INTEGRAL FOLIATED SIMPLICIAL VOLUME AND S^1-ACTIONS

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ABSTRACT. The simplicial volume of oriented closed connected smooth manifolds that admit a non-trivial smooth S^1-action vanishes. In the present work we prove a version of this result for the integral foliated simplicial volume of aspherical manifolds: The integral foliated simplicial volume of aspherical oriented closed connected smooth manifolds that admit a non-trivial smooth S^1-action vanishes. Our proof uses the geometric construction of Yano's proof for ordinary simplicial volume as well as the parametrised uniform boundary condition for S^1.

1. INTRODUCTION

It is a long standing question of Gromov whether all L^2-Betti numbers of an aspherical oriented closed connected manifold with trivial simplicial volume are zero [4, p. 232]. For such manifolds with non-trivial S^1-action it is known that all L^2-Betti numbers vanish [8, Corollary 1.43]. Moreover, Gromov and Yano independently showed that the simplicial volume of oriented closed connected smooth manifolds with non-trivial smooth S^1-action is zero [5, 14]. The integral foliated simplicial volume (see Subsection 3.4) yields an upper bound for the L^2-Betti numbers as well as for the ordinary simplicial volume [10]. This leads to the question whether the integral foliated simplicial volume of an aspherical oriented closed connected smooth manifold with non-trivial smooth S^1-action is also zero. In this work we prove the following result, which answers this question in the positive.

Theorem 1.1 (integral foliated simplicial volume and S^1-actions)**.** *Let M be an oriented compact connected smooth manifold that admits a smooth S^1-action without fixed points such that the inclusion of every orbit into M is π_1-injective. Then $|M, \partial M| = 0$.*

More precisely: If $\alpha\colon \pi_1(M) \curvearrowright (Z, \mu)$ is an essentially free standard $\pi_1(M)$-space, then $|M, \partial M|^{\alpha} = 0$.

Corollary 1.2. *Let M be an aspherical oriented closed connected smooth manifold that admits a non-trivial smooth S^1-action. Then $|M| = 0$.*

More precisely: If $\alpha\colon \pi_1(M) \curvearrowright (Z, \mu)$ is an essentially free standard $\pi_1(M)$-space, then $|M|^{\alpha} = 0$.

Corollary 1.2 directly follows from Theorem 1.1 and a result on the structure of non-trivial S^1-actions on aspherical closed manifolds of Lück [8, Corollary 1.43].

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For the proof of Theorem 1.1, we will combine Yano's construction with the parametrised uniform boundary condition for S^1 [2].

Sauer established an upper bound on a related invariant in terms of minimal volume [9, Section 3] and the minimal volume of compact smooth manifolds with locally free S^1-action is zero [5, Appendix 2]. However, it is not known whether one can adapt Sauer's result such that it fits to the case of integral foliated simplicial volume.

Applications. Apart from having a new proof for vanishing of all L^2-Betti numbers of aspherical manifolds with non-trivial smooth S^1-action, Theorem 1.1 yields applications to gradient invariants: We have a new approximation result for simplicial volume and hence vanishing results for the Betti number gradient, the torsion homology gradient and the rank gradient of (fundamental groups of) aspherical manifolds with non-trivial smooth S^1-action.

We first recall some definitions: The *stable integral simplicial volume* of an oriented closed connected manifold M with fundamental group Γ is given by

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf_{\Lambda\in F(\Gamma)} \frac{\|\widetilde{M}/\Lambda\|_{\mathbb{Z}}}{[\Gamma:\Lambda]},$$

where $F(\Gamma)$ is the set of all finite index subgroups of $\pi_1(M)$. A *residual chain* in a finitely generated group Γ is a descending sequence $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ of normal finite index subgroups whose intersection is trivial.

Corollary 1.3 (stable integral simplicial volume and S^1-actions)**.** *Let M be an aspherical oriented closed connected smooth manifold with residually finite fundamental group Γ that admits a non-trivial smooth S^1-action. Then, we have*

$$\|M\|_{\mathbb{Z}}^{\infty} = 0.$$

More generally: If $(\Gamma_n)_{n\in\mathbb{N}}$ is a Farber chain [1] *(e.g., a residual chain) in Γ, then*

$$\inf_{n\in N} \frac{\|M_n\|_{\mathbb{Z}}}{[\Gamma:\Gamma_n]} = 0,$$

where $M_n \longrightarrow M$ denotes the covering of M associated to Γ_n for all $n \in \mathbb{N}$ and $\|\cdot\|_{\mathbb{Z}}$ is the integral simplicial volume.

Proof. A Farber chain in a finitely generated group Γ yields a standard Γ-space that is essentially free by definition. Now, the result follows from the relation between integral foliated simplicial volume and stable integral simplicial volume [3, Theorem 2.6] and Corollary 1.2. □

Let $\operatorname{tors} A$ denote the torsion of a finitely generated abelian group A and rk_R denote the R-dimension of the free part of finitely generated R-modules. The *rank gradient* of a residually finite group Γ (with respect to a Farber chain $(\Gamma_n)_{n\in\mathbb{N}}$ in Γ) is defined as

$$\operatorname{rg}(\Gamma) := \inf_{\Lambda\in F(\Gamma)} \frac{d(\Lambda)-1}{[\Gamma:\Lambda]} \quad \text{and} \quad \operatorname{rg}\big(\Gamma,(\Gamma_n)_{n\in\mathbb{N}}\big) := \inf_{n\in N} \frac{d(\Gamma_n)-1}{[\Gamma:\Gamma_n]}$$

respectively, where $d(\cdot)$ denotes the minimal number of generators of a finitely generated group.

Corollary 1.4 (gradient invariants and S^1-actions)**.** *Let M be an aspherical oriented closed connected smooth manifold with residually finite fundamental group Γ that admits a non-trivial smooth S^1-action. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a Farber chain in Γ and let $M_n \longrightarrow M$ denote the covering of M associated to Γ_n for all $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$ and for every principal ideal domain R, we have*

$$\limsup_{n\to\infty} \frac{\operatorname{rk}_R H_k(M_n; R)}{[\Gamma : \Gamma_n]} = 0 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\log|\operatorname{tors} H_k(M_n; \mathbb{Z})|}{[\Gamma : \Gamma_n]} = 0.$$

Moreover, we have

$$\operatorname{rg}\bigl(\Gamma, (\Gamma_n)_{n\in\mathbb{N}}\bigr) = 0;$$

in particular, $\operatorname{rg}(\Gamma) = 0$.

Proof. The first part of the corollary follows from a result on homology bounds by Frigerio, Löh, Pagliantini and Sauer [3, Theorem 1.6]. The second part follows from the fact that stable integral simplicial volume yields an upper bound for the rank gradient [6, Theorem 1.1]. □

Organisation of this article. We briefly recall Yano's construction in Section 2. In Section 3 we introduce relative integral foliated simplicial volume and the parametrised uniform boundary condition for S^1. Additional prerequisites are provided in Section 4. In Section 5 we will construct parametrised chains of small norm that we adjust in Section 6 to get parametrised fundamental cycles of small norm.

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2. Yano's construction

In this section, we briefly recall the definitions and notations of Yano's construction [14, Section 2] that he used for the proof of vanishing of the simplicial volume of smooth manifolds with non-trivial smooth S^1-action.

Let M be an oriented closed connected smooth n-manifold that admits a smooth S^1-action without fixed points. Using so-called hollowings, Yano defines a sequence

$$M_{n-2} \xrightarrow{p_{n-3}} M_{n-3} \xrightarrow{p_{n-4}} \cdots \xrightarrow{p_1} M_1 \xrightarrow{p_0} M_0 = M$$

of compact manifolds with smooth S^1-action and S^1-equivariant maps such that M_{n-2} splits as $N \times S^1$ with N an oriented compact connected manifold (possibly) with boundary. Note that the assumption on the S^1-action to be fixed point free allows us to skip the first $n-1$ steps in the sequence that Yano defined originally.

The idea of Yano's proof then is the following: We know that the (relative) simplicial volume of $M_{n-2} \cong N \times S^1$ is zero. So we can choose a fundamental cycle of M_{n-2} of small ℓ^1-norm. The pushforward of this relative cycle to M unfortunately is in general no cycle in M anymore, but one can adjust this pushforward by fillings to get a fundamental cycle of M without changing the norm too much.

One can easily generalise Yano's construction to compact manifolds with boundary by allowing hollowings not just transversal to but also along the boundary.

We come back to the sequence of manifolds above (where we allow M to be compact, possibly with boundary): First, we choose a triangulation on $\overline{M}$ as follows: For all $r \in \mathbb{N}_{\geq 2}$ let $L_r \subset M$ be the set of points whose stabilisers contain the set

$$\left\{0, \frac{1}{r}, \ldots, \frac{r-1}{r}\right\} \subset S^1 \cong \mathbb{R}/\mathbb{Z}$$

and let $L := \bigcup_{r \in \mathbb{N}_{\geq 2}} L_r$. Without loss of generality we may assume that the S^1-action is *effective*, i.e., there exists no element in S^1 that fixes every point in M. Then every L_r is a smooth submanifold of M of dimension less than $n-1$ that admits a smooth S^1-action. Let $\pi \colon M \longrightarrow \overline{M}$ be the projection of M onto its orbit space. Then, $\overline{M}$ is triangulable [11, Corollary 3.8] and we choose a triangulation on $\overline{M}$ with the following property: For all $r \in \mathbb{N}_{\geq 2}$ the orbit space $\overline{L}_r$ of L_r is a subcomplex of the triangulation.

Now, we want to extend the sequence by $M_{-1} := M$ and $p_{-1} := \mathrm{id}_M$ for notational reasons concerning the case with boundary. We inductively define the map p_j for all $j \in \{0, \ldots, n-3\}$ to be the *hollowing at* $X_j \subset M_j$, i.e., M_{j+1} is obtained from M_j by removing a (small) open tubular neighbourhood T of X_j and p_j is the map that projects the boundary of T to X_j, where X_j is the pullback of the j-skeleton of the orbit space $\overline{M}$ along $\pi \circ p_0 \circ \cdots \circ p_{j-1}$. We write

$$p_{l,l'} := p_{l'} \circ \cdots \circ p_{l-1} \colon M_l \longrightarrow M_{l'},$$

for all $l, l' \in \{-1, 0, \ldots, n-2\}$ with $l' < l$. We set $X_{-1} := \partial M \subset M_{-1}$. For all $j \in \{-1, \ldots n-3\}$ the *hollow wall of* p_j is given by

$$N_j := p_j^{-1}(X_j) \subset M_{j+1}$$

and $\widetilde{N}_j \subset M_{n-2}$ is the pullback of N_j along $p_{n-2,j}$. Let $k \in \{1, \ldots, n-1\}$ and let $j_1, \ldots, j_k \in \{-1, \ldots, n-3\}$ be pairwise distinct. We define

$$\widetilde{N}_{j_1,\ldots,j_k} := \widetilde{N}_{j_1} \cap \cdots \cap \widetilde{N}_{j_k} \subset M_{n-2}$$

and

$$X_{j_1,\ldots,j_k} := p_{n-2,j_1}(\widetilde{N}_{j_1,\ldots,j_k}) \subset M_{j_1}.$$

We write

$$\overline{N}_{j_1,\ldots,j_k} \quad \text{and} \quad \overline{X}_{j_1,\ldots,j_k}$$

for the orbit space of $\widetilde{N}_{j_1,\ldots,j_k}$ and $X_{j_1,\ldots,j_k}$ respectively. Then, Yano shows the following [14, Lemma 4, Lemma 6 and Lemma 7]:

Lemma 2.1. *Let $k \in \{1, \ldots, n-2\}$ and let $j_1, \ldots, j_k \in \{0, \ldots, n-3\}$ be pairwise distinct. Then, each connected component of the orbit space $\overline{X}_{j_1,\ldots,j_k}$ of $X_{j_1,\ldots,j_k}$ is contractible and we have*

$$X_{j_1,\ldots,j_k} \cong \overline{X}_{j_1,\ldots,j_k} \times S^1 \quad \textit{and} \quad M_{n-2} \cong \overline{M}_{n-2} \times S^1.$$

For the case with boundary we need in addition the following observation:

Proposition 2.2. *For all pairwise distinct $j_1, \dots, j_k \in \{0, \dots, n-3\}$ we have that $X_{j_1,\dots,j_k,-1}$ is the union of the connected components Y in $X_{j_1,\dots,j_k}$ with*

$$Y \subset p_{n-2,j_1}(\widetilde{N}_{-1}).$$

Proof. Let $j \in \{0, \dots, n-3\}$. We only show the statement for $X_{j,-1} \subset X_j$. The general case can be proven similarly. Let $Y \subset \overline{X}_j$ be a connected component. As in Yano's proof of Lemma 2.1 we observe that Y is homeomorphic to $\Delta^j_{j'}$, where Δ^j_j is obtained from the standard simplex Δ^j by hollowing inductively along the l-skeleton for all $l \in \{0, \dots, j-1\}$. From this it follows easily that we are in one of the following cases:

(1) We have $Y \subset \overline{p}_{n-2,j}(\overline{N}_{-1})$, or
(2) we have $Y \cap \overline{p}_{n-2,j}(\overline{N}_{-1}) = \emptyset$.

In the first case, we have

$$\begin{aligned} Y &\subset \overline{X}_j \cap \overline{p}_{n-2,j}(\overline{N}_{-1}) = \overline{p}_{n-2,j}(\overline{N}_j) \cap \overline{p}_{n-2,j}(\overline{N}_{-1}) \\ &\subset \overline{p}_{n-2,j}(\overline{N}_{j,-1}) = \overline{X}_{j,1}, \end{aligned}$$

where the last inclusion follows from

$$\overline{p}_{n-2,j}(\overline{N}_j \setminus \overline{N}_{-1}) \cap \overline{p}_{n-2,j}(\overline{N}_{-1} \setminus \overline{N}_j) = \emptyset.$$

In the second case, we have $Y \cap \overline{X}_{j,-1} = \emptyset$. □

Remark 2.3. It follows that each $\widetilde{N}_{j_1,\dots,j_k}$ decomposes as $\overline{N}_{j_1,\dots,j_k} \times S^1$. We choose a simplicial structure on $\partial\overline{M}_{n-2}$ that is compatible with the decompositions

$$\partial\overline{M}_{n-2} = \bigcup_{i=-1}^{n-3} \overline{N}_i \quad \text{and} \quad \partial\overline{N}_{j_1,\dots,j_k} = \bigcup_j \overline{N}_{j_1,\dots,j_k,j},$$

where j ranges over $\{-1, \dots, n-3\} \setminus \{j_1, \dots, j_k\}$. Then each $\overline{N}_{j_1,\dots,j_k}$ is an $(n-2-k)$-dimensional subcomplex of the $(n-2)$-dimensional complex $\partial\overline{M}_{n-2}$.

3. Relative integral foliated simplicial volume and the parametrised uniform boundary condition

3.1. The ℓ^1-norm on the singular chain complex and simplicial volume.

We recall the definition of the relative simplicial volume introduced by Gromov [5].

Definition 3.1 (ℓ^1-norm on the singular chain complex)**.** Let $R \in \{\mathbb{Z}, \mathbb{R}\}$. Let M be a topological space and let $n \in \mathbb{N}$. For a singular chain $c = \sum_{j=1}^k a_j \cdot \sigma_j \in C_n(M; R)$ written in reduced form (i.e., the singular simplices $\sigma_1, \dots, \sigma_k$ are pairwise distinct) we define the *ℓ^1-norm of c* by

$$|c|_1 := \sum_{j=1}^k |a_j|.$$

Remark 3.2 (functoriality). The ℓ^1-norm on the singular chain complex is *functorial* in the following sense: Let $f\colon M \longrightarrow N$ be a continuous map between topological spaces M and N. Then $\|C_n(f;R)\| \leq 1$, where $\|\cdot\|$ denotes the operator norm.

Definition 3.3 (relative simplicial volume)**.** Let M be an oriented compact connected n-manifold. A *relative* $\mathbb{R}$*-fundamental cycle of* M is a chain $c \in C_n(M;\mathbb{R})$ of the form

$$c = c_{\mathbb{Z}} + \partial b + d$$

where $c_{\mathbb{Z}} \in C_n(M;\mathbb{Z}) \subset C_n(M;\mathbb{R})$ is an ordinary relative fundamental cycle of M, $b \in C_{n+1}(M;\mathbb{R})$ and $d \in C_n(\partial M;\mathbb{R}) \subset C_n(M;\mathbb{R})$. In other words, c is a cycle in $C_n(M,\partial M;\mathbb{R})$ representing $[M,\partial M]_{\mathbb{R}}$. Then the *relative simplicial volume of* M is defined by

$$\|M,\partial M\| := \inf\bigl\{|c|_1 \bigm| c \in C_n(M,\mathbb{R}) \text{ represents } [M,\partial M]_{\mathbb{R}}\bigr\}.$$

If $\partial M = \emptyset$, we write $\|M\| := \|M,\partial M\|$.

3.2. **The parametrised ℓ^1-norm.** The parametrised ℓ^1-norm is given as the ℓ^1-norm on the singular chain complex with twisted coefficients that are induced by actions of the fundamental group on probability spaces. This leads to the (relative) integral foliated simplicial volume (Subsection 3.4).

Definition 3.4 (standard Γ-space)**.** Let Γ be a countable group. A *standard* Γ*-space* $\alpha = \Gamma \curvearrowright (Z,\mu)$ is a standard Borel probability space (Z,μ) together with a measurable probability measure preserving left-Γ-action.

Definition 3.5 (parametrised ℓ^1-norm)**.** Let M be a path-connected, locally path-connected topological space that admits a universal covering space $\widetilde{M}$, let $\Gamma := \pi_1(M)$, and let $\alpha = \Gamma \curvearrowright (Z;\mu)$ be a standard Γ-space. For $n \in \mathbb{N}$, we define the *parametrised* ℓ^1*-norm*

$$\begin{aligned} |\cdot|_1\colon C_n(M;\alpha) &\longrightarrow \mathbb{R}_{\geq 0} \\ \sum_{j=1}^{k} f_j \otimes \sigma_j &\longmapsto \sum_{j=1}^{k} \int_Z |f_j| \, d\mu \end{aligned}$$

on the chain complex

$$C_n(M;\alpha) := L^\infty(Z,\mu;\mathbb{Z}) \underset{\mathbb{Z}\Gamma}{\otimes} C_n(\widetilde{M};\mathbb{Z}),$$

where we assume that $\sum_{j=1}^{k} f_j \otimes \sigma_j$ is in reduced form, i.e., all the singular simplices $\sigma_j \in \operatorname{map}(\Delta^n,\widetilde{M})$ belong to different Γ-orbits. We consider the right-Γ-action on $L^\infty(Z,\mu;\mathbb{Z})$ given by

$$(f\cdot\gamma)(x) := f(\gamma\cdot x)$$

for all $f \in L^\infty(Z,\mu;\mathbb{Z})$, $\gamma \in \Gamma$ and $x \in Z$ and the left-Γ-action on $C_n(\widetilde{M};\mathbb{Z})$ induced by the deck transformation action of Γ on $\widetilde{M}$. In the following, we also write $L^\infty(Z;\mathbb{Z})$ or $L^\infty(\alpha;\mathbb{Z})$ for $L^\infty(Z,\mu;\mathbb{Z})$.

3.3. **Parametrised fundamental cycles.** In this subsection we recall the definition of the parametrised relative fundamental cycles [2, Section 10.1].

Let $n \in \mathbb{N}$. Let M be an oriented compact connected n-manifold. We write $\Gamma := \pi_1(M)$ and $q\colon \widetilde{M} \longrightarrow M$ for the universal covering of M. Let $\alpha = \Gamma \curvearrowright (Z, \mu)$ be a standard Γ-space. Since the Γ-action on $\widetilde{M}$ restricts to a Γ-action on $q^{-1}(\partial M)$, we can define

$$D_* := L^\infty(Z;\mathbb{Z}) \underset{\mathbb{Z}\Gamma}{\otimes} C_*\bigl(q^{-1}(\partial M);\mathbb{Z}\bigr)$$

as a subcomplex of $C_*(M;\alpha)$. We set

$$C_*(M,\partial M;\alpha) := C_*(M;\alpha)/D_* \cong L^\infty(Z;\mathbb{Z}) \underset{\mathbb{Z}\Gamma}{\otimes} C_*\bigl(\widetilde{M}, q^{-1}(\partial M);\mathbb{Z}\bigr)$$

and

$$H_*(M,\partial M;\alpha) := H_*\bigl(C_*(M,\partial M;\alpha)\bigr).$$

Definition 3.6 (parametrised relative fundamental cycle)**.** An *α-parametrised relative fundamental cycle of M* is a chain $c \in C_n(M;\alpha)$ of the form

$$c = c_{\mathbb{Z}} + \partial b + d$$

with a relative fundamental cycle $c_{\mathbb{Z}} \in C_n(M;\mathbb{Z}) \subset C_n(M;\alpha)$, a chain $b \in C_{n+1}(M;\alpha)$ and a chain $d \in D_n$. In other words, c is a cycle in $C_n(M,\partial M;\alpha)$ representing $[M,\partial M]^\alpha$, i.e., the image of $[M,\partial M]$ under the induced map of the inclusion

$$\begin{aligned} C_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z} \underset{\mathbb{Z}\Gamma}{\otimes} C_n\bigl(\widetilde{M}, q^{-1}(\partial M);\mathbb{Z}\bigr) &\longrightarrow C_*(\widetilde{M},\partial M;\alpha) \\ 1 \otimes \sigma &\longmapsto \mathrm{const}_1 \otimes \sigma. \end{aligned}$$

Lemma 3.7. *Let Γ be a countable group and let $\alpha\colon \Gamma \curvearrowright (Z,\mu)$ be a standard Γ-space. We write $A := L^\infty(Z,\mu;\mathbb{Z})$. Then the canonical map $A^\Gamma \longrightarrow A_\Gamma$ from the Γ-invariants to the Γ-coinvariants of A is injective.*

Proof. Let $f \in A^\Gamma \setminus \{0\}$. We want to show that $[f] \neq 0$ in A_Γ. Since f is non-zero there exists a measurable subset $B \subset X$ with $\Gamma \cdot B = B$, $\mu(B) > 0$, and

$$f|_B \geq 1 \quad \text{or} \quad f|_B \leq -1.$$

On the one hand, we have

$$\Bigl| \int_B f \, d\mu \Bigr| \geq \mu(B) \neq 0.$$

On the other hand, integration $\int_B \cdot \, d\mu\colon A^\Gamma \longrightarrow \mathbb{R}$ over the Γ-invariant set B factors through $A^\Gamma \longrightarrow A_\Gamma$ since μ is Γ-invariant and it follows that $[f] \neq 0$ in A_Γ. ☐

Definition 3.8 (local parametrised fundamental cycles)**.** Let M be an oriented compact connected n-manifold and let $\Gamma := \pi_1(M)$. Let $U \subset M^\circ$ be an embedded n-ball D^n in the interior of M. Let $\alpha\colon \Gamma \curvearrowright (Z,\mu)$ be a standard Γ-space. We write $A := L^\infty(Z,\mu;\mathbb{Z})$. Let $q\colon \widetilde{M} \longrightarrow M$ denote the universal covering of M. Consider the following chain map

$$g\colon A \underset{\mathbb{Z}\Gamma}{\otimes} C_n\bigl(\widetilde{M}, q^{-1}(\partial M);\mathbb{Z}\bigr) \longrightarrow A_\Gamma \underset{\mathbb{Z}\Gamma}{\otimes} C_n\bigl(\widetilde{M}, q^{-1}(M \setminus U);\mathbb{Z}\bigr)$$

induced by the inclusion

$$\big(\widetilde{M}, q^{-1}(\partial M)\big) \longrightarrow \big(\widetilde{M}, q^{-1}(M \setminus U)\big)$$

and the change of coefficients map corresponding to $A \longrightarrow A_\Gamma$. Here, the Γ-coinvariants A_Γ of A are equipped with the trivial Γ-action. Rewriting g gives us a chain map

$$f\colon C_n(M, \partial M; \alpha) \longrightarrow A_\Gamma \underset{\mathbb{Z}}{\otimes} C_n(M, M \setminus U; \mathbb{Z}).$$

Let $i\colon C_n(M;\alpha) \longrightarrow C_n(M, \partial M; \alpha)$ be the canonical map.

Let $c \in C_n(M;\alpha)$ be a relative cycle. Then c is called a *U-local α-parametrised relative fundamental cycle of M* if

$$F\big([i(c)]\big) = \mathrm{const}_1 \in A_\Gamma \cong H_n(M, M \setminus U; A_\Gamma),$$

where F denotes the induced map of f in homology.

Proposition 3.9 (locality of parametrised fundamental cycles)**.** *Let M be an oriented compact connected n-manifold and let $\Gamma := \pi_1(M)$. Let $U \subset M^\circ$ be an embedded n-ball D^n. Let $\alpha\colon \Gamma \curvearrowright (Z, \mu)$ be a standard Γ-space and let $c \in C_n(M;\alpha)$ be a relative cycle. Then the following are equivalent:*

- *The relative cycle c is an α-parametrised relative fundamental cycle of M.*
- *The relative cycle c is a U-local α-parametrised relative fundamental cycle of M.*

Proof. We write $A := L^\infty(Z, \mu; \mathbb{Z})$. Let

$$f\colon C_n(M, \partial M; \alpha) \longrightarrow A_\Gamma \underset{\mathbb{Z}}{\otimes} C_n(M, M \setminus U; \mathbb{Z})$$

be given as in Definition 3.8 and let

$$i\colon C_n(M;\alpha) \longrightarrow C_n(M, \partial M; \alpha)$$

be the canonical map.

By Lefschetz duality with twisted coefficients, we have

$$H_n(M, \partial M; \alpha) \cong H^0(M;\alpha) \cong A^\Gamma,$$

where A^Γ denotes the Γ-invariants in A (the pair $(M, \partial M)$ is a connected simple Poincaré pair [12, Theorem 2.1] and connected Poincaré pairs satisfy Lefschetz duality with twisted coefficients [13, Lemma 1.2]).

It follows that in homology the map $F := f_*$ is injective, since it induces the canonical projection $A^\Gamma \longrightarrow A_\Gamma$, which is injective by Lemma 3.7. Furthermore, note that

$$F\big([M, \partial M]^A\big) = \mathrm{const}_1 \in A_\Gamma.$$

Since F is injective, it follows that $i(c)$ represents $[M, \partial M]^A$ if and only if $F\big([i(c)]\big) = \mathrm{const}_1$. This finishes the proof. ☐

3.4. **The relative integral foliated simplicial volume.** We will now introduce (relative) integral foliated simplicial volume.

Definition 3.10 (relative integral foliated simplicial volume)**.** The *relative integral foliated simplicial volume of M* is defined by

$$|\!|\!| M, \partial M |\!|\!| := \inf\{ |\!|\!| M, \partial M |\!|\!|^{\alpha} \mid \alpha = \Gamma \curvearrowright (Z, \mu) \text{ is a standard } \Gamma\text{-space}\},$$

where $|\!|\!| M, \partial M |\!|\!|^{\alpha} := \inf\{|c|_1 \mid c \text{ represents } [M, \partial M]^{\alpha}\}$. If $\partial M = \emptyset$, we write $|\!|\!| M |\!|\!|^{\alpha} := |\!|\!| M, \partial M |\!|\!|^{\alpha}$ and $|\!|\!| M |\!|\!| := |\!|\!| M, \partial M |\!|\!|$.

The integral foliated simplicial volume is an upper bound for the L^2-Betti numbers; more precisely

$$\sum_{k=0}^{n} b_k^{(2)}(\widetilde{M}) \leq (n+1) \cdot |\!|\!| M |\!|\!|$$

holds [10] (the original constant 2^{n+1} can easily be improved to $n+1$). Furthermore, it fits into the following sandwich [10]

$$\|M\| \leq |\!|\!| M |\!|\!| \leq \|M\|_{\mathbb{Z}},$$

where $\|M\|_{\mathbb{Z}}$ is the *integral simplicial volume* which is given by the minimal ℓ^1-norm of integral fundamental cycles of M. The integral foliated simplicial volume is known to be equal to the simplicial volume in the case of oriented closed connected hyperbolic 3-manifolds [7]. However, it is strictly greater than the simplicial volume for oriented closed connected hyperbolic k-manifolds with $k \in \mathbb{N}_{\geq 4}$ [3, Theorem 1.8]. Moreover, the integral foliated simplicial volume of oriented closed connected aspherical manifolds with amenable fundamental group is zero [3, Theorem 1.9].

3.5. **Normed chain complexes and the parametrised uniform boundary condition for S^1.** In the following we discuss our main tool for the proof of Theorem 1.1, the parametrised uniform boundary condition for S^1.

Definition 3.11 ((semi-)normed abelian groups)**.** Let A be an abelian group.

- A *semi-norm on A* is a map $|\cdot| \colon A \longrightarrow \mathbb{R}_{\geq 0}$ with the following properties:
  - We have $|0| = 0$.
  - For all $a \in A$ we have $|-a| = |a|$.
  - For all $a, b \in A$ we have $|a+b| \leq |a| + |b|$.
- A *norm on A* is a semi-norm $|\cdot|$ on A such that for all $a \in A$ we have $|a| = 0$ if and only if $a = 0$.
- A *(semi-)normed abelian group* is an abelian group equipped with a (semi-)norm.
- A homomorphism $f \colon A \longrightarrow B$ of (semi-)normed abelian groups is called *bounded* if there exists a constant $C \in \mathbb{R}_{>0}$ such that for all $a \in A$ we have
$$|f(a)| \leq C \cdot |a|.$$

Definition 3.12 (normed chain complex)**.** A *normed chain complex* is a chain complex in the category of normed abelian groups and bounded homomorphisms.

Example 3.13. The singular chain complex together with the ℓ^1-norm as well as the singular chain complex with twisted coefficients together with the parametrised ℓ^1-norm are normed chain complexes.

Definition 3.14 (UBC). Let $n \in \mathbb{N}$. A normed chain complex C_* satisfies the *uniform boundary condition in degree n*, short n-UBC, if there exist a constant $C \in \mathbb{R}_{>0}$ such that for every null-homologous cycle $c \in C_n$ there exists an efficient filling, i.e., a chain $b \in C_{n+1}$ with $\partial b = c$ and $|b| \leq C \cdot |c|$.

Proposition 3.15 (UBC and homotopy). *Let $f_* \colon C_* \longrightarrow D_*$ be a chain homotopy equivalence of normed chain complexes with chain homotopy inverse g_* and chain homotopies h_*^C from id_C to $g \circ f$ and h_*^D from id_D to $f \circ g$. If f_n, g_{n+1} and h_n^C are bounded for some $n \in \mathbb{N}$, then the following holds: If D_* satisfies n-UBC, then so does C_*.*

Proof. Let $n \in N$ and let K_n be the maximum of the bounds of f_n, g_{n+1}, h_n^C and the n-UBC constant of D_*. Let $c \in C_n$ be a null-homologous cycle. Then $f_n(c) \in D_n$ is a null-homologous cycle and therefore there exists an efficient filling $b' \in D_{n+1}$ of c. We set $b := g_{n+1}(b') - h_n^C(c)$. Then we have

$$\partial b = \partial g_{n+1}(b') - \partial h_n^C(c) = g_n \circ f_n(c) + h_{n-1}^C(\partial c) - g_n \circ f_n(c) + c = c.$$

and

$$|b| \leq K_n^3 \cdot |c| + K_n \cdot |c|.$$

Hence, the chain b is an efficient filling of c. □

In our proof of Theorem 1.1 (Section 6), the following result will play an important role. It is a special case of the parametrised uniform boundary condition for tori [2, Theorem 1.3].

Theorem 3.16 (parametrised UBC for S^1). *Let $\Gamma := \pi_1(S^1) \cong \mathbb{Z}$, and let $\alpha = \Gamma \curvearrowright (Z, \mu)$ be an essentially free standard Γ-space. Then $C_*(S^1; \alpha)$ satisfies the uniform boundary condition in every degree.*

4. Yano's construction in the parametrised world

In addition to Yano's setup (Section 2), we need some technical prerequisites that we cover in the following section.

Let M be an oriented compact connected smooth n-manifold that admits a smooth S^1-action without fixed points and such that the inclusion of every orbit is π_1-injective. We write $\Gamma := \pi_1(M)$.

Proposition 4.1. *Let $k \in \{1, \ldots, n-2\}$ and let $j_1, \ldots, j_k \in \{0, \ldots, n-3\}$ be pairwise distinct. Then, the inclusions $X_{j_1,\ldots,j_k} \subset M_{j_1}$ and $X_{j_1,\ldots,j_k,-1} \subset M_{j_1}$ are π_1-injective.*

Proof. By Proposition 2.2 it suffices to show, that $X_{j_1,\ldots,j_k} \subset M_{j_1}$ is π_1-injective. By Lemma 2.1 we have

$$X_{j_1,\ldots,j_k} \cong \overline{X}_{j_1,\ldots,j_k} \times S^1$$

and each component of $\overline{X}_{j_1,\ldots,j_k}$ is contractible. With this in mind, observe that the composition of maps

$$X_{j_1,\ldots,j_k} \subset M_{j_1} \xrightarrow{p_{j_1,0}} M$$

is the inclusion of S^1-orbits into M and thus π_1-injective. Hence, the inclusion $X_{j_1,\dots,j_k} \subset M_{j_1}$ is also π_1-injective. □

Setup 4.2. Let $\alpha\colon \Gamma \curvearrowright (Z,\mu)$ be an essentially free standard Γ-space. We define a sequence

$$L^\infty(\alpha_{n-2},\mathbb{Z}) \underset{\mathbb{Z}\Gamma_{n-2}}{\otimes} C_*(\widetilde{M}_{n-2};\mathbb{Z}) \xrightarrow{P_{n-3}} \cdots \xrightarrow{P_0} L^\infty(\alpha_0,\mathbb{Z}) \underset{\mathbb{Z}\Gamma_0}{\otimes} C_*(\widetilde{M}_0;\mathbb{Z})$$

of chain maps: We write $\alpha_0 := \alpha$, $\alpha_{-1} := \alpha$ and $\Gamma_j := \pi_1(M_j)$. For all $j \in \{1,\dots,n-2\}$ let $\alpha_j\colon \Gamma_j \curvearrowright (Z,\mu)$ be the Γ_j-space obtained by restricting α along $\pi_1(p_{j,0})$, i.e., we consider the Γ_j-action on Z given by

$$\gamma \cdot z := \pi_1(p_{j,0})(\gamma)\cdot z.$$

Let $x_0 \in M_{n-2}$. Then, for all $j \in \{0,\dots,n-3\}$ the map p_j induces a homomorphism

$$P_j\colon L^\infty(\alpha_{j+1},\mathbb{Z}) \underset{\mathbb{Z}\Gamma_{j+1}}{\otimes} C_*(\widetilde{M}_{j+1};\mathbb{Z}) \longrightarrow L^\infty(\alpha_j,\mathbb{Z}) \underset{\mathbb{Z}\Gamma_j}{\otimes} C_*(\widetilde{M}_j;\mathbb{Z}),$$
$$f\otimes\sigma \longmapsto f\otimes \widetilde{p}_j\circ\sigma$$

where $\widetilde{p}_j$ denotes the lift of p_j with respect to the base point $p_{n-2,j}(x_0)$. Observe that $\|P_j\| \leq 1$ holds for all $j \in \{0,\dots,n-3\}$ with respect to the parametrised ℓ^1-norm. For all $j,j' \in \{0,\dots,n-2\}$ with $j' < j$ we write

$$P_{j,j'} := P_{j'}\circ\cdots\circ P_{j-1}\colon C_*(M_j,\alpha_j) \longrightarrow C_*(M_{j'},\alpha_{j'}).$$

For all $j \in \{0,\dots,n-2\}$ let $q_j\colon \widetilde{M}_j \longrightarrow M_j$ denote the universal covering of M_j. Since $q_j^{-1}(X_j)$ is closed under the Γ_j-action on $\widetilde{M}_j$, we can consider the subcomplex

$$L^\infty(\alpha_j,\mathbb{Z}) \underset{\mathbb{Z}\Gamma_j}{\otimes} C_*(q_j^{-1}(X_j);\mathbb{Z})$$

of $C_*(M_j;\alpha_j)$. We write $\Lambda_j := \pi_1(X_j) \subset \Gamma_j$ and $\alpha_j' := \operatorname{res}^{\Gamma_j}_{\Lambda_j}\alpha_j$. Let $\widetilde{X}_j$ be a connected component of $q_j^{-1}(X_j)$. Then there is a canonical (isometric) chain isomorphism

$$\begin{aligned} L^\infty(\alpha_j,\mathbb{Z}) \underset{\mathbb{Z}\Gamma_j}{\otimes} C_*(q_j^{-1}(X_j);\mathbb{Z}) &\cong L^\infty(\alpha_j,\mathbb{Z}) \underset{\mathbb{Z}\Gamma_j}{\otimes} \mathbb{Z}\Gamma_j \underset{\mathbb{Z}\Lambda_j}{\otimes} C_*(\widetilde{X}_j;\mathbb{Z}) \\ &\cong L^\infty(\alpha_j',\mathbb{Z}) \underset{\mathbb{Z}\Lambda_j}{\otimes} C_*(\widetilde{X}_j;\mathbb{Z}) \\ &= C_*(X_j;\alpha_j'). \end{aligned}$$

The analogous statements with $X_{j_1,\dots,j_k}$ (or $X_{j_1,\dots,j_k,-1}$) replacing X_j also hold for all $k \in \{1,\dots,n-2\}$ and pairwise distinct $j_1,\dots,j_k \in \{0,\dots,n-3\}$, where we define

$$\Lambda_{j_1,\dots,j_k(,-1)} := \pi_1(X_{j_1,\dots,j_k(,-1)}) \subset \Gamma_{j_1} \quad\text{and}\quad \alpha'_{j_1,\dots,j_k(,-1)} := \operatorname{res}^{\Gamma_{j_1}}_{\Lambda_{j_1,\dots,j_k(,-1)}}\alpha_{j_1}.$$

5. Constructing parametrised chains with small norm

In this section we construct parametrised chains of small norm that we adjust in Section 6 to get parametrised fundamental cycles of small norm.

Proposition 5.1. *Let M be an oriented compact connected smooth n-manifold that admits a smooth S^1-action without fixed points and such that the inclusion of every orbit is π_1-injective. We write $\Gamma := \pi_1(M)$. Let $\alpha\colon \Gamma \curvearrowright (Z,\mu)$ be an essentially free standard Γ-space. Then, we have*

$$\left| M_{n-2}, \partial M_{n-2} \right|^{\alpha_{n-2}} = 0,$$

where M_{n-2} is defined as in Section 2 and α_{n-2} as in Setup 4.2.

In fact, the proof will give an explicit construction of efficient parametrised relative fundamental cycles:

Proof. Recall that we have $M_{n-2} \cong \overline{M}_{n-2} \times S^1$ by Lemma 2.1. In particular, $\overline{M}_{n-2}$ is an orientable compact connected $(n-1)$-manifold. We choose an orientation on $\overline{M}_{n-2}$ such that the homeomorphism $M_{n-2} \cong \overline{M}_{n-2} \times S^1$ is orientation-preserving. Let $\Lambda \subset \Gamma_{n-2}$ be the subgroup corresponding to the S^1-factor of M_{n-2}. Then, the composition of maps

$$\Lambda \subset \Gamma_{n-2} \longrightarrow \Gamma$$

is injective as it is induced by the inclusion of an S^1-orbit. Hence, $\alpha' := \operatorname{res}_{\Lambda}^{\Gamma_{n-2}} \alpha_{n-2}$ is an essentially free standard Λ-space and the result follows [2, Lemma 10.8]. More precisely, let $\overline{K}$ be a triangulation of $\overline{M}_{n-2}$ that extends the simplicial structure of $\partial\overline{M}_{n-2}$ from Remark 2.3. Since $\overline{M}_{n-2}$ is an oriented compact connected manifold, we can construct a relative fundamental cycle $\overline{z} \in C_{n-1}(\overline{M}_{n-2};\mathbb{Z})$ out of the triangulation $\overline{K}$ of $\overline{M}_{n-2}$. Then, for all $\varepsilon \in \mathbb{R}_{>0}$ we can find an α'-parametrised fundamental cycle $c_{S^1} \in C_1(S^1;\alpha')$ such that the α_{n-2}-parametrised relative fundamental cycle given by

$$z := \overline{z} \times c_{S^1} \in C_n(M_{n-2};\alpha_{n-2})$$

has ℓ^1-norm less than ε. ☐

6. Proof of Theorem 1.1

In this last section we prove Theorem 1.1. We basically transfer Yano's original proof [14, Section 3] to the parametrised setting with the difference that we use the uniform boundary condition for S^1 (Theorem 3.16) to get efficient fillings.

In the following, we use the same notation as in Section 2 and Setup 4.2. Let $\varepsilon \in \mathbb{R}_{>0}$. We start with an α_{n-2}-parametrised relative fundamental cycle $z = \overline{z} \times c_{S^1}$ of M_{n-2} as in the proof of Proposition 5.1 with ℓ^1-norm less than ε. For all $j \in \{-1,\dots,n-3\}$ we define

$$\overline{z}_j := (\partial\overline{z})|_{\overline{N}_j} \in C_{n-2}(\overline{N}_j;\mathbb{Z})$$

as the sum of all simplices in $\partial\overline{z}$ that belong to the subcomplex $\overline{N}_j \subset \partial\overline{M}_{n-2}$. We set

$$z_j := \overline{z}_j \times c_{S^1} \in L^\infty(\alpha_{n-2},\mathbb{Z}) \underset{\mathbb{Z}\Gamma_{n-2}}{\otimes} C_{n-1}\big(q_{n-2}^{-1}(\widetilde{N}_j);\mathbb{Z}\big).$$

Analogously, for all $k \in \{1, \dots, n-1\}$ and all $j_1, \dots, j_k \in \{-1, \dots, n-3\}$ that are pairwise distinct we define inductively

$$\overline{z}_{j_1,\dots,j_k} := (\partial \overline{z}_{j_1,\dots,j_{k-1}})|_{\overline{N}_{j_1,\dots,j_k}} \in C_{n-1-k}(\overline{N}_{j_1,\dots,j_k}; \mathbb{Z})$$

and we set $\overline{z}_{j_1,\dots,j_k} := 0$ if $j_1, \dots, j_k$ are not pairwise distinct. We define

$$z_{j_1,\dots,j_k} := \overline{z}_{j_1,\dots,j_k} \times c_{S^1} \in L^\infty(\alpha_{n-2}, \mathbb{Z}) \underset{\mathbb{Z}\Gamma_{n-2}}{\otimes} C_{n-k}\big(q_{n-2}^{-1}(\widetilde{N}_{j_1,\dots,j_k}); \mathbb{Z}\big).$$

Lemma 6.1. *We have*

$$\partial z = \sum_{j=-1}^{n-3} z_j \quad \text{and} \quad \partial z_{j_1,\dots,j_k} = \sum_{j=-1}^{n-3} z_{j_1,\dots,j_k,j}$$

for all $k \in \{1, \dots, n-1\}$ and pairwise distinct $j_1, \dots, j_k \in \{-1, \dots, n-3\}$.

Proof. It is is enough to show the analogous statements for $\overline{z}$ and $\overline{z}_{j_1,\dots,j_k}$. Since $\partial \overline{M}_{n-2}$ is a subcomplex of $\overline{K}$ it follows from Remark 2.3 that we have

$$\partial \overline{z} = (\partial \overline{z})|_{\partial \overline{M}_{n-2}} = \sum_{j=-1}^{n-3} (\partial \overline{z})|_{\overline{N}_j} = \sum_{j=-1}^{n-3} \overline{z}_j$$

and for all $k \in \{1, \dots, n-1\}$ and all $j_1, \dots, j_k \in \{-1, \dots, n-3\}$ that are pairwise distinct, we have

$$\partial \overline{z}_{j_1,\dots,j_k} = (\partial \overline{z}_{j_1,\dots,j_k})|_{\partial \overline{N}_{j_1,\dots,j_k}} = \sum_i (\partial \overline{z}_{j_1,\dots,j_k})|_{\overline{N}_{j_1,\dots,j_k,i}} = \sum_{j=-1}^{n-3} \overline{z}_{j_1,\dots,j_k,j}$$

where i ranges over $\{-1, \dots, n-3\} \setminus \{j_1, \dots, j_k\}$. □

Lemma 6.2. *Let $k \in \{1, \dots, n-1\}$ and let τ be a permutation of $\{1, \dots, k\}$. Then we have*

$$z_{j_1,\dots,j_k} = \operatorname{sign}(\tau) \cdot z_{j_{\tau(1)},\dots,j_{\tau(k)}}.$$

Proof. We may assume that τ is a transposition. In fact, it is enough to consider the case of swapping the last two indices, i.e., to show that

$$z_{j_1,\dots,j_k} = -z_{j_1,\dots,j_{k-2},j_k,j_{k-1}}.$$

By Lemma 6.1 we have

$$0 = \partial\partial z_{j_1,\dots,j_{k-2}} = \sum_{j=-1}^{n-3} \partial z_{j_1,\dots,j_{k-2},j} = \sum_{j=-1}^{n-3} \sum_{i=-1}^{n-3} z_{j_1,\dots,j_{k-2},j,i}.$$

Because $\partial \overline{M}_{n-2}$ is a subcomplex of $\overline{K}$ and by Remark 2.3 it follows from the definition of $z_{j_1,\dots,j_k}$ that the only term that can cancel $z_{j_1,\dots,j_k}$ out is a term that has the same indices as $z_{j_1,\dots,j_k}$, namely $z_{j_1,\dots,j_{k-2},j_k,j_{k-1}}$, and therefore,

$$z_{j_1,\dots,j_k} = -z_{j_1,\dots,j_{k-2},j_k,j_{k-1}}.$$ □

Lemma 6.3. *There exist families of chains*

$$w_{j_1,\dots,j_k} \in C_{n-k+1}(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k})$$

and

$$w_{j_1,\dots,j_k,-1} \in C_{n-k}(X_{j_1,\dots,j_k,-1}; \alpha'_{j_1,\dots,j_k,-1})$$

with $k \in \{1, \dots, n-2\}$ and $j_1, \dots, j_k \in \{0, \dots, n-3\}$ satisfying the following conditions:

(1) *The chains $w_{j_1,\dots,j_k}$ and $w_{j_1,\dots,j_k,-1}$ are alternating with respect to permutations of the indices $\{j_1,\dots,j_k\}$.*

(2) *We have*
$$\partial w_{j_1,\dots,j_{n-2},-1} = P_{n-2,j_1}(z_{j_1,\dots,j_{n-2},-1})$$
and
$$\partial w_{j_1,\dots,j_k} = P_{n-2,j_1}(z_{j_1,\dots,j_k}) - \sum_{j=-1}^{n-3} w_{j_1,\dots,j_k,j}$$
and for $k < n-2$ we have
$$\partial w_{j_1,\dots,j_k,-1} = P_{n-2,j_1}(z_{j_1,\dots,j_k,-1}) + \sum_{j=0}^{n-3} w_{j_1,\dots,j_k,j,-1}.$$

(3) *Let $C \in \mathbb{R}_{>0}$ be the maximum of UBC-constants for all $X_{j_1,\dots,j_k}$ and all $X_{j_1,\dots,j_k,-1}$ in all degrees from 0 to n obtained from Theorem 3.16 together with Proposition 3.15 since all $X_{j_1,\dots,j_k}$ (and therefore all $X_{j_1,\dots,j_k,-1}$ by Proposition 2.2) are homotopy equivalent to S^1 by Lemma 2.1. Then, we have*
$$|w_{j_1,\dots,j_k}|_1 \leq C \cdot B^{n-k-1} \cdot (n+1)! \cdot |z|_1$$
where $B := 1 + C \cdot (n-1)$ and j_k might be -1.

Proof. We prove the lemma by downward induction on k. Let $k = n-2$. By Lemma 6.1 we have
$$\partial z_{n-3,\dots,0} = \sum_{j=-1}^{n-3} z_{n-3,\dots,0,j} = z_{n-3,\dots,0,-1}$$
We even have
$$\partial\big(P_{n-3}(z_{n-3,\dots,0})|_{X_{n-3,\dots,-1}}\big) = P_{n-3}\big(z_{n-3,\dots,0,-1}\big)$$
since $X_{n-3,\dots,-1}$ is a union of connected components of $X_{n-3,\dots,0}$ by 2.2, and therefore,
$$P_{n-3}(z_{n-3,\dots,-1}) \in C_1(X_{n-3,\dots,-1}; \alpha'_{n-3,\dots,-1})$$
is a null-homologous cycle.

We can apply the parametrised uniform boundary condition for S^1 (Theorem 3.16) on each connected component of $X_{n-3,\dots,-1}$. Then, there exists a chain $w_{n-3,\dots,-1} \in C_2(X_{n-3,\dots,-1}; \alpha'_{n-3,\dots,-1})$ with
$$\partial w_{n-3,\dots,-1} = P_{n-3}(z_{n-3,\dots,-1}) \quad \text{and} \quad |w_{n-3,\dots,-1}|_1 \leq C \cdot |z_{n-3,\dots,-1}|_1.$$
For each permutation τ of $\{0,\dots,n-3\}$ we set
$$w_{\tau(n-3),\dots,\tau(0),-1} := \operatorname{sign}(\tau) \cdot P_{n-3,\tau(n-3)}(w_{n-3,\dots,0,-1})$$
and we set $w_{j_1,\dots,j_{n-2},j} = 0$ for $\{j_1,\dots,j_{n-2},j\} \neq \{-1,\dots,n-3\}$.

Now, let $k \in \{1,\dots,n-2\}$ such that $w_{j_1,\dots,j_k,j}$ is defined for all $j_1,\dots,j_k \in \{0,\dots,n-3\}$ and all $j \in \{-1,\dots,n-3\}$. Let $j_1,\dots,j_k \in \{0,\dots,n-3\}$ with $j_1 > \dots > j_k$. We want to define $w_{j_1,\dots,j_k}$. We observe that
$$\widetilde{z}_{j_1,\dots,j_k} := P_{n-2,j_1}(z_{j_1,\dots,j_k}) - \sum_{j=-1}^{n-3} w_{j_1,\dots,j_k,j}$$

is a cycle in $C_{n-k}(X_{j_1,\dots,j_k};\alpha'_{j_1,\dots,j_k})$. Furthermore, the cycle $\widetilde{z}_{j_1,\dots,j_k}$ is null-homologous: By Lemma 2.1, each component of $\overline{X}_{j_1,\dots,j_k}$ is contractible and we have $X_{j_1,\dots,j_k} \cong \overline{X}_{j_1,\dots,j_k} \times S^1$ which implies

$$H_l(X_{j_1,\dots,j_k};\alpha'_{j_1,\dots,j_k}) \cong 0$$

for all $l \in \mathbb{N}_{\geq 2}$. We can apply the parametrised uniform boundary condition for S^1 (Theorem 3.16) on each connected component of $X_{j_1,\dots,j_k}$. Then, there exists a chain $w_{j_1,\dots,j_k} \in C_{n-k+1}(X_{j_1,\dots,j_k};\alpha'_{j_1,\dots,j_k})$ with

$$\partial w_{j_1,\dots,j_k} = \widetilde{z}_{j_1,\dots,j_k} = P_{n-2,j_1}(z_{j_1,\dots,j_k}) - \sum_{j=-1}^{n-3} w_{j_1,\dots,j_k,j}$$

and

$$\begin{aligned} |w_{j_1,\dots,j_k}|_1 &\leq C \cdot |z_{j_1,\dots,j_k}|_1 + C \cdot \sum_{j=-1}^{n-3} |w_{j_1,\dots,j_k,j}|_1 \\ &\leq C \cdot (n+1)! \cdot |z|_1 + (n-1) \cdot C^2 \cdot B^{n-k-2} \cdot (n+1)! \cdot |z|_1 \\ &\leq C \cdot B^{n-k-1} \cdot (n+1)! \cdot |z|_1. \end{aligned}$$

For arbitrary $j_1,\dots,j_k \in \{0,\dots,n-3\}$ we define $w_{j_1,\dots,j_k} := 0$ if $j_1,\dots,j_k$ are not pairwise distinct and otherwise we define

$$w_{j_1,\dots,j_k} := \operatorname{sign}(\tau) \cdot P_{\tau(j_1),j_1}(w_{\tau(j_1),\dots,\tau(j_k)}),$$

where τ is the unique permutation on $\{j_1,\dots,j_k\}$ with $\tau(j_1) > \dots > \tau(j_k)$.

Let now $k \in \{1,\dots,n-3\}$ such that $w_{j_1,\dots,j_k,j}$ is defined for all $j_1,\dots,j_k \in \{0,\dots,n-3\}$ and all $j \in \{0,\dots,n-3\}$. Let $j_1,\dots,j_k \in \{0,\dots,n-3\}$ with $j_1 > \dots > j_k$. We want to define $w_{j_1,\dots,j_k,-1}$. We consider

$$\hat{z}_{j_1,\dots,j_k} := P_{n-2,j_1}(z_{j_1,\dots,j_k}) - \sum_{j=0}^{n-3} w_{j_1,\dots,j_k,j} \in C_{n-k}(X_{j_1,\dots,j_k};\alpha'_{j_1,\dots,j_k})$$

and verify that

$$\partial \hat{z}_{j_1,\dots,j_k} = P_{n-2,j_1}(z_{j_1,\dots,j_k,-1}) + \sum_{j=0}^{n-3} w_{j_1,\dots,j_k,j,-1}.$$

Since $X_{j_1,\dots,j_k,-1}$ is the union of components of $X_{j_1,\dots,j_k}$ that lie in $p_{n-2,j_1}(\widetilde{N}_{-1})$, we have that

$$\partial \hat{z}_{j_1,\dots,j_k} \in C_{n-k}(X_{j_1,\dots,j_k,-1};\alpha'_{j_1,\dots,j_k,-1})$$

is a null-homologous cycle and by Theorem 3.16 there exists an efficient filling

$$w_{j_1,\dots,j_k,-1} \in C_{n-k}(X_{j_1,\dots,j_k,-1};\alpha'_{j_1,\dots,j_k,-1}),$$

i.e., we have

$$\partial w_{j_1,\dots,j_k,-1} = \partial \hat{z}_{j_1,\dots,j_k} = P_{n-2,j_1}(z_{j_1,\dots,j_k,-1}) + \sum_{j=0}^{n-3} w_{j_1,\dots,j_k,j,-1}$$

and

$$\begin{aligned}|\partial w_{j_1,\dots,j_k,-1}|_1 &\le C\cdot|z_{j_1,\dots,j_k,-1}|_1 + C\cdot\sum_{j=0}^{n-3}|w_{j_1,\dots,j_k,j,-1}|_1\\ &\le C\cdot(n+1)!\cdot|z|_1+(n-2)\cdot C^2\cdot B^{n-k-3}(n+1)!\cdot|z|_1\\ &\le C\cdot B^{n-k-2}\cdot(n+1)!\cdot|z|_1.\end{aligned}$$

For arbitrary $j_1,\dots,j_k\in\{0,\dots,n-3\}$ we define $w_{j_1,\dots,j_k,-1}:=0$ if $j_1,\dots,j_k$ are not pairwise distinct and otherwise we define

$$w_{j_1,\dots,j_k,-1}:=\operatorname{sign}(\tau)\cdot P_{\tau(j_1),j_1}(w_{\tau(j_1),\dots,\tau(j_k),-1}),$$

where τ is the permutation on $\{j_1,\dots,j_k\}$ with $\tau(j_1)>\dots>\tau(j_k)$. □

Finally, we are prepared to prove Theorem 1.1:

Proof of Theorem 1.1. We set

$$z' := P_{n-2,0}(z)-\sum_{j=0}^{n-3}P_{j,0}(w_j)\in C_n(M;\alpha).$$

By construction, z' is a relative cycle in $C_n(M;\alpha)$ with norm

$$|z'|_1\le C\cdot B^{n-3}\cdot(n+1)!\cdot|z|_1,$$

where z can be chosen with arbitrary small norm.

It is left to show that z' is an α-parametrised relative fundamental cycle of M: We write $p:=p_{n-2,0}$. Let $x\in M\setminus p(\partial M_{n-2})$ and let $U\subset M\setminus p(\partial M_{n-2})$ be an open neighbourhood of x with $U\cong D^n$. By Proposition 3.9 it is sufficient to prove that z' is a U-local α-parametrised relative fundamental cycle of M, i.e., we have

$$F\big([i(z')]\big)=\text{const}_1\in A_\Gamma\cong H_n(M,M\setminus U,A_\Gamma),$$

where $F:=f_*$ is the induced map in homology of the chain map

$$f\colon C_n(M,\partial M;\alpha)\longrightarrow A_\Gamma\underset{\mathbb{Z}}{\otimes}C_n(M,M\setminus U;\mathbb{Z})$$

which is defined as in Definition 3.8 and

$$i\colon C_n(M;\alpha)\longrightarrow C_n(M,\partial M;\alpha).$$

is the canonical map. Here, we write $A:=L^\infty(\alpha;\mathbb{Z})$.

We write $\beta:=\alpha_{n-2}$, $B:=L^\infty(\beta;\mathbb{Z})$, $N:=M_{n-2}$ and $\Lambda:=\pi_1(N)$. Since by construction p is the identity on $M\setminus p(\partial M_{n-2})$ we also write U for $p^{-1}(U)$. Let

$$g\colon C_n(N,\partial N;\alpha)\longrightarrow B_\Lambda\underset{\mathbb{Z}}{\otimes}C_n(N,N\setminus U;\mathbb{Z})$$

be the analogous map to f from Definition 3.8 and let

$$j\colon C_n(N;\alpha)\longrightarrow C_n(N,\partial N;\alpha)$$

be the canonical map.

Consider the following diagram

$$\begin{array}{ccc}
C_n(N;B) & \xrightarrow{\quad p_* \quad} & C_n(M;A) \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\
C_n(N,\partial N;B) & & C_n(M,\partial M;A) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
B_\Lambda \underset{\mathbb{Z}}{\otimes} C_n(N, N\setminus U;\mathbb{Z}) & \xrightarrow{\ t\ } & A_\Gamma \underset{\mathbb{Z}}{\otimes} C_n(M, M\setminus U;\mathbb{Z})
\end{array}$$

In the following capital letters denote the induced maps in homology. Since by construction, z' and $p_*(z)$ coincide on U, we have

$$f \circ i(z') = f \circ i\big(p_*(z)\big).$$

Moreover, z is a β-parametrised fundamental cycle of N, so by Proposition 3.9, we have

$$G\big([j(z)]\big) = \mathrm{const}_1 \in B_\Lambda.$$

Putting all together, it follows that

$$F\big([i(z')]\big) = [f \circ i \circ p_*(z)] = T \circ G\big([j(z)]\big) = T(\mathrm{const}_1) = \mathrm{const}_1 \in A_\Gamma$$

and it follows from Proposition 3.9 that z' is an α-parametrised fundamental cycle. □

Remark 6.4 (essentially free). Note that we never used that the whole action $\Gamma \curvearrowright Z$ is essentially free, but only that the restrictions of the action to every (π_1-injective) orbit $S^1 \cdot x$ on Z are essentially free (in Proposition 5.1 as well as the inductive filling argument using UBC for S^1 (Theorem 3.16) in the proof of Theorem 1.1).

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