Prof. Dr. Walter Gubler
Thomas Fenzl
Christian Vilsmeier
Veronika Wanner

Universität Regensburg
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Arakelov Geometry

Problem Set 4

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Let K be a number field and $\Sigma := \left\{\sigma : K \hookrightarrow \mathbb{C} \,\middle|\, \sigma \text{ is a homomorphism}\right\}$.

Exercise 1 Prove Proposition 2.1.8, i.e. show that there is a canonical short exact sequence

$$0 \to \mathbb{R}^{r+s} / \log \left|\mathcal{O}_K^\times\right| \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathcal{O}_K)) \to \mathrm{CH}^1(\mathrm{Spec}(\mathcal{O}_K)) \to 0$$

where

$$\mathbb{R}^{r+s} \cong \mathbb{R}^\Sigma \Big/ \left\{(\alpha_\sigma)_{\sigma\in\Sigma} \,\middle|\, \alpha_\sigma = \alpha_{\bar{\sigma}} \; \forall \sigma \in \Sigma\right\}$$

and

$$\log\left|\mathcal{O}_K^\times\right| := \left\{\Big(\log|\sigma(\alpha)|\Big)_{\sigma\in\Sigma} \,\middle|\, \alpha \in \mathcal{O}_K^\times\right\}.$$

Exercise 2 Prove Proposition 2.1.10, i.e. show that there is a canonical isomorphism

$$\widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathcal{O}_K)) \xrightarrow{\sim} \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathcal{O}_K)).$$

Exercise 3 Let $\mathfrak{a}$ be a fractional ideal in $\mathcal{O}_K$ endowed with the metric $||\cdot||_\mathfrak{a}$ induced by the trivial hermitian metric on $\mathcal{O}_K$ (cf. 2.2.4). Prove that the arithmetic Euler-Poincaré characteristic is given by

$$\chi(\mathfrak{a}, ||\cdot||_\mathfrak{a}) = 2^{-s}\sqrt{|D_K(\mathfrak{a})|} = 2^{-s}\sqrt{|D_K|}N(\mathfrak{a})$$

where $2s$ is the number of complex embeddings of K, $D_K(\mathfrak{a})$ (resp. D_K) is the discriminant of $\mathfrak{a}$ (resp. K) and $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$.

Exercise 4 Let $\bar{L} = (L, ||\cdot||)$ be a hermitian line bundle on $\mathfrak{X} = \mathrm{Spec}(\mathcal{O}_K)$ and let $\hat{H}^0(\bar{L}) := \left\{s \in L \,\middle|\, ||s||_\sigma \leq 1 \; \forall \sigma \in \Sigma\right\}$. Prove that under the assumption $\widehat{\deg}(\bar{L}) < 0$, we have $\hat{H}^0(\bar{L}) = \{0\}$.