# MILNOR $K$-THEORY AND MOTIVIC COHOMOLOGY

MORITZ KERZ

Abstract.
These are the notes of a talk given at the Oberwolfach Workshop *K-Theory* 2006. We sketch a proof of Beilinson's conjecture relating Milnor $K$-theory and motivic cohomology. For detailed proofs see [4].

## 1. Introduction

$A$ — semi-local commutative ring with infinite residue fields
$k$ — field
$\mathbb{Z}(n)$ — Voevodsky's motivic complex [8]

**Definition 1.1.**

$$K_*^M(A) = \bigoplus_n (A^\times)^{\otimes n} / (a \otimes (1-a)) \quad a, 1-a \in A^\times$$

Beilinson conjectured [1]:

**Theorem 1.2.** *$A/k$ essentially smooth, $|k| = \infty$. Then:*

$$\eta : K_n^M(A) \longrightarrow H^n_{zar}(A, \mathbb{Z}(n))$$

*is an isomorphism for $n > 0$.*

The formerly known cases are:

**Remark 1.3.**
— *A=k a field (Nesterenko-Suslin* [6], *Totaro* [10]*)*
— *surjectivity of $\eta$ (Gabber* [3], *Elbaz-Vincent/Müller-Stach* [2], *Kerz/Müller-Stach* [5]*)*
— *$\eta \otimes \mathbb{Q}$ is isomorphic (Suslin)*
— *injectivity for $A$ a DVR, $n = 3$ (Suslin-Yarosh* [9]*)*

## 2. General idea of proof

$X = Spec\, A$
We have a morphism of Gersten complexes which we know to be exact except possibly at $K_n^M(A)$:

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(1)

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n^M(A) & \longrightarrow & \oplus_{x\in X^{(0)}} K_n^M(x) & \longrightarrow & \oplus_{x\in X^{(1)}} K_{n-1}^M(x) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^n_{zar}(A),\mathbb{Z}(n)) & \longrightarrow & \oplus_{x\in X^{(0)}} H^n_{zar}(x,\mathbb{Z}(n)) & \longrightarrow & \oplus_{x\in X^{(1)}} H^{n-1}_{zar}(x,\mathbb{Z}(n-1)) \end{array}$$

So it suffices to prove:

**Theorem 2.1** (Main Result). *$A/k$ regular, connected, infinite residue fields, $F = Q(A)$. Then:*

$$i : K_n^M(A) \longrightarrow K_n^M(F)$$

*is (universally) injective.*

## 3. Applications

**Corollary 3.1** (Equicharacteristic Gersten conjecture).
*$A/k$ regular, local, $X = Spec\, A$, $|k| = \infty$. Then:*

$$0 \longrightarrow K_n^M(A) \longrightarrow \oplus_{x\in X^{(0)}} K_n^M(x) \longrightarrow \oplus_{x\in X^{(1)}} K_{n-1}^M(x) \longrightarrow \cdots$$

*is exact.*

Proof. Case $A/k$ essentially smooth: Use (1) + Main Result.
Case $A$ general: Use smooth case + Panin's method [7] + Main Result. □

$\mathcal{K}_*^M$ — Zariski sheaf associated to $K_*^M$

**Corollary 3.2** (Bloch formula). *$X/k$ regular excellent scheme, $|k| = \infty$, $n \geq 0$. Then:*

$$H^n_{zar}(X, \mathcal{K}_n^M) = CH^n(X) \ .$$

Levine and Kahn conjectured:

**Corollary 3.3** (Gerneralized Bloch-Kato conjecture). *Assume the Bloch-Kato conjecture. $A/k$, $|k| = \infty$, $char(k)$ prime to $l > 0$. Then the galois symbol*

$$\chi_n : K_n^M(A)/l \longrightarrow H^n_{et}(A, \mu_l^{\otimes n})$$

*is an isomorphism for $n > 0$.*

Idea of proof. Case $A/k$ essentially smooth: Use Gersten resolution.
Case $A/k$ general: Use Hoobler's trick + Gabber's rigidity for étale cohomology. □

**Corollary 3.4** (Generalized Milnor conjecture). *$A/k$ local, $|k| = \infty$, $char(k)$ prime to 2. Then there exists an isomorphism*

$$K_n^M(A)/2 \longrightarrow I_A^n / I_A^{n+1}$$

*where $I_A \subset W(A)$ is the fundamental ideal in the Witt ring of $A$.*

## 4. New methods in Milnor $K$-theory

The first new result for $K$-groups used in the proof of the Main Result states:

**Theorem 4.1** (COCA). *$A \subset A'$ local extension of semi-local rings, i.e. $A^\times = A \cap A'^\times$. $A, A'$ factorial, $f \in A$ such that $A/(f) = A'/(f)$. Then:*

$$\begin{array}{ccc} K_n^M(A) & \longrightarrow & K_n^M(A_f) \\ \downarrow & & \downarrow \\ K_n^M(A') & \longrightarrow & K_n^M(A'_f) \end{array}$$

*is co-Cartesian.*

**Remark 4.2.** *COCA was proposed by Gabber who proved the surjectivity part at the lower right corner.*

**Theorem 4.3** (Local Milnor Theorem). *$q \in A[t]$ monic. There is a split short exact sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^t(A,q) \longrightarrow \oplus_{(\pi,q)=1} K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0$$

**Explanation:** The abelian group $K_n^t(A,q)$ is generated by symbols $\{p_1, \dots, p_n\}$ with $p_1, \dots, p_n \in A[t]$ pairwise coprime, $(p_i, q) = 1$ and highest non-vanishing coefficients invertible.
For $A = k$ a field $K_n^t(k,1) = K_n^M(k(t))$.
The standard technique gives:

**Theorem 4.4** (Norm Theorem). *Assume $A$ has big residue fields (depending on $n$). $A \subset B$ finite, étale. Then there exists a norm*

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

*satisfying projection formula, base change.*

## 5. Proof of Main Result

1st step: Reduce to $A$ semi-local with respect to closed points $y_1, \dots, y_l \in Y/k$ smooth, $|k| = \infty$ and k perfect. For this use Norm Theorem + Popescu desingularization.

2nd step: Induction on $d = dim\, A$ for all $n$ at once.

$$i : K_n^M(A) \longrightarrow K_n^M(F)$$

If $x \in K_n^M(A)$ with $i(x) = 0$ then there exists $f \in A$ such that $i^f(x) = 0$ where

$$i^f : K_n^M(A) \longrightarrow K_n^M(A_f)\ .$$

Gabber's presentation theorem produces $A' \subset A$ a local extension and $f' \in A'$ such that $f'/f \in A^*$ and $A'/(f') = A/(f)$. Here $A'$ is a semi-local ring with respect to closed points $y_1, \dots, y_l \in \mathbb{A}_k^d$.

The COCA Theorem gives that

$$\begin{array}{ccc} K_n^M(A') & \longrightarrow & K_n^M(A'_f) \\ \downarrow & & \downarrow \\ K_n^M(A) & \longrightarrow & K_n^M(A_f) \end{array}$$

is co-Cartesian. So it suffices to prove that

$$i' : K_n^M(A') \longrightarrow K_n^M(k(t_1, \dots, t_d))$$

is injective. Let $x \in ker(i')$ and $p_1, \dots, p_m \in k[t_1, \dots, t_d]$ be the irreducible, different polynomials appearing in $x$, $p_i \in A'^{\times}$.
Let

$$W = \bigcup_i \text{sing. loc.}(V(p_i)) \cup \bigcup_{i,j} V(p_i) \cap V(p_j)$$

Then $dim(W) < d - 1$ since $k$ is perfect.
There exists a linear projection

$$p : \mathbb{A}_k^d \longrightarrow \mathbb{A}_k^{d-1}$$

such that $p|_{V(p_i)}$ is finite and $p(y_i) \notin p(W)$ for all $i$.
Let now $A''$ be the semi-local ring with respect to $p(y_1), \dots, p(y_l) \in \mathbb{A}_k^{d-1}$

$$A'' \subset A''[t] \subset A'$$

Let $q \in A''[t]$ be monic such that

$$V(q) \cap p^{-1}(p(y_i)) = \{y_1, \dots, y_l\} \cap p^{-1}(p(y_i))$$

Under the natural map $K_n^t(A'', q) \to K_n^M(A')$ there exists a preimage $x' \in K_n^t(A'', q)$ of $x$.
We have a commutative diagram, $F = Q(A'')$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(A'') & \longrightarrow & K_n^M(A'', q) & \longrightarrow & \oplus_\pi K_{n-1}^M(A''[t]/(\pi)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_n^M(F) & \longrightarrow & K_n^M(F(t)) & \longrightarrow & \oplus_\pi K_{n-1}^M(F[t]/(\pi)) & \longrightarrow & 0 \end{array}$$

But since the important summands in the right vertical arrow are injective, a simple diagram chase gives $x' = 0$ and finally $x = 0$. □

## References


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Moritz Kerz
NWF I-Mathematik
Universität Regensburg
93040 Regensburg
Germany

moritz.kerz@mathematik.uni-regensburg.de