Oberseminar Winter 13/14:

Pro-étale cohomology

Monday, 12:15 - 14:00 (1st meeting 21.10.)

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The theory of étale topology in Algebraic Geometry was developed by Grothendieck, Artin et al. to define an analog of singular cohomology for a variety X over a field k of positive characteristic. In particular, one is interested in cohomology with coefficients in $\mathbb{Z}_\ell$, where ℓ is invertible in $\mathcal{O}_X$. It turns out that the naive construction of defining such a cohomology group '$H^i(X_{\mathrm{et}}, \mathbb{Z}_l)$' as sheaf cohomology with coefficients in the constant $\mathbb{Z}_\ell$, does not work, because there are too few étale covers. Instead Grothendieck suggested the ad hoc definition

$$H^i(X_{\mathrm{et}}, \mathbb{Z}_l) := \varprojlim_n H^i(X_{\mathrm{et}}, \mathbb{Z}/l^n\mathbb{Z}), \tag{1}$$

which is well-behaved only if the ground field k is 'nice', e.g. algebraically closed. Later a construction of continuous cohomology [J] was given, which replaces definition (1) and gives well-behaved cohomology groups for arbitrary ground field. This approach was later used to define the important derived category of constructible $\mathbb{Z}_\ell$-sheaves.

The problem with these continuous cohomology approaches is that they do not change the étale topology geometrically, but rather mix it with the internal topology of pro-systems of sheaves, which makes constructions cumbersome.

Recently, Scholze suggested to work with a new, better behaved topology, the so called pro-étale topology. This approach was worked out by Bhatt and Scholze in full generality. The relation of the pro-étale topology to usual étale topology is roughly the same as that of infinite Galois theory to finite Galois theory.

For example in the pro-étale topology one can simply define the sheaf $\mathbb{Z}_l$ as

$$U \mapsto \mathrm{Map}_{\mathrm{cont}}(U, \mathbb{Z}_l)$$

and take its 'naive' cohomology. This gives the same cohomology group as with continuous cohomology.

Talks:

1: Overview (Moritz Kerz)

2: General topology

Recall the definition of constructible sets in topological spaces [SP, 04ZC]. Define spectral spaces and describe with as many details as possible their basic properties, in particular [H] Sec. 0 and 2, Prop. 4 and 10, see also [SP, 09YF].

3: w-local spaces and rings I

First half of [BS] Sec. 2.

4: w-local spaces and rings II

Second half of [BS] Sec. 2.

5: Topos theory and fpqc topology
Recall the definition of sites and topoi and state basic properties, use e.g. [Gi] Ch. 0 as a reference. Recall the definition of fpqc topology and in particular that it is coarser than the canonical topology [SP, 022A,023P].

6: Replete topoi
General properties of replete topoi [BS] Sec. 3.1 - 3.3

7: Derived completion in replete topoi
[BS] Sec. 3.4 - 3.5

8: Pro-étale site I
Define pro-étale topology as subtopology of fpqc topology and prove basic properties [BS] Sec. 4, you can omit 4.3.

9: Pro-étale site II
Compare the pro-étale topology with the classical étale topology [BS] Sec. 5

10: Constructible sheaves I
Explain functoriality of the pro-étale site under open and closed immersions and recall properties of the classical constructible sheaves over discrete rings [BS] Sec. 6.1 - 6.4, omit 6.4. if there is not enough time.

11: Constructible sheaves II
Explain the general definition of constructible sheaves over topological rings and the their nice properties over a noetherian scheme [BS] Sec. 6.5-6.6.

12: 6 functors and local systems
Explain the 6 functor formalism for constructible sheaves and lisse sheaves [BS] 6.7 - 6.8.

13: Pro-étale fundamental group
Define infinite Galois categories and the pro-étale fundamental group. Explain the relation with Grothendieck's pro-finite fundamental group. Keep the first part on topological groups short [BS] Sec. 7.

Literatur

[BS] Bhatt, Scholze The pro-etale topology for schemes, preprint 2013, arXiv:1309.1198
[Gi] Giraud, Jean Cohomologie non abélienne. Springer-Verlag, Berlin-New York, 1971. ix+467 pp.
[H] Hochster, M. Prime ideal structure in commutative rings. Trans. Amer. Math. Soc. 142 1969 43–60.
[J] Jannsen, Uwe Continuous etale cohomology. Math. Ann. 280 (1988), no. 2, 207–45.
[SP] de Jong et al.: Stack Project, see http://stacks.math.columbia.edu

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