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# Analysis on complex manifolds
## Problem set 8

for 1.6.2016

**Exercise 1:** Let $E$ be a smooth $\mathbb{C}$-vector bundle on the smooth manifold $X$ with sheaf of sections $\mathcal{E}_E$. Let $(U_i)_{i\in I}$ be an open covering of $X$ by trivializations of $E$, i.e. we have a given frame of $E$ over every $U_i$.

a) Suppose that $D$ is connection on $E$. Let $\theta_i$ be the connection matrix of $D$ and let $\Theta_i := d\theta_i + \theta_i \wedge \theta_i$ be the curvature matrix of $D$ with respect to the trivialization over $U_i$. Prove that

$$\theta_i = g_{ij} dg_{ji} + g_{ij}\theta_j g_{ji} \tag{1}$$

and

$$\Theta_i = g_{ij}\Theta_j g_{ji}$$

where $g_{ij}$ is the transition matrix from the trivialization over $U_j$ to the one over $U_i$.

b) Conversely, assume that $\theta_i \in M_{r\times r}(\mathcal{E}^1_{\mathbb{C}}(U_i))$ are given for all $i \in I$ and that (1) holds. Show that there is a unique connection $D$ on $E$ with connection matrix $\theta_i$ on $U_i$.

**Exercise 2:** Under the hypotheses above, let $D$ be a connection on $E$ and let $h$ be a hermitian metric on $E$. Show that $D$ is compatible with $h$ if and only if for every open subset $U$ of $E$ with a frame $e = (e_1, \ldots, e_r)$ over $U$ we have

$$dH = \theta^t H + H\bar{\theta}$$

where $H$ is the matrix of the hermitian metric and $\theta$ is the connection matrix wrt $e$.

**Exercise 3:** Let $e$ be a frame of $E$ over $U$ and let $D$ be a connection on $E$. Using the connection matrix $\theta$ and the curvature matrix $\Theta$ wrt $e$ over $U$, show that

$$d\Theta = [\Theta, \theta]$$

where we use the Lie bracket $[\omega, \eta] := \omega \wedge \eta - (-1)^{pq}\eta \wedge \omega$ for $\omega \in \mathcal{E}^p_{\mathbb{C}}(U)$ and $\eta \in \mathcal{E}^q_{\mathbb{C}}(U)$.

**Exercise 4:** Let $D$ be the canonical connection of a hermitian holomorphic vector bundle $E$ over a complex manifold $X$. Show that the connection matrix $\theta$ and the curvature matrix $\Theta$ with respect to a holomorphic frame of $E$ over an open subset $U$ satisfy the following properties:

a) $\theta$ is of type $(1,0)$;

b) $\partial\theta = -\theta \wedge \theta$;

c) $\Theta$ is of type $(1,1)$;

d) $\Theta = \bar{\partial}\theta$;

e) $\bar{\partial}\Theta = 0$;

f) $\partial\Theta = [\Theta, \theta]$.