# ERRATUM: LEFSCHETZ THEOREM FOR ABELIAN FUNDAMENTAL GROUP WITH MODULUS

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The proof of Lemma 3.6 in [KS] is based on Claim 3.7 whose proof is not correct. In fact instead of (3-4) in the proof of Claim 3.7 one would need the vanishing of the analogous first cohomology group.

Below we give a corrected (and simplified) proof of Lemma 3.6. We would like to thank V. Srinivas for pointing out the mistake to us.

Let $X, D, U$ be as in the beginning of §3 but we don't assume $X$ projective. Recall

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} = \mathrm{Cone}\big(\mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X \xrightarrow{1-F} \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X\big)[-1].$$

We have a distinguished triangle in $D^b(X)$:

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} \to \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X \xrightarrow{1-F} \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X \xrightarrow{+} . \tag{1}$$

**Lemma 3.6** There is a canonical isomorphism

$$\phi_{X|D} : H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \xrightarrow{\sim} \mathrm{fil}^{\log}_{D} H^1(U)[p^n].$$

*Proof.* In the (correct) first part of the proof of Lemma 3.6 in [KS], we have constructed a canonical map $\phi_{X|D}$. In what follows we prove that it is an isomorphism. For simplicity of notation we omit the sheaf $(\mathbb{Z}/p^n\mathbb{Z})_{X|D}$ from our notation of cohomology and we just write $H^i(X, D)$ for $H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D})$. For any open covering $X = X_1 \cup X_2$ Mayer-Vietoris gives in degree one the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & H^1(X,D) & \to & H^1(X_1,D_1)\oplus H^1(X_2,D_2) & \to & H^1(X_1\cap X_2, D_1\cap D_2) \\
 & \downarrow{\scriptstyle \phi_{X|D}} & & \downarrow{\scriptstyle \phi_{X_1|D_1}\oplus\phi_{X_2|D_2}} & & \downarrow{\scriptstyle \phi_{X_1\cap X_2|D_1\cap D_2}} \\
0 \to & \mathrm{fil}^{\log}_{D} H^1(U)[p^n] & \to & \mathrm{fil}^{\log}_{D_1} H^1(U_1)[p^n] \oplus \mathrm{fil}^{\log}_{D_2} H^1(U_2)[p^n] & \to & \mathrm{fil}^{\log}_{D_1\cap D_2} H^1(U)[p^n]
\end{array}$$

where $U_i = U \cap X_i$, $D_i = D \cap X_i$ for $i \in \{1, 2\}$. For the zero on the upper left side we use that for any non-empty open $V \subset X$ we have $H^0(V, D \cap V) = \mathbb{Z}/p^n\mathbb{Z}$. By a standard reduction based on this diagram we can assume that $X$ is affine. For an affine $X$ the cohomology $H^2(X, D) = 0$ by vanishing of coherent cohomology on affine schemes. So using Lemma 3.5 we can assume without loss of generality that $n = 1$.

Then, from (1) we get the following Artin-Schreier isomorphism

$$H^1(X, D) = \mathrm{Coker}(H^0(X, \mathcal{O}_X(D/p)) \xrightarrow{1-F} H^0(X, \mathcal{O}(D))), \tag{2}$$

which implies in particular that

$$\mathrm{colim}_D H^1(X, D) = H^1(U, \mathbb{Z}/p\mathbb{Z}). \tag{3}$$

Let $D'$ be as in the proof of Claim 3.7. Let $\mathcal{W} = \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(D/p)^p$ if $p|m_\lambda$ and $\mathcal{W} = \mathcal{O}_{C_\lambda}(D)$ otherwise, and let $\mathcal{W}_\lambda$ be the stalk of $\mathcal{W}$ at the generic point $\lambda$ of $C_\lambda$.

We get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(X,D') & \longrightarrow & H^1(X,D) & \longrightarrow & H^0(C_\lambda,\mathcal{W}) \\
 & & \downarrow{\scriptstyle \phi_{X|D'}} & & \downarrow{\scriptstyle \phi_{X|D}} & & \downarrow \\
0 & \longrightarrow & \mathrm{fil}^{\log}_{D'}H^1(U)[p] & \longrightarrow & \mathrm{fil}^{\log}_{D}H^1(U)[p] & \longrightarrow & \mathcal{W}_\lambda
\end{array}$$

The right vertical arrow is the obvious injective map. The upper row is exact by (2) and the commutative diagram in the proof of Claim 3.7 (for the zero on the upper left side we use the injectivity of $F:\mathcal{L}\to\mathcal{O}_{C_\lambda}(D)$ in the diagram). The lower row is exact by the definition of $\mathrm{fil}^{\log}_{D}H^1(U)[p]$. This implies

$$H^1(X,D') = \mathrm{fil}^{\log}_{D'}H^1(U)[p] \cap H^1(X,D)[p] \subset H^1(U,\mathbb{Z}/p\mathbb{Z}).$$

In combination with (3) this completes the proof of the lemma. □

## References

[KS] Kerz, M., Saito, S. *Lefschetz theorem for abelian fundamental group with modulus*, Algebra Number Theory **8** (2014), no. 3, 689–701.

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