D^- (A) . \]

By [Pst, Lem. 2.58], there is an equivalence \( D^- (A)_{\geq 0} \rightarrow \mathcal{P}_{\Sigma} (A) \) which fits into the following commutative diagram:

\[ \begin{array}{ccc}
D^- (A)_{\geq 0} \& \xrightarrow{\approx} \& \mathcal{P}_{\Sigma} (A) \\
\xrightarrow{\mathcal{A}_{\text{proj}}} \& \xrightarrow{y_A} \& \xrightarrow{\text{Yoneda}} A
\end{array} \]

Define \( \mathcal{C}^- _{\geq 0} (A) \) as the smallest prestable subcategory of \( \mathcal{C}^- (A) \) containing the essential image of \( z_A \). Then (7.36) induces a fully faithful functor \( \mathcal{C}^- _{\geq 0} (A) \rightarrow \mathcal{P}_{\Sigma} (A) \) whose essential image is by definition \( \mathcal{P}_{\Sigma, f} (A) \), and which fits into the following commutative diagram:

\[(7.37)\]

\[ \begin{array}{ccc}
\mathcal{C}^- _{\geq 0} (A) \& \xrightarrow{\approx} \& \mathcal{P}_{\Sigma, f} (A) \\
\xrightarrow{z_A} \& \xrightarrow{\text{Yoneda}} \& A
\end{array} \]

Since \( \mathcal{C}^- (A) \approx \mathbf{S} \mathbb{W} (\mathcal{C}^- _{\geq 0} (A) \approx) \), the proposition follows by applying \( \mathbf{S} \mathbb{W} \) to the upper horizontal equivalence in (7.37). \( \square \)

Let \( A \) be a small additive category, and let \( C \) be a stable \( \infty \)-category.

**Theorem 7.56.** We have an equivalence of \( \infty \)-categories

\[(7.38)\]

\[ - \circ z_A : \text{Fun}_{\text{Cat}} (\mathcal{C}^- (A) \rightarrow \text{Fun}_{\Sigma}(A, C) . \]

\[10\] [Pst] formulates the result for the unbounded derived category \( D(A) \), but note that the connective parts of \( D^- (A) \) and \( D(A) \) agree.
Proof. We can factorise the morphism in question as follows:

\[
\text{Fun}_{\text{Cat}^{\infty}}(\text{Ch}_b(A)_{\infty}, C) \xrightarrow{!} \text{Fun}_{\text{Cat}^{\infty}}(\text{SW}(P_{\Sigma f}(A), C) \\
\cong \text{Fun}_{\text{Cat}_{\text{ex}}^{\infty}}(\mathcal{P}_{\Sigma f}(A), C) \\
\cong \text{Fun}_{\text{Cat}_{\text{rex}}^{\infty}}(\mathcal{P}_{\Sigma}(A), \text{Ind}_\omega(C)) \\
\cong \text{Fun}_{\Sigma}(A, C)
\]

For the equivalence marked by ! we use Proposition 7.55. The equivalence marked by !! follows since \( C \) is stable. \( \text{Fun}^1 \) stands for small colimit-preserving functors which in addition send representable presheaves to the essential image of the canonical functor \( C \to \text{Ind}_\omega(C) \). The equivalences !!! and !!!! are then clear. □

Remark 7.57. One can prove Proposition 7.55 (hence also Theorem 7.56) by using directly the version of the Dold–Kan correspondence provided by [Lura, Prop. 1.3.2.23] and by adapting the proof of [Lura, Prop. 1.3.2.22] in the case where \( A \) is an idempotent-complete additive category, and then by embedding \( A \) into its idempotent completion to reach the general case.

Although this will not be needed in these notes, we add the following natural consequences of Theorem 7.56, since this gives a natural way to see bounded derived categories which does not seem to be documented in the published literature.

For a small exact category \( E \), we recall that the derived category of \( D^b(E) \) is the quotient of \( \text{Ch}_b(E)_{\infty} \) by the thick subcategory of bounded acyclic complexes with coefficients in \( E \). In the case \( A \) is a small abelian category, seen as an exact category for which the admissible short exact sequences are all short exact sequences, the stable \( \infty \)-category \( D^b(A) \) simply is the localisation of the category of bounded chain complexes of \( A \) by quasi-isomorphisms (so that the induced 1-category is the usual bounded derived category).

Let \( E \) be a small exact category and let \( f: \text{Ch}_b(E)_{\infty} \to C \) be an exact functor with values in a stable \( \infty \)-category.

**Proposition 7.58.** We assume that, for any admissible short exact sequence

\[
0 \to x \to y \to z \to 0
\]

in \( E \), the induced commutative square

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & f(z) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{f(x)} & f(y)
\end{array}
\]

is cocartesian. Then, for any acyclic \( k \) in \( \text{Ch}_b(E) \) we have \( f(k) \simeq 0 \).

**Proof.** We first prove that, for any admissible short exact sequence of bounded chain complexes

\[
0 \to k' \to k \to k'' \to 0
\]
the induced square

\[
\begin{array}{ccc}
f(k') & \rightarrow & f(k) \\
\downarrow & & \downarrow \\
0 & \rightarrow & f(k'')
\end{array}
\]

is cocartesian in $C$. We do this by induction on the amplitude $N$ of $k$ (i.e., the
biggest integer $N$ so that there are integers $a \leq b$ with $b-a \geq N$ and the components
of $k$ in degrees $a$ and $b$ are non zero). If $N \leq 1$, this holds by assumption. If $N > 1$,
using “troncation bête”, we see that the admissible short exact sequence above fits
in a homotopy cofibre sequence in $\text{Ch}^b(E)_\infty$ (written vertically below) of admissible
short exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & k' & \rightarrow & k & \rightarrow & k'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & k' & \rightarrow & k & \rightarrow & k'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & k' & \rightarrow & k & \rightarrow & k'' & \rightarrow & 0
\end{array}
\]

where $k_i$ has amplitude $< N$ for $i = 1, 2$, and we conclude by induction.

Now, if $k$ is a bounded acyclic complex of amplitude $N$, then there is an admissible
short exact sequence of the form

\[
0 \rightarrow k' \rightarrow k \rightarrow k'' \rightarrow 0 ,
\]

where both $k'$ and $k''$ are acyclic, $k'$ is of amplitude 1 (i.e., $k'$ is a mapping cone of
an isomorphism of objects of $E$), and $k''$ is of amplitude $< N$; see [TT90, diagram
(1.11.7.6)], for instance. It is clear that $k' \simeq 0$ in $\text{Ch}^b(E)_\infty$, and therefore, that
$f(k) \simeq f(k'')$ in $C$. Therefore, by induction on $N$, we see that $f(k) \simeq 0$. □

Let $E$ be a small exact category.

**Corollary 7.59.** For any stable $\infty$-category $C$, composing with the canonical functor
$E \rightarrow \text{D}^b(E)$ induces an equivalence of $\infty$-categories from the $\infty$-category of exact
functors $F: \text{D}^b(E) \rightarrow C$ to the $\infty$-category of finite coproduct-preserving functors
$f: E \rightarrow C$ which send each admissible short exact sequence

\[
0 \rightarrow x \rightarrow y \rightarrow z \rightarrow 0
\]

in $E$ to a cocartesian square

\[
\begin{array}{ccc}
f(x) & \rightarrow & f(y) \\
\downarrow & & \downarrow \\
0 & \rightarrow & f(z)
\end{array}
\]

in $C$.

**Proof.** We observe that, a functor $f: \text{Ch}^b(E)_\infty \rightarrow C$ sends each admissible short
exact sequence

\[
0 \rightarrow x \rightarrow y \rightarrow z \rightarrow 0
\]
in $E$ to a cocartesian square

\[
\begin{array}{ccc}
 f(x) & \rightarrow & f(y) \\
 \downarrow & & \downarrow \\
 0 & \rightarrow & f(z)
\end{array}
\]

if and only if it factors through $D^b(E)$: this follows right away from the definition of $D^b(E)$ and from Proposition 7.58. This corollary is then a direct consequence of Theorem 7.56.

**Remark 7.60.** Corollary 7.59 explains how to compare a triangulated category equipped with a $t$-structure with the bounded derived category of its heart: if $C$ is a stable $\infty$-category with a $t$-structure ($C_{>0}, C_{\leq 0}$) whose heart is $A = (C_{>0} \cap C_{\leq 0})$, then there is a unique exact functor $D^b(A) \rightarrow C$ extending the inclusion $A \subset C$.

The functor $\text{Ch}^{b}(-)_\infty: \text{Add} \rightarrow \text{Cat}^\infty_{\text{Lex},*}$ preserves equivalences of additive categories and therefore induces a functor (by abuse of notation denoted by the same symbol) $\text{Ch}^{b}(-)_\infty: \text{Add}_\infty \rightarrow \text{Cat}^\infty_{\text{Lex}}$.

**Theorem 7.61.** The functor $\text{Ch}^{b}(-)_\infty: \text{Add}_\infty \rightarrow \text{Cat}^\infty_{\text{Lex}}$ preserves colimits indexed by small groupoids.

The proof of this theorem requires some preparations. We first introduce the free $G$-object functor and study its basic properties.

Let $\text{ADD}^\infty_\infty$ denote the large $\infty$-category of additive categories admitting all small coproducts and coproduct-preserving functors. Furthermore, let $\text{CAT}^\infty_\infty$ denote the $\infty$-category of large $\infty$-categories admitting small coproducts and coproduct-preserving functors. These $\infty$-categories are complete and connected by limit-preserving functors

\[\text{ADD}^\infty_\infty \rightarrow \text{CAT}^\infty_\infty \rightarrow \text{CAT}_\infty.\]

Let $G$ be a group. For $C$ in $\text{Fun}(BG, \text{CAT}^\infty_\infty)$ we have a canonical functor (an instance of (7.5))

\[\eta_C: \lim_{BG} C \rightarrow C.\]

**Lemma 7.62.** There exists a natural transformation

\[\mathcal{F}: \text{id}_{\text{Fun}(BG, \text{CAT}^\infty_\infty)} \rightarrow \lim(-)_{BG}\]

such that $\mathcal{F}: C \rightarrow \lim_{BG} C$ is left adjoint to $\eta_C$ for every $C$ in $\text{Fun}(BG, \text{CAT}^\infty_\infty)$.

**Proof.** Let $ev: \text{Fun}(BG, \text{CAT}^\infty_\infty) \rightarrow \text{CAT}^\infty_\infty$ be the evaluation functor. Consider a morphism $f: C \rightarrow D$ in $\text{Fun}(BG, \text{CAT}^\infty_\infty)$. We claim that the induced diagram

\[
\begin{array}{ccc}
\lim_{BG} C & \xrightarrow{ev(\eta_C)} & ev(C) \\
\downarrow_{\lim f} & & \downarrow_{ev(f)} \\
\lim_{BG} D & \xrightarrow{ev(\eta_D)} & ev(D)
\end{array}
\]

is left adjointable.
We consider \( G \)-sets as \( \infty \)-categories with \( G \)-action. The equivariant projection map \( p : G \to * \) induces a functor
\[
p^* : C \simeq \text{Fun}(*, C) \to \text{Fun}(G, C)
\]
in \( \text{Fun}(BG, \text{CAT}^{\mathbb{I}}_{\text{\infty}}) \). We define \( p^*,G := \lim_{BG} p^* \) and get the commutative diagram
\[
\begin{array}{ccc}
\text{lim}_{BG} C & \xrightarrow{p^*,G} & \text{lim}_{BG} \text{Fun}(G, C) \\
\downarrow & & \downarrow \text{ev}(\eta_{\text{Fun}(G, C)}) \\
\text{ev}(C) & \xrightarrow{\text{ev}(p^*)} & \text{ev}(\text{Fun}(G, C)) \\
\end{array}
\]
where \( i_e : * \to G \) is the inclusion of the identity, the left square commutes by the naturality of \( \eta \), and the lower triangle commutes since \( i_e \circ \text{ev}(p) = \text{id}_* \). The right vertical composition is an equivalence (for example by Lemma 2.58 whose proof works for every \( \infty \)-category \( C \)). In order to show the claim, it thus suffices to show that
\[
\begin{array}{ccc}
\text{lim}_{BG} C & \xrightarrow{p^*,G} & \text{lim}_{BG} \text{Fun}(G, C) \\
\downarrow \text{lim}_{BG} f & & \downarrow \text{lim}_{BG} f_* \\
\text{lim}_{BG} D & \xrightarrow{p^*,G} & \text{lim}_{BG} \text{Fun}(G, D) \\
\end{array}
\]
is left adjointable. Since \( G \) is a discrete \( \infty \)-category and both \( C \) and \( D \) admit small coproducts, the left Kan extension functors \( p_! \) along \( p \) exist and provide left adjoints to \( p^* \). Then \( p^!,G := \lim_{BG} p_! \) is a left adjoint of \( p^*,G \).

Let \( \tau : p^!,G \circ \text{lim}_{BG} f_* \to (\text{lim}_{BG} f) \circ p^!,G \) be the associated base change transformation. Using the pointwise formula for the left Kan extension, \( e(\tau) \) at \( C \) in \( \text{ev}(\text{Fun}(G, C)) \) (see (7.7)) is given by the canonical map \( \coprod_{g \in G} f(C(g)) \to f(\coprod G C(g)) \) in \( \text{ev}(D) \), which is an equivalence since \( f \) is coproduct-preserving. In view of Lemma 7.11, this proves the claim.

If we replace both instances of \( \eta \) in (7.41) by the action map of a group element \( g \), the resulting diagram is also left adjointable since \( g \) is invertible.

Considering the natural transformation \( \eta \) as a functor
\[
\eta : \text{Fun}(BG, \text{CAT}^{\mathbb{I}}_{\text{\infty}}) \to \text{Fun}(\Delta^1 \times BG, \text{CAT}_{\text{\infty}}),
\]
this shows that \( \eta \) takes values in the subcategory \( \text{Fun}^{\text{LAd}}(\Delta^1 \times BG, \text{CAT}_{\text{\infty}}) \) of \( \text{Fun}(\Delta^1 \times BG, \text{CAT}_{\text{\infty}}) \) (see [Lura, Def. 4.7.4.16] for the definition of \( \text{Fun}^{\text{LAd}} \)). By [Lura, Cor. 4.7.4.18], we obtain a functor
\[
\mathcal{F}^r : \text{Fun}(BG, \text{CAT}^{\mathbb{I}}_{\text{\infty}}) \to \text{Fun}((\Delta^1 \times BG)^{\text{op}}, \text{CAT}_{\text{\infty}}).
\]
The inversion map \( \iota : BG \to BG^{\text{op}} \) induces an equivalence
\[
\text{Fun}((\Delta^1 \times BG)^{\text{op}}, \text{CAT}_{\text{\infty}}) \simeq \text{Fun}(\Delta^{1,\text{op}} \times BG, \text{CAT}^{\mathbb{I}}_{\text{\infty}}).
\]
Composing \( \mathcal{F}^r \) with this equivalence gives the desired functor \( \mathcal{F} \).

**Definition 7.63.** We call \( \mathcal{F}_C \) the free \( G \)-object functor.
Corollary 7.64. There exists a natural transformation
\[ F^{\text{add}} : \text{id}_{\text{Fun}(BG, \text{ADD}_{\infty}^\Pi)} \to \lim_{BG} (-) \]
such that \( F : C \to \lim_{BG} C \) is left adjoint to \( \eta_C \) for every \( C \) in \( \text{Fun}(BG, \text{ADD}_{\infty}^\Pi) \).
Moreover, \( \iota F^{\text{add}} \simeq F \iota \).

Proof. Since \( \iota \) is limit-preserving and fully faithful, \( F \iota \) defines the required transformation \( F^{\text{add}} \) in \( \text{ADD}_{\infty}^\Pi \). Then \( \iota F^{\text{add}} \simeq F \iota \) holds by definition. \( \square \)

For an \( \infty \)-category \( C \) and \( C \) in \( \text{Fun}(BG, C) \) we use the notation \( c_C : C \to \colim_{BG} C \) for the canonical morphism in \( \text{Fun}(BG, C) \).

We now consider \( A \) in \( \text{Fun}(BG, \text{Add}_{\infty}) \). Then we have \( \text{Fun}_\Sigma(A^{op}, \text{Ab}) \in \text{Fun}(BG, \text{ADD}_{\infty}^\Pi) \).

Lemma 7.65. The following square commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{Y_0} & \text{Fun}_\Sigma(A^{op}, \text{Ab}) \\
\colim_{BG} A & \xrightarrow{c_A} & \text{Fun}_\Sigma(\colim_{BG} A^{op}, \text{Ab}) & \xrightarrow{\simeq} & \lim_{BG} \text{Fun}(A^{op}, \text{Ab}) \\
& \downarrow F_{\text{Fun}_\Sigma(A^{op}, \text{Ab})} & & \downarrow F_{\text{Fun}(A^{op}, \text{Ab})} & \\
& \text{Fun}_\Sigma(\colim_{BG} A^{op}, \text{Ab}) & \xrightarrow{c_A} & \text{Fun}_\Sigma(A^{op}, \text{Ab}) & \\
\end{array}
\]

Proof. We have a commutative square

\[
\begin{array}{c}
\lim_{BG} \text{Fun}_\Sigma(A^{op}, \text{Ab}) \xrightarrow{\eta_{\text{Fun}_\Sigma(A^{op}, \text{Ab})}} \text{Fun}_\Sigma(A^{op}, \text{Ab}) \\
\simeq \downarrow \downarrow \\
\text{Fun}_\Sigma(\colim_{BG} A^{op}, \text{Ab}) \xrightarrow{c_A} \text{Fun}_\Sigma(A^{op}, \text{Ab})
\end{array}
\]

Consequently, the left adjoint \( F_{\text{Fun}_\Sigma(A^{op}, \text{Ab})} \) of \( \eta_{\text{Fun}_\Sigma(A^{op}, \text{Ab})} \) is equivalent to the left Kan extension functor along \( c_A \). The assertion of the lemma exactly expresses this fact. \( \square \)

We now consider \( C \) in \( \text{Fun}(BG, \text{Cat}_{\infty,a}^{\text{Lex}}) \). Then we have \( \text{Fun}_\Sigma(C^{op}, \text{Sp}) \in \text{Fun}(BG, \text{CAT}_{\infty}^\Pi) \).

Lemma 7.66. The following square commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{Y_0} & \text{Fun}_{\text{Cat}_{\infty,a}^{\text{Lex}}}(C^{op}, \text{Sp}) \\
\colim_{BG} C & \xrightarrow{c_C} & \text{Fun}_{\text{Cat}_{\infty,a}^{\text{Lex}}}(\colim_{BG} C^{op}, \text{Sp}) & \xrightarrow{\simeq} & \lim_{BG} \text{Fun}(C^{op}, \text{Sp}) \\
& \downarrow F_{\text{Fun}(C^{op}, \text{Sp})} & & \downarrow F_{\text{Fun}(C^{op}, \text{Sp})} & \\
& \text{Fun}_{\text{Cat}_{\infty,a}^{\text{Lex}}}(\colim_{BG} C^{op}, \text{Sp}) & \xrightarrow{c_C} & \text{Fun}_{\text{Cat}_{\infty,a}^{\text{Lex}}}(C^{op}, \text{Sp}) & \\
\end{array}
\]

Proof. The proof is completely analogous to the proof of Lemma 7.65. \( \square \)
We now consider $A$ in $\text{Fun}(BG, \text{Add}_\infty)$ and form $\text{colim}_{BG} A$ in $\text{Add}_\infty$. Furthermore, we form $\text{Ch}^b(A)_\infty$ in $\text{Fun}(BG, \text{Cat}^{\text{ex}}_\infty)$ and consider $\text{colim}_{BG} \text{Ch}^b(A)_\infty$ in $\text{Cat}^{\text{ex}}_\infty$. Let $\text{Cat}^{\text{add}}_\infty$ denote the $\infty$-category of additive $\infty$-categories and finite coproduct-preserving functors. Using the inclusions

$$\text{Add}_\infty \to \text{Cat}^{\text{add}}_\infty \text{ and } \text{Cat}^{\text{ex}}_\infty \to \text{Cat}^{\text{add}}_\infty,$$

we can consider these objects as small additive $\infty$-categories.

**Proposition 7.67.** There exists a morphism in $\text{Fun}(BG, \text{Cat}^{\text{add}}_\infty)$

$$b_A : \text{colim}_{BG} A \to \text{colim} \text{Ch}^b(A)_\infty$$

fitting into a commutative square

$$\begin{array}{ccc}
A & \xrightarrow{c_A} & \text{colim} A \\
\downarrow{z_A} & & \downarrow{b_A} \\
\text{Ch}^b(A)_\infty & \xrightarrow{c_{\text{Ch}^b(A)_\infty}} & \text{colim} \text{Ch}^b(A)_\infty
\end{array}$$

in $\text{Fun}(BG, \text{Cat}^{\text{add}}_\infty)$.

**Proof.** Let $i^\otimes : \text{Ab} \simeq \text{Sp}^\otimes \to \text{Sp}$ be the canonical functor. As a first step, we construct a commutative diagram in $\text{Fun}(BG, \text{CAT}^{\text{add}}_\infty)$:

$$\begin{array}{ccc}
A & \xrightarrow{z_A} & \text{Ch}^b(A)_\infty \\
\text{Yo} & & \text{Yo} \\
\text{Fun}_\Sigma(A, \text{Ab}) & \xrightarrow{i^\otimes} & \text{Fun}_\Sigma(A, \text{Sp}) \\
\text{Fun}_\Sigma(\text{colim}_{BG} A, \text{Ab}) & \xrightarrow{i^\otimes} & \text{Fun}_\Sigma(\text{colim}_{BG} A, \text{Sp}) \\
\text{Fun}_{\text{Cat}^{\text{ex}}_\infty}(\text{colim} \text{Ch}^b(A)_\infty, \text{Sp}) & \xrightarrow{i^\otimes} & \text{Fun}_{\text{Cat}^{\text{ex}}_\infty}(\text{colim} \text{Ch}^b(A)_\infty, \text{Sp})
\end{array}$$

Part $2$ of the diagram commutes by Lemma 7.65, and part $5$ commutes by Lemma 7.66. The morphisms denoted by $z_A^*$ are equivalences by Theorem 7.56. The lower horizontal morphism is defined by the commutativity of part $6$ of the diagram.

In order to see that part $1$ of the diagram commutes, we observe that we have a natural equivalence of spectra

$$i^\otimes(\text{Hom}_A(A, A')) \simeq \text{Map}_{\text{Ch}^b(A)}(z_A(A), z_A(A'))$$

for all objects $A, A'$ of $A$. It is clear that part $3$ of the diagram commutes. It remains to show that part $5$ of the diagram commutes. We expand this square as
follows and use the notation $\iota$ as in (7.39) in order to indicate where the objects live:

$$
\begin{array}{ccc}
\iota(\text{Fun}_{\Sigma}(A^{\text{op}}, \text{Ab})) & \xrightarrow{\iota^\circ} & \text{Fun}_{\Sigma}(A^{\text{op}}, \text{Sp}) \\
\downarrow \iota(\text{Fun}_{\Sigma}(A^{\text{op}}, \text{Ab})) & \downarrow & \downarrow \\
\iota(\lim_{BG} \text{Fun}_{\Sigma}(A^{\text{op}}, \text{Ab})) & \xrightarrow{\iota^\circ} & \lim_{BG} \text{Fun}_{\Sigma}(A^{\text{op}}, \text{Sp})
\end{array}
$$

The left square commutes by Corollary 7.64, and the right square commutes by the naturality of $F$ (Lemma 7.62).

We now expand square (7.42) as follows:

$$
\begin{array}{ccc}
A & \xrightarrow{c_A} & \text{colim}_{BG} A \\
\downarrow z_A & & \downarrow b_A \\
\text{Ch}^b(A)_{\infty} & \xrightarrow{c_{\text{Ch}^b(A)_{\infty}}} & \text{colim}_{BG} \text{Ch}^b(A)_{\infty} \\
\downarrow & & \downarrow \\
\text{Fun}_{\text{Cat}^\infty_{\text{ex}}} \text{Ch}^b(A)_{\infty} & \xrightarrow{\iota^\circ} & \lim_{BG} \text{Fun}_{\text{Cat}^\infty_{\text{ex}}} \text{Ch}^b(A)_{\infty}
\end{array}
$$

The right vertical morphism is the lower horizontal morphism in diagram (7.43).

The filler for the outer square in diagram (7.44) is given by the filler of the outer square of (7.43).

The commutativity of (7.43) shows that the right-down composition in (7.44) takes values in the full subcategory given by the essential image of $Y_0$. Since $\text{ev}(c_A)$ is essentially surjective (by the explicit description of colimits in $\text{Add}_{\infty}$, see [BEKWh]) and the Yoneda embeddings are fully faithful, we obtain the desired factorisation $b_A$. \hfill \Box

**Proof of Theorem 7.61.** It suffices to show the theorem for colimits indexed by discrete categories and groupoids of the form $BG$ for a group $G$. The first case of discrete index categories, i.e., that $\text{Ch}^b(-)_{\infty}$ preserves coproducts, is easy and left to the reader. In the following, we give the details for the second case.\(^\dagger\)

The canonical morphism $c_A$ induces a morphism $\text{Ch}(c_A) : \text{Ch}^b(A)_{\infty} \to \text{Ch}^b(\text{colim } A)_{\infty}$ in $\text{Fun}(BG, \text{Cat}^\infty_{\text{ex}})$. By the universal property of $c_{\text{Ch}^b(A)_{\infty}}$, the morphism $\text{Ch}(c_A)$ fits into the following commutative diagram:

$$
\begin{array}{ccc}
\text{Ch}^b(A)_{\infty} & \xrightarrow{c_{\text{Ch}^b(A)_{\infty}}} & \text{colim } \text{Ch}^b(A)_{\infty} \\
\downarrow \text{Ch}(c_A) & & \downarrow \kappa \\
\text{Ch}^b(\text{colim } A)_{\infty}
\end{array}
$$

The assertion of Theorem 7.61 is that $\kappa$ is an equivalence.

\(^\dagger\)In the application Corollary 7.68 below, we only need this second case.
By the naturality of the transformation \( z_- : (-) \to \text{Ch}^b(-)_{\infty} \), we have a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{c_A} & \text{colim}_{BG} A \\
\downarrow z_A & & \downarrow z_{\text{colim}_{BG} A} \\
\text{Ch}^b(A)_{\infty} & \xrightarrow{\text{Ch}(c_A)} & \text{Ch}^b(\text{colim} A)_{\infty}
\end{array}
\]

in \( \text{Fun}(BG, \text{Cat}^{\text{add}}_{\infty}) \).

Proposition 7.67, (7.46) and (7.45) combine to a commutative diagram

\[
\begin{array}{ccc}
\text{colim} A & \xrightarrow{\text{colim}_{BG} A} & \text{Ch}^b(A)_{\infty} \\
\downarrow c_A & \downarrow \kappa \circ b_A & \downarrow \text{Ch}(c_A) \circ \kappa \\
\text{colim}_{BG} A & \xrightarrow{b_A} & \text{Ch}^b(\text{colim} A)_{\infty}
\end{array}
\]

in \( \text{Fun}(BG, \text{Cat}^{\text{add}}_{\infty}) \), where the essentially unique factorisation \( b^*_A \) is justified by Theorem 7.56. We will show that \( b^*_A \) is an inverse of \( \kappa \).

We first show that \( \kappa \circ b_A \) is equivalent to the identity. Since \( c_A \) is essentially surjective (by the explicit description of colimits in \( \text{Add}_{\infty} \)), the middle part of (7.47) implies that the essential image of \( \kappa \circ b_A \) is contained in \( z_{\text{colim}_{BG} A}(\text{colim}_{BG} A) \), i.e., \( \kappa \circ b_A \) has a factorisation over a morphism \( \sigma \) as in the commutative diagram

\[
\begin{array}{ccc}
\text{colim} A & \xrightarrow{\kappa \circ b_A} & \text{Ch}^b(\text{colim} A)_{\infty} \\
\downarrow \sigma & & \downarrow z_{\text{colim}_{BG} A} \\
\text{colim} A & \xrightarrow{\text{colim}_{BG} A} & \text{Ch}^b(\text{colim} A)_{\infty}
\end{array}
\]

By the commutativity of the whole diagram (7.47) and since the canonical functor \( \text{Add}_{\infty} \to \text{Cat}^{\text{add}}_{\infty} \) is fully faithful, we conclude that

\[
\begin{array}{ccc}
A & \xrightarrow{c_A} & \text{colim}_{BG} A \\
\downarrow c_A & & \downarrow \sigma \\
\text{colim}_{BG} A & \xrightarrow{\text{colim}_{BG} A} & \text{colim}_{BG} A
\end{array}
\]

commutes in \( \text{Fun}(BG, \text{Add}_{\infty}) \). Using the universal property of \( c_A \), we see that \( \sigma \) is equivalent to the identity. By Theorem 7.56, the resulting equivalence \( \kappa \circ b_A \simeq z_{\text{colim}_{BG} A} \) (see (7.48)) implies that \( \kappa \circ b_A^* \simeq \text{id} \).
It remains to show that $b^s_A \circ \kappa$ is equivalent to the identity. By the universal property of $c_{\text{Ch}^b(A)_\infty}$, it suffices to check that
\[ b^s_A \circ \kappa \circ c_{\text{Ch}^h(A)_\infty} \simeq c_{\text{Ch}^h(A)_\infty} . \]

Note that
\[ b^s_A \circ \kappa \circ c_{\text{Ch}^h(A)_\infty} \simeq b^s_A \circ \text{Ch}(c_A) \]
by (7.45). By Theorem 7.56, it suffices to show that
\[ b^s_A \circ \text{Ch}(c_A) \circ z_A \simeq c_{\text{Ch}^h(A)_\infty} \circ z_A . \]
This equivalence follows from a combination of (7.46), (7.47), and Proposition 7.67, giving the chain of equivalences
\[ b^s_A \circ \text{Ch}(c_A) \circ z_A \simeq b^s_A \circ z_{\text{colim}_BG \bullet A} \circ c_A \simeq b_A \circ c_A \simeq c_{\text{Ch}^h(A)_\infty} \circ z_A . \]
\qed

Recall that $\text{Ind}^G$ denotes the left Kan extension functor along the inclusion $BG \to G\text{Orb}$ (see (1.2)). Let $A$ be in $\text{Fun}(BG, \text{Ad}_{\infty})$.

**Corollary 7.68.** The canonical transformation
\[ (7.49) \quad \text{Ind}^G(\text{Ch}^b(A)_\infty) \to \text{Ch}^b(\text{Ind}^G(A))_\infty \]
is an equivalence.

**Proof.** For a subgroup $H$ of $G$ we have an equivalence $BH \xrightarrow{\cong} G\text{Orb}_{/BH}$ which sends $*_{BH}$ to $G \to G/H$ in $G\text{Orb}_{/BH}$ and $h$ in $H \cong \text{Aut}_{BH}(*)_{BH}$ to the right multiplication by $h$ on $G$. The evaluation of the transformation (7.49) at $G/H$ becomes equivalent to the morphism $\text{colim}_{BH} \text{Ch}^b(A)_\infty \to \text{Ch}^b(\text{colim}_{BH} A)_\infty$ which is an equivalence by Theorem 7.61. \qed

### 7.7. $A$-theory as a $G\text{Orb}$-spectrum

In Example 1.9, we claimed that the functor $A_P : G\text{Orb} \to \text{Sp}$ from \cite{BKW} admits an equivalent description in terms of the $G\text{Orb}$-spectra considered in this article. This section supplies a proof of this claim, see Corollary 7.71 below.

Let us first recall the construction of the functor $A_P$ (see also \cite[Sec. 5.1]{BKW}). Associated to any topological space $Q$, there is the Waldhausen category $R(Q)$ of retractive spaces over $Q$, by which we mean the category of CW-complexes relative $Q$ which are equipped with a retraction to $Q$, and all cellular maps over and under $Q$.

Its subcategory $hR(Q)$ of weak equivalences is given by those morphisms which are homotopy equivalences under $Q$. The assignment $Q \mapsto R(Q)$ defines a functor from topological spaces to Waldhausen categories via cobase change: Given a continuous map $f : Q \to Q'$, the induced functor $R(f)$ sends a retractive space $Q \cong X$ to the retractive space $Q' \cong X \cup_Q Q'$ determined by the pushout along $f$. If we require the base space to be an actual subspace of every retractive space, there is a strictly functorial choice of this construction.

Denote by $R_f(Q)$ the full subcategory of finite retractive spaces over $Q$ and by $R_{fd}(Q)$ the full subcategory of finitely dominated retractive spaces. Both of these define full subfunctors of $R$.

The localisation $R_f(Q)[h^{-1}] := R_f(Q)[hR_f(Q)^{-1}]$ at the subcategory of weak equivalences defines a functor
\[ R_f(-)[h^{-1}] : \text{Top} \to \text{Cat}_{\text{Rex}}^\infty, \]
to the category of right-exact $\infty$-categories (see e.g. \cite[Prop. 7.5.6]{Cis19}; note that the opposite of every Waldhausen category satisfying the two-out-of-three axiom
is an $\infty$-category with fibrations and weak equivalences in the sense of [Cis19, Def. 7.4.12]). The same is true for the analogous localisation of $R_{fd}(-)$. Applying the algebraic $K$-theory functor from Section 6.4 to the opposite $\infty$-category gives rise to the $A$-theory functor

$$A: \text{Top} \xrightarrow{R_{fd}(-)[h^{-1}]} \text{Cat}_{\infty,*}^{\text{Rex}} \xrightarrow{(-)^{op}} \text{Cat}_{\infty,*}^{\text{Lex}} \xrightarrow{K} \text{Sp}$$

considered in [BKW]. If $P$ is a principal $G$-bundle for some discrete group $G$, define $A_P: G\text{Orb} \to \text{Sp}, \ S \mapsto A(P \times_G S)$.

Since $\text{Cat}_{\infty,*}^{\text{Rex}}$ is cocomplete, every right-exact $\infty$-category $C$ determines a colimit-preserving functor $- \otimes C: \text{Spc} \to \text{Cat}_{\infty,*}^{\text{Rex}}$.

**Proposition 7.69.** There exists an equivalence

$$R_{fd}(-)[h^{-1}] \simeq - \otimes \text{Spc}^{\text{cp}}_{\infty,*}.$$  

Let $\ell: \text{Top} \to \text{Spc}$ denote the canonical functor.

**Corollary 7.70.** There exists an equivalence of functors $\text{Top} \to \text{Sp}$

$$A(-) \simeq K((\ell(-) \otimes \text{Spc}^{\text{cp}}_{\infty,*})^{op})$$.

**Proof.** By the cofinality theorem [BKW, Thm. 2.30] (see e.g. [BKW, Constr. 4.13] for the “mapping cylinder argument”), we may replace $R_{fd}(-)[h^{-1}]$ by the functor $R_{fd}(-)[h^{-1}]$ in the definition of $A$. Consequently, the corollary is an immediate consequence of Proposition 7.69. \(\square\)

Let $P$ be a principal $G$-bundle. Recall the functor $K_C: G\text{Orb} \to \text{Sp}$ associated to any left-exact $\infty$-category $C$ with $G$-action (see (1.9) or Definition 6.22).

**Corollary 7.71.** There is an equivalence

$$A_P \simeq K(\ell(P) \otimes \text{Spc}^{\text{cp}}_{\infty,*})^{op}_{G}.$$  

**Proof.** Let $j: BG \to G\text{Orb}$ be the inclusion functor (see e.g. (1.2)), and let $o_P := \ell(P \times_G -): G\text{Orb} \to \text{Spc}.$

Note that $\ell(P)$ defines an object in $\text{Fun}(BG, \text{Spc})$. We claim that $j_\ell(P) \simeq o_P,$

where $j_\ell$ denotes the left Kan extension functor. By the pointwise characterisation of left Kan extensions, it suffices to check that the canonical map

$$\text{colim}_{(G \to S) \in BG / S} \ \ell(P) \to \ell(P \times_G S)$$

is an equivalence. Any choice of base point $s$ in $S$ induces an equivalence $B(G_s) \simeq BG / S$ (compare Remark 5.1) and an equivalence $\ell(P \times_G S) \simeq \ell(P/G_s)$. Since restricting $\ell(P)$ along the equivalence $B(G_s) \simeq BG / S$ is given by the localisation of $P$ equipped with the right $B(G_s)$-action, it suffices to check that

$$\text{colim}_{B(G_s)} \ell(P) \to \ell(P/G_s)$$

is an equivalence for $s$ in $S$. Since $P$ is a principal $G$-bundle, the orbit space $P/G_s$ is equivalent to the homotopy orbit space, so this map is an equivalence.
By Corollary 7.70, there are equivalences
\[ A_P \simeq K \circ (\ell_P(-) \otimes \text{Spc}_\mathcal{P})^{\mathbb{L}} \simeq K \circ (j_!(P) \otimes \text{Spc}_\mathcal{P})^{\mathbb{L}}. \]
Since \(- \otimes \text{Spc}_\mathcal{P}\) is colimit-preserving, we have
\[ j_!(P) \otimes \text{Spc}_\mathcal{P} \simeq j_!(P) \otimes \text{Spc}_\mathcal{P}. \]
By definition, \(K((\ell(P) \otimes \text{Spc}_\mathcal{P})^{\mathbb{L}}) \simeq K \circ (\ell(P) \otimes \text{Spc}_\mathcal{P})^{\mathbb{L}}\), so we are done.

**Proof of Proposition 7.69.** Let \((\text{Top}/Q)_*\) denote the category of pointed topological spaces over \(Q\), equipped with the model structure transferred from \text{Top} via the forgetful functor \((\text{Top}/Q)_* \to \text{Top}\). Denote the class of weak equivalences in \((\text{Top}/Q)_*\), by \(W\). Note that \(R(Q)\) is canonically a subcategory of the full subcategory \((\text{Top}/Q)_*\) of cofibrant objects in \((\text{Top}/Q)_*\). The inclusion functor induces an equivalence \(R(Q)[hR(Q)^{-1}] \simeq (\text{Top}/Q)_*^{w}[W^{-1}]\) since there exists a functorial cofibrant replacement in \((\text{Top}/Q)_*\) which takes values in \(R(Q)\).

This equivalence restricts to an equivalence
\[(7.50)\]
\[ R_{\text{id}}(Q)[h^{-1}] \simeq (\text{Top}/Q)_*^{w}[W^{-1}]^{\mathbb{L}} \]
of right-exact \(\infty\)-categories: Every finitely dominated retractive space is compact as an object in \((\text{Top}/Q)_*^{w}[W^{-1}]\). Conversely, consider a retractive space \(B \simeq X\) which is compact in \((\text{Top}/Q)_*^{w}[W^{-1}]\). Since every retractive space is the (homotopy) colimit of its finite subcomplexes, compactness implies that the identity on \(X\) factors (up to homotopy) through a finite subcomplex of \(X\). Hence \(B \simeq X\) is finitely dominated.

Consider \((\text{sSet}/\text{Sing}(Q))_*\), with the projective model structure. Taking singular complexes induces an equivalence
\[(7.51)\]
\[ (\text{Top}/Q)_*^{w}[W^{-1}] \simeq (\text{sSet}/\text{Sing}(Q))_*^{w}[W^{-1}] \]
since all objects in \((\text{sSet}/\text{Sing}(Q))_*\) are cofibrant. Note that the equivalences from (7.50) and (7.51) are natural under cobase change. Hence we obtain an equivalence
\[(7.52)\]
\[ R_{\text{id}}(-)[h^{-1}] \simeq (\text{sSet}/\text{Sing}(-))_*^{w}[W^{-1}]^{\mathbb{L}} \]
of functors \(\text{Top} \to \text{Cat}^{\mathbb{L}}_{\infty,x}\).

Consider now the functor \((\text{sSet}/\text{Sing}(-))_*^{w}[W^{-1}]\) as a contravariant functor on topological spaces (via pullback). As in the case of \(\text{Top}/Q\), equip \((\text{sSet}/\text{Sing}(Q))_*\) with the model structure transferred from the Quillen model structure on \text{sSet} via the forgetful functor. Any fibrant replacement functor on \((\text{sSet}/\text{Sing}(Q))_*\) induces a natural equivalence
\[(7.53)\]
\[ (\text{sSet}/\text{Sing}(Q))_*^{w}[W^{-1}] \simeq N((\text{sSet}/\text{Sing}(Q))_*^{w}) \]
by [Lura, Prop. 1.3.4.7], where \(N\) denotes the homotopy coherent nerve. Note that the cofibrant-fibrant objects in \((\text{sSet}/\text{Sing}(-))_*\) are precisely the Kan fibrations. Let \((\text{sSet}/\text{Sing}(Q))_*^{\mathbb{L}}\) denote the category \((\text{sSet}/\text{Sing}(Q))_*\) equipped with the cartesian model structure (see [Lur09, Rem. 2.1.4.12], where it is called the contravariant model structure). This category is also contravariantly functorial via pullback. Since the right fibrations are precisely the cofibrant-fibrant objects in the cartesian model structure ([Lur09, Cor. 2.2.3.12]), and since every right fibration over a Kan complex is a Kan fibration ([Lur09, Lem. 2.1.3.3]), we have
\[(7.54)\]
\[ N((\text{sSet}/\text{Sing}(-))_*^{w}) \simeq N((\text{sSet}/\text{Sing}(-))_*^{w}) \simeq (\text{Spc}/\ell(-))_* \]
By [GHN17, Cor. A.32], there exists a functor
\[ \text{Cat}^{\text{op}}_{\infty} \to \text{Fun}(\Delta^1, \text{CAT}) \]
whose value at \( C \) is given by the unstraightening equivalence \( \text{Fun}(\text{C}^{\text{op}}, \text{Cat}_{\infty}) \cong \text{Cat}_{\infty/C}^{\text{cart}} \). Note that the unstraightening equivalence restricts to an equivalence of full subcategories
\[ \text{Fun}(\text{C}^{\text{op}}, \text{Spc}) \cong \text{Cat}_{\infty/C}^{\text{fib}} \]
where \( \text{Cat}_{\infty/C}^{\text{fib}} \) denotes the full subcategory of cartesian fibrations whose fibres are objects in \( \text{Spc} \). Since \( \text{Cat}_{\infty/C}^{\text{fib}} \cong \text{Spc}/C \) when \( C \) is in \( \text{Spc} \), there is an induced natural equivalence
\[ (\text{7.55}) \quad \text{Fun}((-)^{\text{op}}, \text{Spc}_*) \simeq \text{Fun}((-)^{\text{op}}, \text{Spc})_* \simeq (\text{Spc}/-)_* \]
of functors \( \text{Spc} \to \text{Pr}_C^{\text{R}} \). From (7.53), (7.54) and (7.55) we obtain an equivalence
\[ (\text{7.56}) \quad (\text{sSet}/\text{Sing}(-))_*[W^{-1}] \simeq \text{Fun}(\ell(-)^{\text{op}}, \text{Spc}) \]
of functors \( \text{Top} \to \text{Pr}_C^{\text{R}} \). We can now pass to left adjoints and restrict to the full subfunctors on compact objects to obtain an equivalence
\[ (\text{7.57}) \quad \text{R}_{\text{Ad}}((-)[-h^{-1}]) \simeq \text{ad}^{-1} \text{Fun}(\ell(-)^{\text{op}}, \text{Spc})_*^{\text{cp}} \]
of functors \( \text{Top} \to \text{Cat}_{\infty,*}^{\text{Rex}} \).

Finally, we note that \( \text{Fun}((-)^{\text{op}}, \text{Spc})_*: \text{Spc}^{\text{op}} \to \text{Pr}_C^{\text{R}} \) is limit-preserving. Consequently, \( \text{ad}^{-1} \text{Fun}((-)^{\text{op}}, \text{Spc})_* \) is colimit-preserving as a functor \( \text{Spc} \to \text{Pr}_C^{\text{R}} \). Therefore,
\[ \text{ad}^{-1} \text{Fun}((-)^{\text{op}}, \text{Spc})_*^{\text{cp}}: \text{Spc} \to \text{Cat}_{\infty,*}^{\text{Rex}} \]
is a colimit-preserving functor that sends the terminal object in \( \text{Spc} \) to \( \text{Spc}_*^{\text{cp}} \). It follows that
\[ (\text{7.58}) \quad \text{ad}^{-1} \text{Fun}((-)^{\text{op}}, \text{Spc})_*^{\text{cp}} \simeq - \otimes \text{Spc}_*^{\text{cp}} \]
The proposition follows by combining (7.57) and (7.58). \( \square \)

References


\[12\] This was essentially a repetition of the proof of [ABG18, Prop. B.1] emphasising the naturality of all identifications.


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