Six lectures on analytic torsion

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Abstract

Contents

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1 Analytic torsion - from algebra to analysis - the finite-dimensional case

1.1 Torsion of chain complexes

Let k be a field. If V is a finite-dimensional k-vector space and $A : V \to V$ an isomorphism, then we have the **determinant** $\det A \in k^*$. The **torsion of a chain complex** is a generalization of the determinant as we will explain next.

If V is a finite-dimensional k-vector space of dimension n, then we define **the determinant of** V by

$$\det(V) := \Lambda^n V \ .$$

This is a one-dimensional k-vector space which functorially depends on V. By definition we have

$$\det\{0\} := k \ .$$

Note that det is a functor from the category of finite-dimensional vector spaces over k and isomorphisms to one-dimensional vector spaces over k and isomorphisms.

If L is a one-dimensional vector space, then we have a canonical isomorphism

$$\mathtt{Aut}(L) \cong k^* \ .$$

Under this identification the isomorphism $\det(A) : \det V \to \det V$ and the functor det induced by an isomorphism $A : V \to V$ is exactly mapped to the element $\det(A) \in k^*$.

We let $L^{-1} := \mathtt{Hom}_k(L, k)$ denote the dual k-vector space.

We now consider a **finite chain complex** over k, i.e. a chain complex

$$\mathcal{C} : \cdots \to C^{n-1} \to C^n \to C^{n+1} \to \dots$$

of finite-dimensional k-vector spaces which is bounded from below and above.

Definition 1.1. *We define the* ***determinant*** *of the chain complex $\mathcal{C}$ to be the one-dimensional k-vector space*

$$\det \mathcal{C} := \bigotimes_n (\det C^n)^{(-1)^n}$$

The determinant $\det(\mathcal{C})$ only depends on the underlying $\mathbb{Z}$-graded vector space of $\mathcal{C}$ and not on the differential.

The **cohomology** of the chain complex $\mathcal{C}$ can be considered as a chain complex $H(\mathcal{C})$ with trivial differentials. Hence the one-dimensional k-vector space $\det H(\mathcal{C})$ is well-defined.

Proposition 1.2. *We have a canonical isomorphism*

$$\tau_{\mathcal{C}} : \det \mathcal{C} \xrightarrow{\cong} \det H(\mathcal{C}) \ .$$

This isomorphism is called the **torsion isomorphism**.

Proof. For two finite-dimensional k-vector spaces U, W we have a canonical isomorphism

$$\det(U \oplus W) \cong \det U \otimes \det W \ .$$

More generally, given a short exact sequence (with U in degree 0)

$$\mathcal{V} : 0 \to U \to V \to W \to 0$$

we can choose a split. It induces a decomposition $V \cong U \oplus W$ and therefore an isomorphism

$$\det U \otimes \det W \cong \det V \ .$$

The main observation is that this isomorphism does not depend on the choice of the split. The torsion of the short exact sequence $\mathcal{V}$ is the induced isomorphism

$$\tau_{\mathcal{V}} : \det U \otimes (\det V)^{-1} \otimes \det W \to k \ .$$

Finally, we can decompose a chain complex $\mathcal{C}$ into short exact sequences and construct the torsion inductively by the length of the chain complex. Assume that $\mathcal{C}$ starts at $n \in \mathbb{Z}$. We consider the two short exact sequences

$$\mathcal{A} : 0 \to H^n(C) \to C^n \to B^{n+1} \to 0 \ ,$$
$$\mathcal{B} : 0 \to B^{n+1} \to C^{n+1} \to C^{n+1}/B^{n+1} \to 0 \ ,$$

and the sequence

$$\mathcal{C}' : 0 \to C^{n+1}/B^{n+1} \to C^{n+2} \to \cdots \to \ .$$

Here $B^{n+1} := d(C^n) \subset C^{n+1}$ denotes the subspace of boundaries. Note that $\mathcal{C}'$ starts in degree $n+1$. By induction, the isomorphism $\tau_{\mathcal{C}}$ is defined by

$$\begin{aligned}
\det(\mathcal{C}) &\cong (\det C^n)^{(-1)^n} \otimes (\det C^{n+1})^{(-1)^{n+1}} \otimes \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
&\overset{\tau_{\mathcal{A}}}{\cong} (\det H^n(C))^{(-1)^n} \otimes (\det B^{n+1})^{(-1)^n} \otimes (\det C^{n+1})^{(-1)^{n+1}} \otimes \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
&\overset{\tau_{\mathcal{B}}}{\cong} (\det H^n(C))^{(-1)^n} \otimes (\det(C^{n+1}/B^{n+1}))^{(-1)^{n+1}} \otimes \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
&\cong (\det H^n(C))^{(-1)^n} \otimes \det \mathcal{C}' \\
&\overset{\tau_{\mathcal{C}'}}{\cong} (\det H^n(C))^{(-1)^n} \otimes \det H(\mathcal{C}') \\
&\cong \det H(\mathcal{C}) \ .
\end{aligned}$$

□

Example 1.3. If $A : V \to W$ is an isomorphism of finite-dimensional vector spaces, then we can form the acyclic complex

$$\mathcal{A} : V \to W$$

with W in degree 0. Its torsion is an isomorphism

$$\tau_{\mathcal{A}} : (\det V)^{-1} \otimes \det W \to k^*$$

which corresponds to the generalization of the determinant of A as a morphism $\det(A) : \det V \to \det W$. Only for $V = W$ we can interpret this as an element in k^*.

Example 1.4. Let $\mathcal{C}$ be a finite chain complex and W be a finite-dimensional k-vector space. Then we form the chain complex

$$\mathcal{W} : W \stackrel{\mathtt{id}_W}{\to} W$$

starting at 0 and let $n \in \mathbb{Z}$. We say that the chain complex $\mathcal{C}' := \mathcal{C} \oplus \mathcal{W}[n]$ is obtained from $\mathcal{C}$ by a **simple expansion**.

If $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a simple expansion, then we have canonical isomorphisms $\det \mathcal{C} \cong \det \mathcal{C}'$ and $H(\mathcal{C}) \cong H(\mathcal{C}')$. Under these isomorphisms we have the equality of torsion isomorphisms $\tau_{\mathcal{C}} = \tau_{\mathcal{C}'}$.

1.2 Torsion and Laplace operators

We now assume that $k = \mathbb{R}$ or $k = \mathbb{C}$.

A **metric** h^V on V is a (hermitean in the case $k = \mathbb{C}$) scalar product on V. It induces a metric $h^{\det V}$ on $\det V$. This metric is fixed by the following property. Let $(v_i)_{i=1,\dots,n}$ be an orthonormal basis of V with respect to h^V, then $v_1 \wedge \dots \wedge v_n$ is a normalized basis vector of $\det V$ with respect to $h^{\det V}$.

Example 1.5. Let (V, h^V) and (W, h^W) be finite-dimensional k-vector spaces with metrics and $A : V \to W$ be an isomorphism of k-vector spaces. Then we can choose an isometry $U : W \to V$. We have the number $\det(UA) \in k^*$ which depends on the choice of U. We now observe that

$$|\det A| := |\det(UA)| \in \mathbb{R}^+ \tag{1}$$

does not depend on the choice of U. The analytic torsion generalizes this idea to chain complexes.

A metric $h^{\mathcal{C}}$ on a chain complex is a collection of metrics $(h^{C_n})_n$. Such a metric induces a metric on $\det \mathcal{C}$ and therefore, by push-forward, a metric $\tau_{\mathcal{C},*}h^{\mathcal{C}}$ on $\det H(\mathcal{C})$.

Definition 1.6. *Let $\mathcal{C}$ be a finite chain complex and $h^{\mathcal{C}}$ and $h^{H(\mathcal{C})}$ metrics on $\mathcal{C}$ and its cohomology $H(\mathcal{C})$. Then the* ***analytic torsion***

$$T(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})}) \in \mathbb{R}^+$$

is defined by the relation

$$h^{\det H(\mathcal{C})} = T(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})})\, \tau_{\mathcal{C},*} h^{\det \mathcal{C}} \ .$$

Example 1.7. In general the analytic torsion depends non-trivially on the choice of metrics. For example, if $t, s \in \mathbb{R}^+$, then we have the relation

$$T(\mathcal{C}, sh^{\mathcal{C}}, th^{H(\mathcal{C})}) = (\frac{t}{s})^{\chi(\mathcal{C})} \ ,$$

where $\chi(\mathcal{C}) \in \mathbb{Z}$ denotes the Euler characteristic of $\mathcal{C}$. But observe that if $\mathcal{C}$ is acyclic, then $\chi(\mathcal{C}) = 0$ and $T(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})})$ does not depend on the scale of the metrics.

Example 1.8. In this example we discuss the dependence of the analytic torsion on the choice of $h^{H(\mathcal{C})}$. Assume that $h_i^{H(\mathcal{C})}$, $i = 0, 1$ are two choices. Then we define numbers $v_k(h_0^{H(\mathcal{C})}, h_1^{H(\mathcal{C})}) \in \mathbb{R}^+$ uniquely such that

$$v_k(h_0^{H(\mathcal{C})}, h_1^{H(\mathcal{C})})\, h_1^{\det H^k(\mathcal{C})} = h_0^{\det H^k(\mathcal{C})} \ .$$

We further set

$$v(h_0^{H(\mathcal{C})}, h_1^{H(\mathcal{C})}) := \prod_k v_k(h_0^{H(\mathcal{C})}, h_1^{H(\mathcal{C})})^{(-1)^k} \ .$$

Then we have

$$T(\mathcal{C}, h^{\mathcal{C}}, h_0^{H(\mathcal{C})}) = v(h_0^{H(\mathcal{C})}, h_1^{H(\mathcal{C})})\, T(\mathcal{C}, h^{\mathcal{C}}, h_1^{H(\mathcal{C})}) \ .$$

We consider the $\mathbb{Z}$-graded vector space

$$C := \bigoplus_{n \in \mathbb{Z}} C^n$$

and the differential $d : C \to C$ as a linear map of degree one. The metric $h^{\mathcal{C}}$ induces a metric h^C such that the graded components are orthogonal. Using $h^{\mathcal{C}}$ we can define the adjoint $d^* : C \to C$ which has degree -1. We define the **Laplace operator**

$$\Delta := (d + d^*)^2 \ .$$

Since $d^2 = 0$ and $(d^*)^2 = 0$ we have $\Delta = dd^* + d^*d$. Hence the Laplace operator preserves degree and therefore decomposes as

$$\Delta = \oplus_{n \in \mathbb{Z}} \Delta_n \ .$$

As in **Hodge theory** we have an orthogonal decomposition

$$C \cong \mathtt{im}(d) \oplus \mathtt{ker}\Delta \oplus \mathtt{im}(d^*) \ , \quad \mathtt{ker}(d) = \mathtt{im}(d) \oplus \mathtt{ker}\Delta \ .$$

In particular, we get an isomorphism isomorphism of graded vector spaces

$$H(\mathcal{C}) \cong \mathtt{ker}(\Delta) \ .$$

This isomorphism induces the **Hodge metric** $h^{H(\mathcal{C})}_{Hodge}$ on $H(\mathcal{C})$.

We let Δ'_n be the restriction of Δ_n to the orthogonal complement of $\mathtt{ker}(\Delta_n)$.

Lemma 1.9. *We have the equality*

$$T(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})}_{Hodge}) = \sqrt{\prod_{n\in\mathbb{Z}} \det(\Delta'_n)^{(-1)^n n}} \ .$$

Proof. The differential d induces an isomorphism of vector space

$$d_k : C^k \supseteq \mathtt{ker}(d_{|C^k})^{\perp} \cong \mathtt{im}(d_{|C^k}) \subseteq C^{k+1} \ .$$

First show inductively that

$$\prod_k |\det\ d_k|^{(-1)^{k+1}} \tau_{\mathcal{C}} : \det \mathcal{C} \to \det H(\mathcal{C})$$

is an isometry (see (1) for notation), hence

$$T(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})}_{Hodge}) = \left[\prod_k |\det\ d_k|^{(-1)^{k+1}}\right] \ . \tag{2}$$

We repeat the construction of the torsion isomorphism. But in addition we introduce factors to turn each step into an isometry. Note that

$$|\det(d_k)|^{-1} \det(d_k) : \det(\mathtt{ker}(d_{|C^k})^{\perp}) \to \det(\mathtt{im}(d_{|C^k}))$$

is an isometry. This accounts for the first correction factor in the following chain of isometries.

$$
\begin{aligned}
\det(\mathcal{C}) &\cong (\det C^n)^{(-1)^n} \otimes (\det C^{n+1})^{(-1)^{n+1}} \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
&\stackrel{|\det d_n|^{(-1)^{n+1}}\tau_{\mathcal{A}}}{\to} (\det H^n(C))^{(-1)^n} \otimes (\det B^{n+1})^{(-1)^n} \otimes (\det C^{n+1})^{(-1)^{n+1}} \otimes \\
&\qquad \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
&\stackrel{\tau_{\mathcal{B}}}{\cong} (\det H^n(C))^{(-1)^n} \otimes (\det(C^{n+1}/B^{n+1}))^{(-1)^{n+1}} \otimes \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
&\cong (\det H^n(C))^{(-1)^n} \otimes \det(\mathcal{C}') \\
&\stackrel{\prod_{k=n+1}^{\infty} |\det d_k|^{(-1)^{k+1}}\tau_{\mathcal{C}'}}{\cong} (\det H^n(C))^{(-1)^n} \otimes \det H(\mathcal{C}') \\
&\cong \prod_{k=n+1}^{\infty} \det H(\mathcal{C}) \ .
\end{aligned}
$$

We now observe that

$$|\det d_k| = (\det d_k^* d_k)^{1/2} = (\det d_k d_k^*)^{1/2} \ .$$

Furthermore, we have

$$\det \Delta_k' = \det(d_k^* d_k) \det(d_{k-1} d_{k-1}^*) \ .$$

This gives

$$\prod_{k\in\mathbb{Z}} \det(\Delta_k')^{(-1)^k k} = \prod_{k\in\mathbb{Z}} \det(d_k^* d_k)^{(-1)^{k+1}} = \left[\prod_{k\in\mathbb{Z}} |\det d_k|^{(-1)^{k+1}}\right]^2 \ . \tag{3}$$

The Lemma now follows from (2). □

1.3 Torsion and Whitehead torsion

Let now G be a group and $\mathcal{X}$ be an acyclic based complex of free $\mathbb{Z}[G]$-modules. Its **Whitehead torsion** is an element

$$\tau(\mathcal{X}) \in Wh(G) \ .$$

Let $\rho : G \to SL(N, \mathbb{C})$ be a finite-dimensional representation of G. It induces a ring homomorphism

$$\rho : \mathbb{Z}[G] \to \mathtt{End}(\mathbb{C}^N) \ .$$

Then we can form the complex

$$\mathcal{C} := \mathcal{X} \otimes_{\mathbb{Z}[G]} \mathbb{C}^N \ .$$

Lemma 1.10. *This complex is acyclic.*

Proof. The acyclic complex of free (or more generally, of projective) $\mathbb{Z}[G]$-modules $\mathcal{X}$ admits a chain contraction. We get an induced chain contraction of $\mathcal{C}$. □

The basis of $\mathcal{X}$ together with the standard orthonormal basis of $\mathbb{C}^N$ induces a basis of $\mathcal{C}$ which we declare to be orthonormal, thus defining a metric $h^{\mathcal{C}}$. Since $H(\mathcal{C}) = 0$ we have a **canonical metric** $h^{H(\mathcal{C})}$ on $H(\mathcal{C}) = \mathbb{C}$ and the analytic torsion $T(\mathcal{C}, h^{\mathcal{C}}) := T(\mathcal{C}, h^{\mathcal{C}}, h^{\mathcal{H}(\mathcal{C})})$ is defined.

The representation ρ induces a homomorphism

$$K_1(\mathbb{Z}[G]) \to K_1(\mathtt{End}(\mathbb{C}^N)) \cong K_1(\mathbb{C}) \cong \mathbb{C}^* \stackrel{\|\cdot\|}{\to} \mathbb{R}^+ \ .$$

Under this homomorphism

$$K_1(\mathbb{Z}[G]) \ni [\pm g] \mapsto |\det(\pm\rho(g))| = 1 \in \mathbb{R}^+ \ .$$

Therefore, by passing through the quotient, we get a well-defined homomorphism

$$\chi_{\rho:} : Wh(G) = K_1(\mathbb{Z}[G])/(\pm[g]) \to \mathbb{R}^+ \ . \qquad (4)$$

Proposition 1.11. *The Whitehead torsion and the analytic torsion are related by*

$$T(\mathcal{C}, h^{\mathcal{C}}) = \chi_\rho(\tau(\mathcal{X})) \ .$$

Proof. We can define the Whitehead torsion of based complexes $\mathcal{X}$ over $\mathbb{Z}[G]$ again inductively by the length. We make the simplifying assumption that the complements of the images of the differentials are free. Assume that the complex $\mathcal{X}$ starts with X_n. We consider the short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to X_n \to X_{n+1} \stackrel{p}{\to} X_{n+1}/X_n \to 0$$

and set

$$\mathcal{X}' : 0 \to X_{n+1}/X_n \stackrel{i}{\to} X_{n+2} \to \dots \ .$$

Let c_n be the chosen basis of X_n. We choose a basis c'_{n+1} of X_{n+1}/X_n. Lifting its elements and combining it with the images of the elements of c_n and get a basis b' of X_{n+1}. Note that $\mathcal{X}'$ is again based (by $c'_k := c_k$ for $k \geq n+2$ and the basis c'_{n+1} chosen above) and starts at $n+1$. Then by definition of the Whitehead torsion

$$\tau(\mathcal{X}) = [c_{n+1}/b']^{(-1)^{n+1}} \tau(\mathcal{X}') \in Wh(G) \ .$$

Note that

$$T(\mathcal{C}, h^{\mathcal{C}}) = T(\mathcal{D}, h^{\mathcal{D}})\, T(\mathcal{C}', h^{\mathcal{C}'}) \ ,$$

where

$$\mathcal{D} : 0 \to X_n \otimes_{\mathbb{Z}[G]} \mathbb{C}^N \to X_{n+1} \otimes_{\mathbb{Z}[G]} \mathbb{C}^N \stackrel{p}{\to} X_{n+1}/X_n \otimes_{\mathbb{Z}[G]} \mathbb{C}^N \to 0$$

starting at n, and we use the metrics induced by c_n, c_{n+1} and c'_{n+1}. Here we use (2) and that $d_{n+1} = i \circ p$ and hence $|\det\, d_{n+1}| = |\det\, p||\det\, i|$. Therefore we must check that

$$\chi_\rho([c_{n+1}/b'])^{(-1)^{n+1}} = T(\mathcal{D}, h^{\mathcal{D}}) \ . \tag{5}$$

The choice of the lift in the definition of b' induces a split of this sequence $\mathcal{D}$. On its middle vector space we have two metrics, one defined by the split, and the other defined by the basis c_{n+1}. If we take the split metric, then its torsion is trivial. Hence $T(\mathcal{D}, h^{\mathcal{D}})$ is equal to the determinant of the base change from b' to c_{n+1}, i.e. (5) holds true, indeed.□

Example 1.12. We consider the group $\mathbb{Z}/5\mathbb{Z}$ and the complex

$$\mathcal{X} : \mathbb{Z}[\mathbb{Z}/5\mathbb{Z}] \stackrel{1-[1]-[4]}{\to} \mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]$$

starting at 0. This complex is acyclic since $1 - [2] - [3]$ is an inverse of the differential. We consider the representation $\mathbb{Z}/5\mathbb{Z} \to U(1)$ which sends [1] to $\exp(\frac{2\pi i}{5})$. Its Whitehead torsion is represented by $1 - [1] - [4] \in \mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]^*$. Then

$$\tau_\rho([1 - [1] - [4]) = \|1 - \exp(\frac{2\pi i}{5}) - \exp(\frac{8\pi i}{5})\| = 2\cos(\frac{2\pi}{5}) - 1 \neq 1 \ .$$

2 Zeta regularized determinants of operators - Ray-Singer torsion

2.1 Motivation

Let $(\mathcal{C}, h^{\mathcal{C}})$ be a finite chain complex over $\mathbb{R}$ or $\mathbb{C}$ with a metric. Then by Lemma 1.9 we have the following formula for its analytic torsion

$$T(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})}_{Hodge}) = \sqrt{\prod_k (\det \Delta'_k)^{(-1)^k k}} \ .$$

Let now (M, g^{TM}) be a closed Riemannian manifold manifold. Then we can equip the de Rham complex $\Omega(M)$ with a a metric $h^{\Omega(M)}_{L^2}$ given by

$$h^{\Omega(M)}_{L^2}(\alpha, \beta) = \int_M \alpha \wedge *_{g^{TM}} \beta \ ,$$

where $*_{g^{TM}}$ is the Hodge-$*$ operator associated to the metric.

More generally, let (V, ∇^V, h^V) be a vector bundle with a flat connection and a metric. Then we can form the twisted de Rham complex $\Omega(M, V)$. We consider the sheaf $\mathcal{V}$ of parallel sections of (V, ∇). The Rham isomorphism relates the sheaf cohomology of $\mathcal{V}$ with the cohomology of the twisted de Rham complex:

$$H(M, \mathcal{V}) \cong H(\Omega(M, V)) \ .$$

The metric h^V together with the Riemannian metric g^{TM} induce a metric $h_{L^2}^{\Omega(M,V)}$ on the twisted de Rham complex.

Note that to give (V, ∇) is, up to isomorphism, equivalent to give a representation of the fundamental group

$$\pi_1(M) \to \mathtt{End}(\mathbb{C}^{\dim(V)})$$

(we assume M to be connected, for simplicity). Hence (M, V, ∇^V) is **differential-topological data**, while the metrics g^{TM} and h^V are **additional geometric choices**.

In this section we discuss the **definition of analytic torsion**

$$T(M, \nabla^V, g^{TM}, h^V) := \sqrt{\prod_k (\det \Delta'_k)^{(-1)^k k}}$$

which is essentially due to Ray-Singer [RS71]. To this end we must define the determinant of the Laplace operators properly. We will also discuss in detail, how the torsion depends on the metrics.

2.2 Spectral zeta functions

We consider a finite-dimensional vector space with metric (V, h^V) and a linear, invertible, selfadjoint and positive map $\Delta : V \to V$. Then the endomorphism $\log(\Delta)$ is defined by spectral theory and we have the relation

$$e^{\mathtt{Tr} \log \Delta} = \det(\Delta) \ .$$

The **spectral zeta** function of Δ is defined by

$$\zeta_\Delta(s) = \mathtt{Tr}\ \Delta^{-s} \ , \quad s \in \mathbb{C} \ .$$

It is an entire function on $\mathbb{C}$ and satisfies

$$-\zeta'_\Delta(0) = \mathtt{Tr}\ \log(\Delta) \ .$$

So we get the formula for the determinant of Δ in terms of the spectral zeta function

$$\det \Delta = e^{-\zeta'_\Delta(0)} \ .$$

The idea is to use this formula to define the determinant in the case where Δ is a differential operator.

We now consider a closed Riemannian manifold (M, g^{TM}) and a vector bundle (V, ∇^V, h^V) with connection and metric. The metrics induce L^2-scalar products on $\Omega^k(M, V)$ so that we can form the adjoint

$$\nabla^{V,*} : \Omega^1(M, V) \to \Omega^0(M, V)$$

of the connection ∇^V. The **Laplace operator** is the differential operator

$$\Delta := \nabla^{V,*}\nabla^V : \Gamma(M,V) \to \Gamma(M,V) \ .$$

It is symmetric with respect to the metric $\|.\|_{L^2} := h_{L^2}^{\Omega^0(M,V)}$.
More generally, a second order differential operator A on $\Gamma(M,V)$ is called of **Laplace type** of it is of the form $A = \Delta + R$, for Δ defined for certain choices of h^{TM}, h^V, ∇^V such that $R \in \Gamma(M, \texttt{End}(V))$ is a selfadjoint bundle endomorphism with respect to the same metrics

Assume that A is a Laplace type differential operator on $\Gamma(M,V)$ and symmetric with respect to a metric $\|.\|_{L^2}$. We consider (possibly densely defined unbounded) operators on the Hilbert space closure $\overline{\Gamma(M,V)}^{\|.\|_{L^2}}$. The following assertions are standard facts from the analysis of elliptic operators on manifolds.

1. *A is an elliptic differential operator.* Indeed, its principal symbol is that of the Laplace operator and given by $\sigma_A(\xi) = \|\xi\|^2_{g^{TM}}$.

2. *A is essentially selfadjoint on the domain $\Gamma(M,V)$.* It is a general fact for a symmetric elliptic operator A on a closed manifold that its closure $\bar{A}$ coincides with the adjoint A^*. The proof uses elliptic regularity.

3. *The spectrum of A is real, discrete of finite multiplicity and accumulates at $+\infty$.* Since A is essentially selfadjoint the operator $A+i$ is invertible on $L^2(M,V)$. Using again elliptic regularity in the quantitative form

   $$\|\phi\|_{H^2} \leq C(\|A\phi\|_{L^2} + \|\phi\|_{L^2})$$

   we see that its inverse can be factored as a composition of a bounded operator and an inlcusion

   $$L^2(M,V) \stackrel{(A+i)^{-1}}{\to} H^2(M,V) \stackrel{incl}{\to} L^2(M,V) \ .$$

   For a closed manifold the inclusion of the second Sobolev space into the L^2-space is compact by Rellich's theorem. This shows that $(A+i)^{-1}$ is compact as an operator on L^2. We conclude discreteness of the spectrum. Furthermore, using the positivity of the Laplace operator and the fact that R is bounded, we see that the spectrum accumulates at ∞.

4. *The number (with multiplicity) of eigenvalues of A less than $\lambda \in (0,\infty)$ grows as $\lambda^{\dim(M)/2}$. This is called **Weyl's asymptotic**.* This follows from the heat asymptotics stated in Proposition 2.3.

5. *A preserves an orthogonal decomposition of*

   $$\Gamma(M,V) = N \oplus P \ ,$$

   such that $\dim(N) < \infty$, $A_{|N} \leq 0$ *and* $A' := A_{|P} > 0$. This follows from 3.

6. $\zeta_{A'}(s) := \mathtt{Tr}\ A'^{,-s}$ *is holomorphic for* $\mathtt{Re}(s) > \dim(M)/2$. This is a consequence of Weyl's asymptotic stated in 4.

In order to define $\zeta_A'(0)$ we need an **analytic continuation** of the zeta function. Note that for $\lambda > 0$ and $s > 0$ we have the equality

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt \ . \tag{6}$$

We define the **Mellin transform** of a measurable function θ of $t \in (0,\infty) \to \mathbb{C}$ for $s \in \mathbb{C}$ by

$$M(\theta)(s) := \int_0^\infty \theta(t) t^{s-1} dt \ ,$$

provided the integral converges.

Example 2.1. For $\lambda \in (0,\infty)$ we have $M(e^{\lambda t})(s) = \Gamma(s)\lambda^{-s}$.

Lemma 2.2. *Assume that* θ *is exponentially decreasing for* $t \to \infty$, *and that it has an asymptotic expansion*

$$\theta(t) \overset{t\to 0}{\sim} \sum_{n\in\mathbb{N}} a_n t^{\alpha_n}$$

for a monotoneously increasing, unbounded sequence $(\alpha_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$. *Then* $M(\theta)(s)$ *is defined for* $\mathtt{Re}(s) > -\alpha_0$ *and has a meromorphic continuation to all of* $\mathbb{C}$ *with first order poles at the points* $s = -\alpha_n$, $n \in \mathbb{N}$, *such that*

$$\mathtt{res}_{s=-\alpha_n} M(\theta)(s) = a_n \ .$$

Proof. This is an exercise. The idea is to split the integral in the Mellin transformation as $\int_0^1 + \int_1^\infty$. The second summand yields an entire function. In order to study the first summand one decomposes $\theta(t)$ as a sum of the first n terms of its expansion and a remainder. The integral of the asymptotic expansion term can be evaluated explicitly and contributes the singularities for $\mathtt{Re}(s) > -\alpha_{n+1}$, and the remainder gives a holomorphic function on this domain. Since we can choose n arbitrary large we get the assertion. □

In view of Weyl's asymptotics the **heat trace** of a Laplace-type operator A on a closed manifold is defined for $t > 0$ as

$$\theta_A(t) := \mathtt{Tr}\ e^{-tA} \ .$$

Proposition 2.3. *Assume that* A *is a Laplace-type operator on a closed manifold.*

1. *We have an asymptotic expansion*

$$\theta_A(t) \overset{t\to 0}{\sim} \sum_{n\geq 0} a_n(A) t^{n-\dim(M)/2} \ . \tag{7}$$

2. $\theta_{A'}(t)$ *vanishes exponentially as* $t \to \infty$.

Proof. For a proof of 1. we refer e.g. to [**?**, Thm. 2.30]. The second assertion is an exercise. □

Note that the numbers $a_n(A)$ are integrals over M of local invariants of A. Further note that $\theta_{A|_{|N}}(t)$ is smooth at $t = 0$ and therefore

$$\theta_{A'}(t) = \theta_A(t) - \theta_{A|_{|N}}(t)$$

also has an asymptotic expansion at $t \to 0$ whose singular part coincides with the singular part of (7). But also note that in the odd-dimensional case the positive part of the expansion for $\theta_{A'}(t)$ in general has terms with $t^{m/2}$ for all $m \in \mathbb{N}$ (not only for odd m). By (6) the spectral zeta function of A' can be written in the form

$$\zeta_{A'}(s) = \frac{1}{\Gamma(s)} M(\theta_{A'})(s) \ .$$

By Proposition 2.3 and Lemma 2.2 it has a meromorphic continuation. Since $\Gamma(s)$ has a pole at $s = 0$ we further see that $\zeta_{A'}(s)$ is regular it $s = 0$.

Definition 2.4. *We define the* ***zeta-regularized determinant*** *of a Laplace-type operator A on a closed manifold by*

$$\det A' := e^{-\zeta'_{A'}(0)} \ .$$

Remark 2.5. The **value** $\zeta_{A'}(0)$ of the zeta function at zero can be calculated. It is given by the coefficient of the constant term of the asymptotic expansion of $\theta_{A'}(t)$. We get

$$\zeta_{A'}(0) = a_{\dim(M)/2}(A) - \dim(N) \ .$$

It is a combination of a locally computable quantity $a_{\dim(M)/2}$ and information about finitely many eigenvalues. Note that the determinant is a much more difficult quantity.

Example 2.6. For $R > 0$ let M be $S^1_R := \mathbb{R}/R\mathbb{Z}$, i.e. the circle of volume R, and $A := -\partial_t^2$.

Lemma 2.7. *We have* $\det A' = R^2$.

Proof. Then the eigenvalues of A are given by

$$4\pi^2 R^{-2} n^2 \ , \quad n \in \mathbb{Z} \ .$$

We can express the spectral zeta function in terms of the Riemann zeta function as

$$\zeta_{A'}(s) = 2^{1-2s} R^{2s} \pi^{-2s} \zeta(2s) \ .$$

We get

$$\zeta'_{A'}(0) = -4\log(2\pi R^{-1})\zeta(0) + 4\zeta'(0) \ .$$

Using the formulas

$$\zeta(0) = -\frac{1}{2}\ , \quad \zeta'(0) = -\frac{\log(2\pi)}{2}$$

we get

$$\zeta'_{A'}(0) = 2\log(2\pi R^{-1}) - 2\log(2\pi) = -2\log(R)\ .$$

The final formula

$$\det A' = R^2$$

now follows. □

Observe the dependence of the determinant on the geometry.

2.3 Analytic torsion

We consider a closed Riemannian manifold (M, g^{TM}) and a vector bundle (V, ∇^V, h^V) with a flat connection and metric. The connection $\nabla^V : \Omega^0(M,V) \to \Omega^1(M,V)$ extends uniquely to a derivation of $\Omega(M)$-modules

$$d^V : \Omega(M,V) \to \Omega(M,V)$$

of degree one and square zero. Then the Laplace operator

$$\Delta := (d^{V,*} + d^V)^2 \tag{8}$$

preserves degree and its components

$$\Delta_k : \Omega^k(M,V) \to \Omega^k(M,V)\ , \quad k \in \mathbb{N}\ , \tag{9}$$

are of Laplace type.

Definition 2.8. *We define the analytic torsion of* (M, ∇, h^{TM}, h^V) *by*

$$T_{an}(M, \nabla^V, h^{TM}, h^V) := \sqrt{\prod_{k\in\mathbb{N}} (\det \Delta'_k)^{(-1)^k k}}\ .$$

It is the analog of $T(\Omega(M,V), h_{L^2}^{\Omega(M,V)}, h_{Hodge}^{H(M,V)})$.

As a consequence of Poincaré duality the analytic torsion for unitary flat bundles on even-dimensional manifolds is trivial. We say that a hermitean bundle with connection (V, ∇^V, h^V) is **unitary**, if ∇^V preserves h^V. Unitary flat bundles correspond to unitary representations of the fundamental group.

Proposition 2.9. *If* M *is even-dimensional and* ∇^V *is unitary, then*

$$T_{an}(M, \nabla^V, g^{TM}, h^V) = 1\ .$$

Proof. We have

$$\Delta_k = d_k^{V,*} d_k^V \oplus d_{k-1}^V d_{k-1}^{V,*}$$

and therefore, with appropriate definitions and (3),

$$T_{an}(M, \nabla^V, g^{TM}, h^V) = \sqrt{\prod_k (\det(d_k^{V,*} d_k^V)')^{(-1)^{k+1}}} \ . \tag{10}$$

Using that ∇^V is unitary we get the identity $*_{g^{TM}}(d_k^{V,*} d_k^V) *_{g^{TM}}^{-1} = d_{n-k-1}^V d_{n-k-1}^{V,*}$. It implies

$$\det(d_k^{V,*} d_k^V)' = \det(d_{n-k-1}^{V,*} d_{n-k-1}^V)' \ .$$

If $\dim(M)$ is even, then we see that the factors for k and $\dim(M) - k - 1$ in (10) cancel each other. □

2.4 Ray-Singer torsion

Let (M, g^{TM}) be a closed odd-dimensional Riemannian manifold, (V, ∇^V, h^V) be a vector bundle on M with flat connection and metric, and $h^{H(M,\mathcal{V})}$ be a metric on the cohomology.

Definition 2.10. *We define the* ***Ray-Singer torsion*** *of* $(M, \nabla^V, h^{H(M,\mathcal{V})})$ *by*

$$T_{RS}(M, \nabla^V, h^{H(M,\mathcal{V})}) := v(h^{H(M,\mathcal{V})}, h^{H(M,V)}_{Hodge})\, T_{an}(M, \nabla^V, g^{TM}, h^V) \ . \tag{11}$$

It is interesting because of the following theorem (which also justifies the omission of the metrics in the notation).

Theorem 2.11. *The* ***Ray-Singer torsion*** $T_{RS}(M, \nabla^V, h^{H(M,\mathcal{V})})$ *is independent of the choices of metrics* g^{TM} *and* h^V*.*

Proof. We give a sketch. We first assume that $H(M, \mathcal{V}) = 0$. As a consequence all integrands below vanish exponentially at $t \to \infty$. Moreover, the Ray-Singer torsion coincides with the analytic torsion. Let N denote the $\mathbb{Z}$-grading operator on $\Omega(M, V)$. We define

$$F(s) := \frac{1}{\Gamma(s)} \int_0^\infty \mathtt{Tr}\, (-1)^N N e^{-t\Delta} t^{s-1} dt \ .$$

This integral converges for $\mathtt{Re}(s) >> 0$ and by Proposition 2.3 and Lemma 2.2 has, as a function of s, a meromorphic continuation to $\mathbb{C}$. Then

$$\log T_{an}(M, \nabla^V, h^{TM}, h^V) = -\frac{d}{ds}_{|s=0} F(s) \ .$$

Since any two metric data can be connected by a path, it suffices to discuss the variation formula. The derivative of $F(s)$ with respect to the metric data is given by

$$\delta F(s) = \int_0^\infty \mathtt{Tr}\, (-1)^N N \delta(e^{-t\Delta}) t^{s-1} dt = -\int_0^\infty \mathtt{Tr}\, (-1)^N N \delta(\Delta) e^{-t\Delta} t^s dt \ .$$

Here we use that $\delta(\Delta)$ commutes with N and the cyclicity of the trace.
In order to calculate $\delta(\Delta)$ we encode the metric data into a duality map

$$I : \Omega(M,V) \to \overline{\Omega(M,V)}' \ , \quad \langle \alpha, \omega \rangle = I(\alpha)(\omega) \ .$$

We further define its logarithmic derivative $L := I^{-1}\delta(I) \in \Gamma(M, \mathtt{End}(\Lambda^* T^* M \otimes V))$. We write $d^{V,*} = I^{-1} d^{V,\prime} I$, where the adjoint $d^{V,\prime}$ of d^V does not depend on the metrics. Consequently, $\delta(d^{V,*}) = -[L, d^{V,*}]$. Inserting this into (8) we get

$$\delta(\Delta) = -L d^{V,*} d^V + d^V d^{V,*} L - d^V L d^{V,*} + d^{V,*} L d^V \ .$$

Using $[\Delta, d^V] = 0$, $[\Delta, d^{V,*}] = 0$, $[d^V, N] = -d^V$ and $[d^{V,*}, N] = d^{V,*}$ and the cyclicity of the trace we get

$$\mathtt{Tr}\ (-1)^N N \delta(\Delta) e^{-t\Delta} = \mathtt{Tr} (-1)^N L \Delta e^{-t\Delta} = -\frac{d}{dt} \mathtt{Tr}\ (-1)^N L e^{-t\Delta} \ .$$

Here are some more details of the calculation in which we move all differential operators on the left of L to the right. In this process we must commute them with $(-1)^N N$, the heat operator, and we use the cyclicity of the trace.

$$\begin{aligned} \mathtt{Tr}\ (-1)^N N \delta(\Delta) e^{-t\Delta} &= \mathtt{Tr}\ (-1)^N N(-L d^{V,*} d^V + d^V d^{V,*} L - d^V L d^{V,*} + d^{V,*} L d^V) e^{-t\Delta} \\ &= \mathtt{Tr}\ (-1)^N N(-L d^{V,*} d^V + L d^V d^{V,*} + L d^{V,*} d^V - L d^V d^{V,*}) e^{-t\Delta} \\ &\quad + \mathtt{Tr}\ (-1)^N L d^{V,*} d^V e^{-t\Delta} + \mathtt{Tr}\ (-1)^N L d^V d^{V,*} e^{-t\Delta} \end{aligned}$$

□

We get by partial integration for $\mathtt{Re}(s) >> 0$ (in order to avoid a boundary term at $t = 0$)

$$\begin{aligned} \delta F(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{dt} \left[-\mathtt{Tr}\ (-1)^N L e^{-t\Delta} \right] t^s dt \\ &= \frac{s}{\Gamma(s)} \int_0^\infty \mathtt{Tr}\ (-1)^N L e^{-t\Delta}\ t^{s-1} dt \ . \end{aligned}$$

We now use the asymptotic expansion (a generalization of Proposition 2.3, 1. to traces of the form $\mathtt{Tr}\ L e^{-tA}$, where L is some bundle endmorphism)

$$\mathtt{Tr}\ (-1)^N L e^{-t\Delta} \overset{t \to 0}{\sim} \sum_{n \in \mathbb{N}} b_n\, t^{n - \dim(M)/2} \ . \tag{12}$$

In particular it has no constant term. Therefore, by Lemma 2.2 we have

$$\delta F(s) = \frac{s}{\Gamma(s)} \kappa(s) \ ,$$

where κ is meromorphic on $\mathbb{C}$ and regular at $s = 0$. In order to get the logarithmic derivative of the Ray-Singer torsion we must apply $-\frac{d}{ds}_{|s=0}$ to this function. Since $\frac{s}{\Gamma(s)}$ has a second order zero at $s = 0$ we conclude that

$$\delta \log T_{RS}(M, \nabla^V, h^{H(M,\mathcal{V})}) = 0 \ .$$

In the presence of cohomology one uses a similar argument. In the definition of $F(s)$ one replaces $\mathtt{Tr}$ by $\mathtt{Tr}(1-P)$, where P is the projection onto $\mathtt{ker}(\Delta)$. Then (12) has a constant term given by $-\mathtt{Tr}\,(-1)^N PL$. Using that $\mathtt{res}_{s=0}\Gamma(s)=1$ we get

$$\delta \log T_{an}(M,\nabla^V,h^{TM},h^V) = -\mathtt{Tr}(-1)^N PL \ .$$

This is exactly the negative of the logarithmic variation of the volume on the cohomology induced by the Hodge metric, i.e.

$$\delta \log v(h^{H(M,\mathcal{V})},h^{H(M,V)}_{Hodge}) = \mathtt{Tr}(-1)^N PL \ .$$

These two terms cancel in the product defining the Ray-Singer torsion.

Remark 2.12. The arguments $(M,\nabla^V,h^{H(M,\mathcal{V})})$ of the Ray-Singer torsion are differential-topological data. The right-hand side is of global analytic nature and apriori depends on additional geometric choices. The interesting fact is that its actually does not depend on these choices. This is a typical situation in which the natural question is now to provide an explicit description of this quantity in terms of differential topology.

2.5 Torsion for flat bundles on the circle

In this example we give an explicit calculation of the Ray-Singer torsion for $M=S^1$ and the flat line (V,∇^V) bundle with holonomy $1\neq\lambda\in U(1)$. We have $H(S^1,\mathcal{V})=0$ so that we can drop the metric on the cohomology from the notation.

Proposition 2.13. *We have*

$$T_{RS}(S^1,\nabla^V) = \frac{1}{2\sin(\pi q)} \ .$$

Proof. We represent $S^1:=\mathbb{R}/\mathbb{Z}$ in order to fix the geometry. In order to calculate the spectrum of the Laplace operator we work on the universal covering $\mathbb{R}$ and trivialize the bundle $T^*\mathbb{R}$ using the section dt. We further trivialize the pull-back of the flat line bundle using parallel sections. Under these identifications

$$\Omega^1(S^1,L) = \{f\in C^\infty(\mathbb{R}) \mid (\forall t\in\mathbb{R} \mid f(t+1)=\lambda f(t))\} \ .$$

The Laplace operator Δ_1 acts as $-\partial_t^2$.
We now calculate its spectrum. We choose $q\in(0,1)$ such that $\lambda=e^{2\pi i q}$. The eigenvectors of Δ_1 are the functions $t\mapsto e^{2\pi i(q+n)t}$ for $n\in\mathbb{Z}$, and the corresponding eigenvalues are given by

$$4\pi^2(q+n)^2 \ .$$

The zeta function of the Laplace operator is now

$$\zeta_{\Delta_1} = 4^{-s}\pi^{-2s}\sum_{n\in\mathbb{Z}}(q+n)^{-2s} \ .$$

In order to calculate $\det \Delta_1$ we express this zeta function in terms of the Hurwitz zeta function

$$\zeta(s,q) := \sum_{n\in\mathbb{N}} (q+n)^{-s}$$

and then use known properties of the latter. We have

$$\zeta_{\Delta_1}(s) = 4^{-s}\pi^{-2s}\left[\zeta(2s,q) + \zeta(2s,1-q)\right] \ .$$

We have the relation

$$\partial_q \zeta(s,q) = -s\zeta(s+1,q) \ .$$

This gives

$$\partial_s\partial_q[\zeta(s,q)+\zeta(s,1-q)] = [\zeta(s+1,1-q)-\zeta(s+1,q)] + s[\partial_s\zeta(s+1,1-q)-\partial_s\zeta(s+1,q)] \ .$$

We now use the expansion of the Hurwitz zeta function at $s=1$

$$\zeta(s,q) = (s-1)^{-1} - \psi(q) + O(s-1)$$

with

$$\psi(q) := \frac{\Gamma'(q)}{\Gamma(q)} \ .$$

We see that the two differences are regular at $s=1$. The evaluation of the second term at $s=0$ vanishes because of the prefactor s. Hence

$$\partial_{s|s=0}\partial_q[\zeta(s,q)+\zeta(s,1-q)] = \psi(q) - \psi(1-q) \ .$$

Consequently, integrating from $q=1/2$ we get using

$$\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin(\pi q)}$$

that

$$\partial_{s|s=0}[\zeta(s,q)+\zeta(s,1-q)] = \gamma + \log(\Gamma(q)\Gamma(1-q)) = \gamma + \log(\frac{\pi}{\sin \pi q}) \ ,$$

where

$$\gamma := 2\partial_{s|s=0}\zeta(s,1/2) - \log(\pi) \ .$$

In order to determine this number we express the Hurwitz zeta function in terms of the Riemann zeta function

$$\begin{aligned} \zeta(s,1/2) \quad &= \sum_{n\in\mathbb{N}}(n+1/2)^{-s} = 2^s\sum_{n\in\mathbb{N}}(2n+1)^{-s} \\ &= 2^s\sum_{n\in\mathbb{N}}(2n+1)^{-s} + 2^s\sum_{n\in\mathbb{N}}(2n)^{-s} - 2^s\sum_{n\in\mathbb{N}_+}(2n)^{-s} \\ &= (2^s-1)\zeta(s) \ . \end{aligned}$$

We get, using $\zeta(0) = -\frac{1}{2}$,

$$\partial_{s|s=0}\zeta(0,1/2) = (\log(2)2^s\zeta(s) + (2^s - 1)\zeta'(s))_{|s=0} = -\frac{1}{2}\log(2) \ .$$

Finally,

$$\gamma = -\log(2) - \log(\pi) \ .$$

So

$$\partial_{s|s=0}[\zeta(s,q) + \zeta(s,1-q)] = -\log(2) - \log(\sin(\pi q)) \ .$$

We have $\zeta(0,q) = 1/2 - q$. This implies $[\zeta(2s,q) + \zeta(2s,1-q)]_{|s=0} = 0$. We now calculate

$$\zeta'_{\Delta_1}(0) = \partial_{s|s=0}\left(4^{-s}\pi^{-2s}[\zeta(2s,q) + \zeta(2s,1-q)]\right) = -2\log(2) - 2\log(\sin(\pi q)) \ .$$

We thus get

$$\det \Delta_1 = 4\sin^2(\pi q) \ .$$

We finally get

$$T_{RS}(S^1, \nabla^V) = \frac{1}{2\sin(\pi q)} \ .$$

□

We now compare this result with an evaluation of the **Reidemeister-Franz torsion (to be defined later)** $T_{RF}(S^1, \nabla^V)$. Using the standard cell decomposition $S^1 \cong \Delta^1/\partial\Delta^1$ the Reidemeister-Franz torsion is the analytic torsion $T(\mathcal{C}, h^{\mathcal{C}})$ of the chain complex $\mathcal{C}$ given by

$$\mathcal{C} : \mathbb{C} \stackrel{1-\lambda}{\to} \mathbb{C}$$

starting at 0, where $h^{\mathcal{C}}$ is the canonical metric. We have

$$\det \Delta_1 = |(1-\lambda)(1-\lambda^{-1})| = 2 - \lambda - \lambda^{-1} = 2 - 2\cos(2\pi q) \ .$$

We calculate

$$2 - 2\cos(2\pi q) = 2 - 2(1 - 2\sin^2(\pi q)) = 4\sin^2(\pi q) \ .$$

This gives

$$T_{RF}(S^1, \nabla^V) = T(\mathcal{C}, h^{\mathcal{C}}) = \frac{1}{2\sin(\pi q)} \ .$$

We observe that

$$T_{RF}(S^1, \nabla^V) = T_{RS}(S^1, \nabla^V) \ .$$

The equality $T_{RF} = T_{RS}$ in general is the contents of the **Cheeger-Müller theorem**.