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Arakelov Geometry

Problem Set 8

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Let $\mathfrak{X}, \mathfrak{X}'$ be arithmetic varieties over $\mathcal{O}_K$ for a number field K.

Exercise 1 Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism of arithmetic varieties. Let $Z' \in Z_d(\mathfrak{X}')$ and let $\overline{\mathfrak{L}}_0, \ldots, \overline{\mathfrak{L}}_d$ be hermitian line bundles on $\mathfrak{X}$. Prove the projection formula

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathfrak{L}}_0) \cdots \widehat{c}_1(\overline{\mathfrak{L}}_d) \cdot \varphi_*(Z')) = \widehat{\deg}(\widehat{c}_1(\varphi^*\overline{\mathfrak{L}}_0) \cdots \widehat{c}_1(\varphi^*\overline{\mathfrak{L}}_d) \cdot Z').$$

Exercise 2 Give an example of a vertically ample hermitian line bundle on an arithmetic variety which is not ample. Use the arithmetic Nakai–Moishezon criterion to prove that for any vertically ample hermitian line bundle $(\mathfrak{L}, \| \ \|)$, there is a constant $c \in \mathbb{R}_{>0}$ such that $(\mathfrak{L}, c\| \ \|)$ is ample.

Exercise 3 Let $\overline{\mathfrak{L}}$ be a hermitian line bundle on $\mathfrak{X}$. Prove that

$$\widehat{H}^0(\mathfrak{X}, \overline{\mathfrak{L}}) := \{s \in H^0(\mathfrak{X}, \mathfrak{L}) \mid \|s\|_{\sup} \leq 1\}$$

is a lattice in

$$H^0(\mathfrak{X}, \mathfrak{L}) \otimes_{\mathbb{Z}} \mathbb{R} = H^0(\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{C}, \mathfrak{L} \otimes_{\mathbb{Z}} \mathbb{C})^{F_\infty}.$$

Exercise 4 Let $\overline{\mathfrak{L}}, \overline{\mathfrak{M}}$ be hermitian line bundles on $\mathfrak{X}$. If $\overline{\mathfrak{L}} \otimes \overline{\mathfrak{M}}^{-1}$ is an effective hermitian line bundle, then show that $\widehat{\mathrm{vol}}(\overline{\mathfrak{L}}) \geq \widehat{\mathrm{vol}}(\overline{\mathfrak{M}})$.