

EXPLICIT ℓ^1 -EFFICIENT CYCLES AND AMENABLE NORMAL SUBGROUPS

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ABSTRACT. By Gromov's mapping theorem for bounded cohomology, the projection of a group to the quotient by an amenable normal subgroup is isometric on group homology with respect to the ℓ^1 -semi-norm. Gromov's description of the diffusion of cycles also implicitly produces efficient cycles in this situation. We present an elementary version of this explicit construction.

1. INTRODUCTION

The size of classes in group homology with coefficients in a normed module can be measured by the ℓ^1 -semi-norm (Section 2). This semi-norm admits a description in terms of bounded cohomology [5].

By Gromov's mapping theorem for bounded cohomology, the projection $\pi: \Gamma \rightarrow \Gamma/N$ of a group Γ to the quotient by an amenable normal subgroup N induces an isometric isomorphism in bounded cohomology [5, 6]. In particular, the map induced by π on homology (e.g., with real coefficients) is isometric with respect to the ℓ^1 -semi-norm. More concretely, Gromov's description of diffusion of cycles also implicitly produces ℓ^1 -efficient cycles in this situation; similar diffusion arguments have been used in applications to Morse theory [2] and Lipschitz simplicial volume [9]. In this note, we present a slightly simplified version of this explicit construction. Because we focus on the group homology case, we can avoid the use of multicomplexes or model complexes for cycles. More precisely, we consider averaging maps of the following type to produce ℓ^1 -efficient cycles:

Theorem 1.1. *Let Γ be a group, let $N \subset \Gamma$ be a finitely generated amenable normal subgroup, let $n \in \mathbb{N}$, let $c \in C_n(\Gamma; \mathbb{R})$ be a cycle, and let $(F_k)_{k \in \mathbb{N}}$ be a Følner sequence for N . Then the cycles $\psi_k(c)$ are homologous to c and*

$$\lim_{k \rightarrow \infty} |\psi_k(c)|_1 = |\bar{c}|_1.$$

Here, $\bar{c} \in C_n(\Gamma/N; \mathbb{R})$ denotes the push-forward of c under the canonical projection $\Gamma \rightarrow \Gamma/N$ and for $k \in \mathbb{N}$, the averaging map ψ_k is defined by

$$\begin{aligned} \psi_k: C_n(\Gamma; \mathbb{R}) &\longrightarrow C_n(\Gamma; \mathbb{R}) \\ [\gamma_0, \dots, \gamma_n] &\longmapsto \frac{1}{|F_k|^{n+1}} \cdot \sum_{\eta \in F_k^{n+1}} [\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n] \end{aligned}$$

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Standard arguments then let us derive the following statement, which originally is a simple consequence of the mapping theorem in bounded cohomology [5]:

Corollary 1.2. *Let Γ be a group, let $N \subset \Gamma$ be a finitely generated amenable normal subgroup and let $\pi: \Gamma \rightarrow \Gamma/N$ be the canonical projection. Then the map $H_*(\pi; \mathbb{R}): H_*(\Gamma; \mathbb{R}) \rightarrow H_*(\Gamma/N; \mathbb{R})$ is isometric with respect to the ℓ^1 -semi-norm.*

More explicitly, in the situation of Corollary 1.2, we can construct ℓ^1 -efficient cycles out of knowledge on ℓ^1 -efficient cycles for $H_*(\Gamma/N; \mathbb{R})$ and a Følner sequence of N . This procedure is explained in detail in Section 4.

1.1. Twisted coefficients. Our proof of Theorem 1.1 and Corollary 1.2 also carries over to general normed coefficients.

Theorem 1.3. *Let Γ be a group, let $N \subset \Gamma$ be a finitely generated amenable normal subgroup, let A be a normed $\mathbb{R}\Gamma$ -module, let $n \in \mathbb{N}$, let $c \in C_n(\Gamma; A)$ be a cycle, and let $(F_k)_{k \in \mathbb{N}}$ be a Følner sequence for N . Then the cycles $\psi_k(c)$ are homologous to c and*

$$\lim_{k \rightarrow \infty} |\psi_k(c)|_1 = |\bar{c}|_1.$$

Here, $\bar{c} \in C_n(\Gamma/N; A_N)$ denotes the push-forward of c under the canonical projection $\Gamma \rightarrow \Gamma/N$ and for $k \in \mathbb{N}$, the averaging map ψ_k is defined by

$$\begin{aligned} \psi_k: C_n(\Gamma; A) &\rightarrow C_n(\Gamma; A) \\ a \otimes (\gamma_0, \dots, \gamma_n) &\mapsto \frac{1}{|F_k|^{n+1}} \cdot \sum_{\eta \in F_k^{n+1}} a \otimes (\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n). \end{aligned}$$

Corollary 1.4. *Let Γ be a group, let $N \subset \Gamma$ be a finitely generated amenable normal subgroup, let $\pi: \Gamma \rightarrow \Gamma/N$ be the canonical projection, and let A be a normed $\mathbb{R}\Gamma$ -module. Then the map $H_*(\pi; A): H_*(\Gamma; A) \rightarrow H_*(\Gamma/N; A_N)$ is isometric with respect to the ℓ^1 -semi-norm.*

Originally, Corollary 1.4 was proved by Ivanov [6] via an algebraic approach to bounded cohomology. Again, our proof gives a recipe to construct ℓ^1 -efficient cycles out of knowledge on Γ/N and a Følner sequence of N .

1.2. Simplicial volume. Simplicial volume of an oriented closed connected manifold is defined as the ℓ^1 -semi-norm of the \mathbb{R} -fundamental class [5]. As simplicial volume is related to both Riemannian geometry and mapping degrees, one is interested in calculations of simplicial volume and the construction of explicit ℓ^1 -efficient \mathbb{R} -fundamental cycles. The methods for group homology have a canonical counterpart for aspherical spaces and hence for simplicial volume of aspherical manifolds (Section 4.2).

The geometric Følner fillings used in the context of integral foliated simplicial volume of aspherical manifolds with amenable fundamental group [4, 3] are related to the averaging maps above but are not quite the same. The Følner construction in the present article does not use the deck transformation action of the fundamental group, but a slightly different action by tuples on the chain level. As a consequence, it is not clear whether the proof of Theorem 1.1 can be adapted to the integral foliated simplicial volume or

stable integral simplicial volume setting. The wish to understand ℓ^1 -efficient cycles of aspherical manifolds in the presence of amenable normal subgroups derives from the following open problem:

Question 1.5 (Lück). Let M be an oriented closed connected aspherical manifold with the property that $\pi_1(M)$ contains an infinite amenable normal subgroup. Does this imply $\|M\| = 0$?

Organisation of this note. In Section 2, we recall the ℓ^1 -semi-norm on group homology and singular homology. Section 3 contains the basic averaging argument and the proof of Theorems 1.1 and Theorem 1.3. In Section 4, we discuss how Theorem 1.1 can be used to give explicit ℓ^1 -efficient cycles in group homology (which proves Corollaries 1.2 and 1.4) and singular homology.

2. THE ℓ^1 -SEMI-NORM

We recall the definition of the ℓ^1 -semi-norm on group homology and singular homology.

2.1. Normed modules and ℓ^1 -norms. Let Γ be a group. Then a *normed Γ -module* is an $\mathbb{R}\Gamma$ -module A together with a (semi-)norm such that the Γ -action on A is isometric. If X is a free Γ -set, then the free $\mathbb{R}\Gamma$ -module $\bigoplus_X \mathbb{R}$ is a normed $\mathbb{R}\Gamma$ -module with respect to the ℓ^1 -norm associated to the canonical \mathbb{R} -basis $(e_x)_{x \in X}$. If A is a normed Γ -module, then we define the ℓ^1 -norm on the \mathbb{R} -vector space $A \otimes_{\mathbb{R}\Gamma} \bigoplus_X \mathbb{R}$ by

$$\begin{aligned} |\cdot|_1 : A \otimes_{\mathbb{R}\Gamma} \bigoplus_X \mathbb{R} &\longrightarrow \mathbb{R}_{\geq 0} \\ \sum_{x \in X} a_x \otimes e_x &\longmapsto \sum_{\Gamma \cdot x \in \Gamma \backslash X} \left| \sum_{\gamma \in \Gamma} \gamma^{-1} \cdot a_{\gamma \cdot x} \right|. \end{aligned}$$

Here, in order to define the tensor product $A \otimes_{\mathbb{R}\Gamma} \bigoplus_X \mathbb{R}$, we implicitly converted the (left) $\mathbb{R}\Gamma$ -module A to a right $\mathbb{R}\Gamma$ -module via the involution on the group ring $\mathbb{R}\Gamma$ given by inversion of group elements.

Remark 2.1 (rational coefficients). All of our arguments also work in the case of normed $\mathbb{Q}\Gamma$ -modules; however, for the sake of simplicity, we will formulate everything in the more standard case of normed $\mathbb{R}\Gamma$ -modules.

2.2. The ℓ^1 -semi-norm: groups. Let Γ be a group. We write $C_*(\Gamma)$ for the simplicial $\mathbb{R}\Gamma$ -resolution of \mathbb{R} . For $n \in \mathbb{N}$, we consider the ℓ^1 -norm $|\cdot|_1$ on $C_n(\Gamma) = \bigoplus_{\Gamma^{n+1}} \mathbb{R}$, where Γ acts diagonally on Γ^{n+1} . If A is a normed $\mathbb{R}\Gamma$ -module, then $C_n(\Gamma; A) := A \otimes_{\mathbb{R}\Gamma} C_n(\Gamma)$ carries a natural ℓ^1 -norm, as described in Section 2.1. This norm induces the ℓ^1 -semi-norm on group homology $H_n(\Gamma; A) := H_n(A \otimes_{\mathbb{R}\Gamma} C_*(\Gamma))$:

$$\begin{aligned} \|\cdot\|_1 : H_n(\Gamma; A) &\longrightarrow \mathbb{R}_{\geq 0} \\ \alpha &\longmapsto \inf \{ |c|_1 \mid c \in C_n(\Gamma; A), \partial c = 0, [c] = \alpha \}. \end{aligned}$$

If $N \subset \Gamma$ is a normal subgroup of Γ and A is a normed Γ -module, then the N -coinvariants A_N of A form a normed Γ/N -module (with the quotient semi-norm), and the canonical projections $\pi : \Gamma \longrightarrow \Gamma/N$ and $A \longrightarrow A_N$

induce a well-defined chain map $C_*(\pi; A): C_*(\Gamma; A) \longrightarrow C_*(\Gamma/N; A_N)$. In particular, we obtain an induced homomorphism $H_*(\pi; A): H_*(\Gamma; A) \longrightarrow H_*(\Gamma/N; A_N)$. These are the maps occurring in Theorem 1.3 and Corollary 1.4.

2.3. The ℓ^1 -semi-norm: spaces. Let M be a path-connected topological space that admits a universal covering \widetilde{M} with deck transformation action by the fundamental group $\Gamma := \pi_1(M)$ (e.g., M could be a connected manifold or CW-complex). If $n \in \mathbb{N}$, then the singular chain module $C_n(\widetilde{M}; \mathbb{R})$ is a free $\mathbb{R}\Gamma$ -space, as witnessed by the \mathbb{R} -basis map $(\Delta^n, \widetilde{M})$, which is a free Γ -set with respect to the deck transformation action. Hence, $C_n(\widetilde{M}; \mathbb{R})$ is a normed Γ -module with respect to the ℓ^1 -norm. If A is a normed Γ -module, then the twisted chain module $C_n(M; A) := A \otimes_{\Gamma} C_n(\widetilde{M}; \mathbb{R})$ inherits an ℓ^1 -norm (see Section 2.1). In the case that A is trivial Γ -module \mathbb{R} , this description coincides with the classical ℓ^1 -norm on $C_n(M; \mathbb{R})$. The semi-norm on $H_*(M; A)$ induced by the ℓ^1 -norm on $C_*(M; A)$ is the ℓ^1 -semi-norm on singular homology. If M is an oriented closed connected n -manifold, then

$$\|M\| := \|[M]_{\mathbb{R}}\|_1$$

is Gromov's *simplicial volume* of M [5, 8], where $[M]_{\mathbb{R}} \in H_{\dim M}(M; \mathbb{R})$ denotes the \mathbb{R} -fundamental class of M .

In the aspherical case, this ℓ^1 -semi-norm on homology coincides with the ℓ^1 -semi-norm on group homology [5, 6]. More explicitly:

Proposition 2.2. *Let M be a path-connected topological space that admits a weakly contractible universal covering with deck transformation action by the fundamental group $\Gamma := \pi_1(M)$, and let $D \subset \widetilde{M}$ be a set-theoretic fundamental domain of the deck transformation action. Then the Γ -equivariant chain map $C_*(\widetilde{M}; \mathbb{R}) \longrightarrow C_*(\Gamma; \mathbb{R})$ given by*

$$\begin{aligned} C_n(\widetilde{M}; \mathbb{R}) &\longrightarrow C_n(\Gamma; \mathbb{R}) \\ \sigma &\longmapsto (\gamma_0, \dots, \gamma_n) \text{ with } \sigma(e_j) \in \gamma_j \cdot D \end{aligned}$$

admits a Γ -equivariant chain homotopy inverse (given by inductive filling of simplices). In particular, if A is a normed Γ -module, then these chain maps induce mutually inverse isometric isomorphisms $H_(M; A) \cong H_*(\Gamma; A)$. \square*

Actually, even more is known to be true: The classifying map $M \longrightarrow B\Gamma$ is known to induce an isometric map on singular homology with respect to the ℓ^1 -semi-norm [5, 6].

2.4. Survey of known explicit efficient cycles. Explicit constructions of ℓ^1 -efficient cycles are known for spheres and tori [5], cross-product classes (through the homological cross-product and the duality principle) [5], finite coverings (through lifts of cycles) [5, 8], hyperbolic manifolds (through straightening/smearing) [1], manifolds with non-trivial S^1 -action (through an inductive filling/wrapping construction) [10], spaces with amenable fundamental group (through diffusion or Følner filling) [5, 2, 3], and for some amenable glueings [5, 7].

3. AVERAGING CHAINS

3.1. Finite normal subgroups. We will start by giving some context. Namely, we will first recall the case of finite normal subgroups; we will then explain how Følner sequences allow to pass from the finite case to the amenable case. We therefore recall the following well-known statement (whose proof is a straightforward calculation):

Proposition 3.1. *Let Γ be a group, let $N \subset \Gamma$ be a finite normal subgroup, and let $\pi: \Gamma \rightarrow \Gamma/N$ be the projection map. Then*

$$C_n(\Gamma/N; \mathbb{R}) \longrightarrow C_n(\Gamma; \mathbb{R})$$

$$[\gamma_0 \cdot N, \dots, \gamma_n \cdot N] \longmapsto \frac{1}{|N|^{n+1}} \cdot \sum_{\eta \in N^{n+1}} [\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n]$$

defines a well-defined chain map $C_*(\Gamma/N; \mathbb{R}) \rightarrow C_*(\Gamma; \mathbb{R})$ that induces the inverse of $H_*(\pi; \mathbb{R}): H_*(\Gamma; \mathbb{R}) \rightarrow H_*(\Gamma/N; \mathbb{R})$ on homology.

3.2. Averaging maps. In order to pass from the case of finite normal subgroups N to larger amenable normal subgroups, we replace the averaging over the subgroup N by averaging over Følner sets. If the normal subgroup $N \subset \Gamma$ is infinite, in general, there does not exist a splitting of the homomorphism $H_*(\Gamma; \mathbb{R}) \rightarrow H_*(\Gamma/N; \mathbb{R})$ induced by the canonical projection $\Gamma \rightarrow \Gamma/N$ and if $F \subset N$ is finite,

$$C_n(\Gamma/N; \mathbb{R}) \longrightarrow C_n(\Gamma; \mathbb{R})$$

$$[\gamma_0 \cdot N, \dots, \gamma_n \cdot N] \longmapsto \frac{1}{|F|^{n+1}} \cdot \sum_{\eta \in F^{n+1}} [\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n]$$

in general will *not* produce a well-defined map. However, the following “round-trip” averaging map over finite subsets is always well-defined.

Proposition 3.2 (averaging over a finite subset). *Let Γ be a group and let $F \subset \Gamma$ be a non-empty finite subset. Then the map $\psi_*^F: C_*(\Gamma) \rightarrow C_*(\Gamma)$ defined by*

$$\psi_n^F: C_n(\Gamma) \longrightarrow C_n(\Gamma)$$

$$(\gamma_0, \dots, \gamma_n) \longmapsto \frac{1}{|F|^{n+1}} \cdot \sum_{\eta \in F^{n+1}} (\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n)$$

has the following properties:

- (1) The map ψ_*^F is Γ -equivariant.
- (2) The map ψ_*^F is a chain map.
- (3) The chain map ψ_*^F is Γ -equivariantly chain homotopic to the identity.
- (4) We have $\|\psi_*^F\| \leq 1$ (with respect to the ℓ^1 -norm).

Proof. Because left and right multiplication on Γ do not interfere with each other, the map ψ_*^F is Γ -equivariant. Moreover, by definition of the ℓ^1 -norm, we clearly have $\|\psi_*^F\| \leq 1$.

We now show that ψ_*^F is a chain map: Let $n \in \mathbb{N}$ and $(\gamma_0, \dots, \gamma_n) \in C_n(\Gamma)$. By construction, we have

$$\begin{aligned} & \partial \psi_n^F((\gamma_0, \dots, \gamma_n)) \\ &= \sum_{j=0}^n (-1)^j \cdot \frac{1}{|F|^{n+1}} \cdot \sum_{\eta \in F^{n+1}} (\gamma_0 \cdot \eta_0, \dots, \gamma_{j-1} \cdot \eta_{j-1}, \gamma_{j+1} \cdot \eta_{j+1}, \dots, \gamma_n \cdot \eta_n) \end{aligned}$$

and reindexing shows (the last summand does not depend on “ η_j ”)

$$\begin{aligned} & \partial \psi_n^F((\gamma_0, \dots, \gamma_n)) \\ &= \sum_{j=0}^n (-1)^j \cdot \frac{|F|}{|F|^{n+1}} \cdot \sum_{\eta \in F^n} (\gamma_0 \cdot \eta_0, \dots, \gamma_{j-1} \cdot \eta_{j-1}, \gamma_{j+1} \cdot \eta_j, \dots, \gamma_n \cdot \eta_{n-1}) \\ &= \frac{1}{|F|^n} \cdot \sum_{\eta \in F^n} \sum_{j=0}^n (-1)^j \cdot (\gamma_0 \cdot \eta_0, \dots, \gamma_{j-1} \cdot \eta_{j-1}, \gamma_{j+1} \cdot \eta_j, \dots, \gamma_n \cdot \eta_{n-1}) \\ &= \psi_n^F(\partial(\gamma_0, \dots, \gamma_n)). \end{aligned}$$

It remains to prove the third part: Because $C_*(\Gamma)$ is a free $\mathbb{R}\Gamma$ -resolution of the trivial Γ -module \mathbb{R} and because the chain map ψ_*^F extends the identity $\text{id}_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ of the resolved module, the fundamental lemma of homological algebra immediately implies that ψ_*^F is Γ -equivariantly chain homotopic to the identity. \square

Passing to the tensor product, we obtain corresponding chain maps on the standard complexes:

Corollary 3.3. *Let Γ be a group, let A be a normed $\mathbb{R}\Gamma$ -module, and let $F \subset \Gamma$ be a non-empty finite subset. Then $\text{id}_A \otimes_{\Gamma} \psi_*^F: C_*(\Gamma; A) \rightarrow C_*(\Gamma; A)$ is a well-defined chain map that is chain homotopic to the identity and that satisfies $\|\text{id}_A \otimes_{\Gamma} \psi_*^F\| \leq 1$. \square*

3.3. The push-forward estimate. As final ingredient, we estimate the norm of averaged chains in terms of the image in the quotient group. For subsets $F, S \subset N$ of a group N , we use the version

$$\partial_S F := \{\eta \in F \mid \exists \sigma \in S \cup S^{-1} \ \sigma \cdot \eta \notin F\}$$

of the S -boundary of F in N .

Proposition 3.4 (the push-forward estimate). *Let Γ be a group, let A be a normed $\mathbb{R}\Gamma$ -module, let $N \subset \Gamma$ be a normal subgroup, and let $c \in C_n(\Gamma; A)$. For every $\varepsilon \in \mathbb{R}_{>0}$ there exists a finite set $S \subset N$ and a constant $K \in \mathbb{R}_{>0}$ with the following property: For all non-empty finite subsets $F \subset N$ we have*

$$|(\text{id}_A \otimes_{\Gamma} \psi_n^F)(c)|_1 \leq |\bar{c}|_1 + \varepsilon + K \cdot \frac{|\partial_S F|}{|F|}$$

Here, $\bar{c} \in C_n(\Gamma/N; A_N)$ denotes the push-forward of c along the projection $\Gamma \rightarrow \Gamma/N$.

Proof. Let $\varepsilon \in \mathbb{R}_{>0}$. Using the canonical isometric isomorphism between the chain module $A \otimes_{\Gamma} C_n(\Gamma/N)$ (equipped with the semi-norm induced by the ℓ^1 -norm) and $C_n(\Gamma/N; A_N)$, we see that we can decompose

$$c = c_0 + c_1$$

with $c_0, c_1 \in C_n(\Gamma; A)$ and $\bar{c}_0 = \bar{c}$ as well as $|c_0|_1 \leq |\bar{c}|_1 + \varepsilon$. Because the averaging maps of Proposition 3.2 are linear and have norm at most 1, we may assume without loss of generality that $\bar{c} = 0$. Then the canonical isomorphism (where N^{n+1} acts by component-wise multiplication from the right)

$$C_n(\Gamma/N; A_N) \cong (A \otimes_{\Gamma} C_n(\Gamma))_{N^{n+1}}$$

shows that we can write c in the form

$$c = \sum_{j=1}^m (c(j) - c(j) \cdot \sigma(j))$$

with $m \in \mathbb{N}$, $c(1), \dots, c(m) \in C_n(\Gamma; A)$, and $\sigma(1), \dots, \sigma(m) \in N^{n+1}$. We then set

$$\begin{aligned} S &:= \{\sigma(j)_k \mid j \in \{1, \dots, m\}, k \in \{0, \dots, n\}\} \\ K &:= 2 \cdot (n+1) \cdot \max_{j \in \{1, \dots, m\}} |c(j)|_1. \end{aligned}$$

By definition of the ℓ^1 -norm on $C_n(\Gamma; A)$, we may now assume that c is of the form

$$c = a \otimes (\gamma_0, \dots, \gamma_n) - a \otimes (\gamma_0 \cdot \sigma_0, \dots, \gamma_n \cdot \sigma_n)$$

with $a \in A$ and $\gamma_0, \dots, \gamma_n \in \Gamma$, $\sigma_0, \dots, \sigma_n \in N$. Then the definition of the S -boundary shows that for all non-empty finite subsets $F \subset N$ we have

$$\begin{aligned} |(\text{id}_A \otimes_{\Gamma} \psi_n^F)(c)|_1 &= \frac{1}{|F|^{n+1}} \cdot \left| a \otimes \left(\sum_{\eta \in F^{n+1}} (\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n) \right. \right. \\ &\quad \left. \left. - \sum_{\eta \in F^{n+1}} (\gamma_0 \cdot \sigma_0 \cdot \eta_0, \dots, \gamma_n \cdot \sigma_n \cdot \eta_n) \right) \right|_1 \\ &\leq 2 \cdot (n+1) \cdot |a| \cdot \frac{|\partial_S F|}{|F|}. \end{aligned}$$

This gives the desired estimate. \square

Remark 3.5. In the case of trivial \mathbb{R} -coefficients, the same argument shows: For every chain $c \in C_n(\Gamma; \mathbb{R})$ there is a finite set $S \subset N$ with the following property: For all non-empty finite subsets $F \subset N$ we have

$$|(\text{id}_{\mathbb{R}} \otimes_{\Gamma} \psi_n^F)(c)|_1 \leq |\bar{c}|_1 + 2 \cdot (n+1) \cdot \frac{|\partial_S F|}{|F|}.$$

3.4. Proof of Theorems 1.1 and 1.3. As Theorem 1.1 is a special case of Theorem 1.3, it suffices to prove the latter:

Proof of Theorem 1.3. Let $\varepsilon \in \mathbb{R}_{>0}$ and let $S \subset N$ be a finite set that is adapted to c and ε and let $K \in \mathbb{R}_{>0}$ be a constant as provided by Proposition 3.4. Let $k \in \mathbb{N}$. By construction, we have $\psi_k = \text{id}_A \otimes_{\Gamma} \psi_n^{F_k}$ (where $\psi_n^{F_k}$ is defined in Proposition 3.2). In view of Corollary 3.3, $c_k := \psi_k(c)$ is a cycle that represents $[c] \in H_n(\Gamma; A)$. Moreover, Proposition 3.4 shows that

$$|c_k|_1 \leq |\bar{c}|_1 + \varepsilon + K \cdot \frac{|\partial_S F_k|}{|F_k|}.$$

Because $(F_k)_{k \in \mathbb{N}}$ is a Følner sequence for N we have $\lim_{k \rightarrow \infty} |\partial_S F_k|/|F_k| = 0$. Therefore, $\lim_{k \rightarrow \infty} |c_k|_1 \leq |\bar{c}|_1 + \varepsilon$. Letting ε go to 0 gives the inequality $\lim_{k \rightarrow \infty} |c_k|_1 \leq |\bar{c}|_1$.

The converse inequality holds because $\|\psi_k\| \leq 1$ and $|\bar{c}|_1 \leq |c|_1$. \square

Remark 3.6. This argument also generalises to the case that the amenable normal subgroup is not finitely generated; one just has to first take a finite set $S \subset N$ that is adapted to the cycle in question and then to pick a Følner sequence for the finitely generated amenable subgroup $\langle S \rangle_N$ of N .

4. EXPLICIT ℓ^1 -EFFICIENT CYCLES

4.1. Efficient cycles: groups. We can now prove Corollary 1.4 (which implies Corollary 1.2). In fact, we will explain how one can produce explicit ℓ^1 -efficient cycles (using the input specified in the proof).

Proof of Corollary 1.4. Clearly, the homomorphism $H_*(\pi; A): H_*(\Gamma; A) \rightarrow H_*(\Gamma/N; A_N)$ satisfies $\|H_*(\pi; A)\| \leq 1$. Thus, it suffices to prove that the ℓ^1 -semi-norm cannot decrease under $H_*(\pi; A)$:

Let $\alpha \in H_n(\Gamma; A)$ and let $\bar{\alpha} := H_n(\pi; A)(\alpha)$ be the push-forward of α . We will now produce cycles representing α whose ℓ^1 -norm approximates $\|\bar{\alpha}\|_1$. As input for our construction, we need the following:

- A cycle $c \in C_n(\Gamma; A)$ representing α ; let $\bar{c} := C_n(\pi; A)(c)$.
- For every $m \in \mathbb{N}_{>0}$ a cycle $z_m \in C_n(\Gamma/N; A_N)$ that represents $\bar{\alpha}$ and satisfies

$$|z_m|_1 \leq \|\bar{\alpha}\|_1 + \frac{1}{m}.$$

- For every $m \in \mathbb{N}_{>0}$ a chain $\bar{b}_m \in C_{n+1}(\Gamma/N; A_N)$ with

$$\partial \bar{b}_m = z_m - \bar{c};$$

let $b_m \in C_{n+1}(\Gamma; A)$ be some π -lift of \bar{b}_m .

- A Følner sequence $(F_k)_{k \in \mathbb{N}}$ for N ; let $(\psi_k)_{k \in \mathbb{N}}$ the corresponding sequence of averaging maps as in Theorem 1.3.

For $k, m \in \mathbb{N}$ we then set

$$c_{k,m} := (\text{id}_A \otimes_{\Gamma} \psi_k)(c + \partial b_m) \in C_n(\Gamma; A).$$

From Theorem 1.3, we obtain for every $m \in \mathbb{N}$ that

$$\lim_{k \rightarrow \infty} |c_{k,m}|_1 = |C_n(\pi; A)(c + \partial b_m)|_1 = |\bar{z}_k|_1 \leq \|\bar{\alpha}\|_1 + \frac{1}{m}.$$

Moreover, Proposition 3.4 can also be used to get an explicit estimate on the rate of convergence in terms of the Følner sequence $(F_k)_{k \in \mathbb{N}}$. \square

The situation is particularly simple if the push-forward satisfies $\bar{\alpha} = 0$ because we can then take $z_m := 0$ for all $m \in \mathbb{N}$. So, in this case, we only need to find a cycle c representing α , a single filling of \bar{c} , and a Følner sequence for N .

Remark 4.1. The same modifications as in Remark 3.6 allow also to generalise Corollary 1.4 and its explicit proof to the case that the normal subgroup is not finitely generated.

4.2. Efficient cycles: spaces. Using Proposition 2.2, we can translate the explicit constructions in the proof of Theorem 1.3 and Corollary 1.4 to the case of aspherical spaces. At this point it might be helpful to unravel these conversions and spell out the action of tuples on singular simplices (as needed in the averaging maps) in more explicit terms:

Let M be an aspherical space (with universal covering and deck transformation action), let $\Gamma := \pi_1(M)$ be the fundamental group of M , and let $D \subset \widetilde{M}$ be a set-theoretic fundamental domain for the deck transformation action. If $\sigma \in \text{map}(\Delta^n, \widetilde{M})$ and $(\eta_0, \dots, \eta_n) \in \Gamma^{n+1}$, then we construct $\sigma \cdot (\eta_0, \dots, \eta_n)$ as follows:

- We determine the group elements $\gamma_0, \dots, \gamma_n \in \Gamma$ with $\sigma(e_j) \in \Gamma_j \cdot D$.
- We then look at the tuple $(\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n) \in \Gamma^{n+1}$.
- Using the inductive filling procedure alluded to in Proposition 2.2, we reconstruct a singular simplex on \widetilde{M} from $(\gamma_0 \cdot \eta_0, \dots, \gamma_n \cdot \eta_n)$. This is the desired singular simplex $\sigma \cdot (\eta_0, \dots, \eta_n)$.

For a finite subset $F \subset \Gamma$, the averaging map then has the form

$$C_n(\widetilde{M}; \mathbb{R}) \longrightarrow C_n(\widetilde{M}; \mathbb{R})$$

$$\text{map}(\Delta^n, \widetilde{M}) \ni \sigma \longmapsto \frac{1}{|F|^{n+1}} \cdot \sum_{\eta \in F^{n+1}} \sigma \cdot \eta.$$

In particular, we hence obtain explicit ℓ^1 -efficient fundamental cycles of aspherical manifolds with (finitely generated) amenable normal subgroup in terms of knowledge of cycles on the quotient and Følner sequences of this normal subgroup.

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