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Analysis on complex manifolds
Problem set 10

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Exercise 1: Let E be a complex vector bundle on a compact smooth manifold X. Prove that the Sobolov s-norm on $\mathcal{E}_E(X)$ is really a norm. Show also that in the special case $s = 0$, this norm is equivalent to the L^2-norm.

Exercise 2: Let $P : \mathcal{E}_E(X) \to \mathcal{E}_F(X)$ be a differential operator of order d for complex vector bundles E, F over X. Prove that P has a unique continuous extension $H_s(X, E) \to H_s(X, F)$ to Sobolev spaces and that this extension is linear.

Exercise 3: Let E, F, G be complex vector bundles over X and let $P : \mathcal{E}_E(X) \to \mathcal{E}_F(X)$ (resp. $Q : \mathcal{E}_F(X) \to \mathcal{E}_G(X)$) be a differential operator of order d (resp. e). Prove that $Q \circ P$ is a differential operator of order $d + e$ with leading symbol $\sigma_{d+e}(Q \circ P) = \sigma_e(Q) \circ \sigma_d(P)$.

Exercise 4: Show that the exterior derivative $d : \mathcal{E}^j_{\mathbb{C}}(X) \to \mathcal{E}^{j+1}_{\mathbb{C}}(X)$ on the smooth compact manifold X is a differential operator of order 1. Prove that the leading symbol is given by

$$\sigma_1(d)(x, \omega) = i\omega \wedge \cdot : \Lambda^j T^*_{X,x} \otimes \mathbb{C} \to \Lambda^{j+1} T^*_{X,x} \otimes \mathbb{C}$$

for $(x, \omega) \in T^*_{X,x}$.