ON STABILITY OF NON-DOMINATION
UNDER TAKING PRODUCTS

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ABSTRACT. We show that non-domination results for targets that are not dominated by products are stable under Cartesian products.

1. Motivation

If $M$ and $N$ are closed oriented manifolds of the same dimension, we say that $M$ dominates $N$, and we write $M \geq N$, if there is a continuous map $f: M \to N$ of non-zero degree. The existence of such a dominant map is a property of the homotopy types of $M$ and $N$, and it has been known since the pioneering work of Hopf [11] that for such a map $f$ the pullback $f^*$ is an injection of rational cohomology algebras, and that $f_*$ is virtually surjective on the fundamental group. However, the existence of an injective algebra homomorphism $H^*(N; \mathbb{Q}) \to H^*(M; \mathbb{Q})$ and of a virtually surjective homomorphism $\pi_1(M) \to \pi_1(N)$ is usually far from sufficient for $M \geq N$.

Motivated by the work of Gromov [7, 8] in particular, (non-)domination between manifolds has in recent years been studied in several different contexts, using a variety of techniques from topology, geometry, and group theory; see for example [7, 4, 8, 5, 12] and the references given there. An idea due to Thurston [16] and Gromov [7] is to study numerical invariants $I$ of manifolds that are monotone under maps of non-zero degree, so that $M \geq N$ implies $I(M) \geq I(N)$. Then, whenever one can compute or estimate $I$ and prove $I(M) < I(N)$ for some specific manifolds, one concludes that $M$ does not dominate $N$. The simplest example of such an invariant is the cuplength in rational cohomology, which is monotone by the result of Hopf mentioned before. A more subtle monotone invariant – of geometric rather than algebraic origin – is the simplicial volume $\|\cdot\|$ defined by Gromov [7]. In general, monotone invariants are closely connected to functorial semi-norms on homology [8, 6, 15].

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According to Gromov, the simplicial volume has a major deficiency: its lack of multiplicativity. In fact, he proved in [7] that the simplicial volume is approximately multiplicative for Cartesian products, and it is known that it is not strictly multiplicative [3]. However, approximate multiplicativity is not good enough to obtain stable non-domination results. Indeed, suppose that $0 < \|M\| < \|N\|$ for some specific $M$ and $N$. Then $M \not\geq N$, but it is unclear whether the $d$-fold product $M^\times d$ may dominate $N^\times d$ for some $d \geq 2$, or not. The approximate multiplicativity does not rule out the possibility that, as a function of the number of factors, the simplicial volume of direct products of $M$ might grow faster than that of direct products of $N$, so that the former eventually surpasses the latter.

Invariants that are strictly multiplicative – or strictly additive, like the cuplength – do not have this deficiency: if $I(M) < I(N)$, then $I(M^\times d) < I(N^\times d)$, so that $M^\times d \not\geq N^\times d$ for all $d \geq 1$. In this case the non-domination result $M \not\geq N$ is stable under Cartesian products.

Gromov [8] suggested that many manifolds $N$ might have the property that they cannot be dominated by a non-trivial product $M = M_1 \times M_2$. This conjecture has since been verified [12], and there are now lots of examples of manifolds that are known not to be dominated by products [12, 13, 14, 17]. We will see here that in general non-domination results for targets that cannot be dominated by products are stable under Cartesian products. This is interesting in its own right, and also has geometric applications [17].

Conventions. Throughout this paper, the word manifold means a connected closed oriented non-empty topological manifold; we denote the rational fundamental class of a manifold $M$ by $[M]$. A product of manifolds is always a non-trivial product, so no factor is a point.

2. Results

Our first result is that for targets that are not dominated by products, the loss of information in taking products discussed in the previous section does not occur.

**Theorem 2.1.** Suppose $M$ and $N$ are $n$-manifolds, and that $N$ is not dominated by a product. Then for any $d \geq 2$ we have $M^\times d \geq N^\times d$ if and only if $M \geq N$.

In a similar spirit, taking Cartesian products with arbitrary manifolds preserves non-domination for targets that are not dominated by products.

**Theorem 2.2.** Suppose $M$ and $N$ are $n$-manifolds, and that $N$ is not dominated by a product. Then for any manifold $W$, we have $M \times W \geq N \times W$ if and only if $M \geq N$.

Note that $W$ may very well have trivial simplicial volume. Even if one deduces $M \not\geq N$ from $\|M\| < \|N\|$, this theorem shows that multiplying with $W$ preserves non-domination, while killing the simplicial volume if $\|W\| = 0$. 
Finally, controlling the dimensions of the factors in a product, we have the following:

**Theorem 2.3.** Let \( N \) be an \( n \)-manifold that is not dominated by a product. Then there is no manifold \( V \) for which the product \( N \times V \) can be dominated by a product \( P = X_1 \times \ldots \times X_s \) that satisfies \( \dim X_j < n \) for all \( j \in \{1, \ldots, s\} \).

### 3. Proofs

The proofs of the above theorems all use the following lemma, which is a consequence of Thom’s work [18] on the Steenrod problem.

**Lemma 3.1.** Let \( N \) be an \( n \)-manifold that is not dominated by a product. If \( f : M_1 \times M_2 \rightarrow N \) is a continuous map, then for all \( i \in \{1, \ldots, n-1\} \) the map

\[
 f_* : H_i(M_1; \mathbb{Q}) \otimes H_{n-i}(M_2; \mathbb{Q}) \rightarrow H_n(N; \mathbb{Q})
\]

induced by the homological cross-product and \( f \) is the zero map.

**Proof.** Because elements of \( H_i(M_1; \mathbb{Q}) \otimes H_{n-i}(M_2; \mathbb{Q}) \) are finite linear combinations of decomposable elements, and \( f_* \) is linear, it suffices to show \( f_*(\alpha \otimes \beta) = 0 \) for all \( \alpha \in H_i(M_1; \mathbb{Q}) \) and all \( \beta \in H_{n-i}(M_2; \mathbb{Q}) \). Again by the linearity of \( f_* \), there is no loss of generality in replacing \( \alpha \) and \( \beta \) by non-zero multiples. Thus we may assume that these are integral homology classes. By Thom’s result [18], after replacing the integral classes \( \alpha \) and \( \beta \) by suitable non-zero multiples, there are continuous maps \( g_j : X_j \rightarrow M_j \) defined on manifolds \( X_j \) of dimensions \( i \) and \( n-i \) respectively, such that \( (g_1)_*[X_1] = \alpha \) and \( (g_2)_*[X_2] = \beta \). It follows that

\[
 f_*(\alpha \otimes \beta) = (f \circ (g_1 \times g_2))_*[X_1 \times X_2].
\]

This must vanish, because otherwise the map \( f \circ (g_1 \times g_2) : X_1 \times X_2 \rightarrow N \) would have non-zero degree, contradicting the assumption on \( N \).  

Using Lemma 3.1, we now prove the theorems stated in the previous section.

**Proof of Theorem 2.1.** If \( M \geq N \), then clearly \( M^{\times d} \geq N^{\times d} \) for all \( d \geq 2 \). Conversely, suppose that \( g : M^{\times d} \rightarrow N^{\times d} \) has non-zero degree for some \( d \geq 2 \). We consider the composition \( f = p_1 \circ g \), where \( p_1 \) is the projection to the first factor. Then \( f_* \) is surjective in rational homology. Since we assumed that \( N \) is not dominated by a product, Lemma 3.1 tells us that, in degree \( n \), the map \( f_* \) vanishes on tensor products of homology vector spaces of non-zero degree. It follows that for at least one of the inclusions \( i : M \rightarrow M^{\times d} \) of a factor of \( M^{\times d} \), the composition \( f \circ i \) has non-zero degree, and thus \( M \geq N \).
Proof of Theorem 2.2. If \( M \geq N \), then clearly \( M \times W \geq N \times W \) for all manifolds \( W \). Conversely, suppose that \( f: M \times W \rightarrow N \times W \) has non-zero degree for some \( W \). We consider the induced map \( f_*: H_n(\cdot; \mathbb{Q}) \) in terms of the Künneth decompositions of the domain and of the target:

\[
f_*: H_n(M; \mathbb{Q}) \oplus M_1 \oplus H_n(W; \mathbb{Q}) \rightarrow H_n(N; \mathbb{Q}) \oplus M_2 \oplus H_n(W; \mathbb{Q}) ,
\]

where \( M_i \) denotes the direct sum of tensor products of homology vector spaces in non-zero degrees.

Since we assumed that \( N \) is not dominated by a product, Lemma 3.1 tells us that \( f_*(M_1) \) is contained in \( M_2 \oplus H_n(W; \mathbb{Q}) \). If we assume for a contradiction that \( M \not\geq N \), then the same is true for \( f_*(H_n(M; \mathbb{Q})) \).

Because \( f_* \) is surjective, we conclude that there is an \( \alpha_0 \in H_n(W; \mathbb{Q}) \) such that \( f_*(\alpha_0) = [N] \neq 0 \) holds in the quotient vector space

\[
Q = H_n(N \times W; \mathbb{Q}) / f_*(H_n(M; \mathbb{Q}) \oplus M_1) .
\]

Note that \( Q \) is of finite, non-zero, dimension.

Now we think of \( \alpha_0 \) as being in the target of \( f_* \). By surjectivity of \( f_* \), the class \( \alpha_0 \) is in its image, so there exists an \( \alpha_1 \in H_n(W; \mathbb{Q}) \) satisfying \( f_*(\alpha_1) = \alpha_0 \) in \( Q \) (though not necessarily in \( H_n(N \times W; \mathbb{Q}) \)). We proceed inductively to find \( \alpha_i+1 \in H_n(W; \mathbb{Q}) \) with the property that \( f_*(\alpha_{i+1}) = \alpha_i \) in \( Q \). The assumptions that \( N \) is not dominated by a product, or by \( M \), imply at every step that \( \alpha_i \) does not vanish in the quotient \( Q \).

Since \( Q \) is finite-dimensional, there is a minimal \( k \in \mathbb{N} \) such that \( \alpha_0, \ldots, \alpha_k \) are linearly dependent in \( Q \). There are then \( \lambda_i \in \mathbb{Q} \) with \( \lambda_k \neq 0 \) such that

\[
\lambda_k \alpha_k + \ldots + \lambda_0 \alpha_0 = 0 \in Q .
\]

We now take the left-hand-side of this equation, considered as an element of \( H_n(W; \mathbb{Q}) \subset H_n(M \times W; \mathbb{Q}) \), and apply \( f_* \) to it to obtain

\[
\lambda_k \alpha_{k-1} + \ldots + \lambda_1 \alpha_0 + \lambda_0 [N] \in f_*(H_n(M; \mathbb{Q}) \oplus M_1) .
\]

If \( \lambda_0 = 0 \), then this contradicts the minimality of \( k \). If \( \lambda_0 \neq 0 \), then we reach the conclusion that in \( H_n(N \times W; \mathbb{Q}) \) the generator \( [N] \in H_n(N; \mathbb{Q}) \) is a linear combination of \( \lambda_k \alpha_{k-1} + \ldots + \lambda_1 \alpha_0 \in H_n(W; \mathbb{Q}) \) and of elements in

\[
f_*(H_n(M; \mathbb{Q}) \oplus M_1) \subset M_2 \oplus H_n(W; \mathbb{Q}) .
\]

This contradicts the Künneth decomposition, and hence proves \( M \geq N \).

\[\square\]

Proof of Theorem 2.3. Suppose \( g: X_1 \times \ldots \times X_s \rightarrow N \times V \) is a continuous map, and consider the composition \( f = p_1 \circ g \). The assumptions that \( N \) is not dominated by a product and that \( \dim X_j < n \) for all \( j \) imply, as in the proof of Lemma 3.1, that \( f_* \) is the zero map in degree \( n \). Therefore, \( g \) has degree zero.

\[\square\]
4. DISCUSSION

4.1. Applications of the cuplength. It is not clear to what extent the assumption that $N$ is not dominated by a product is necessary in the above theorems. While it is crucial for our proofs, this could be an artefact of our method. Indeed, there are cases of targets $N$ which are dominated by products, and still one can prove our results for them. We now do this for tori, using the cuplength.

Recall that the cuplength of $M$, denoted $\text{cl}(M)$, is the maximal number $k$ for which there are classes $\alpha_1, \ldots, \alpha_k \in H^*(M; \mathbb{Q})$ of positive degrees with the property that $\alpha_1 \cup \ldots \cup \alpha_k \neq 0 \in H^*(M; \mathbb{Q})$. This is monotone under maps of non-zero degree by \cite{11}. The compatibility of the K"unneth decomposition with the cup product implies

\begin{equation}
\text{cl}(M \times W) = \text{cl}(M) + \text{cl}(W).
\end{equation}

The following is easy and well known.

**Lemma 4.1.** An $n$-manifold $M$ dominates $T^n$ if and only if there is an injective algebra homomorphism $H^*(T^n; \mathbb{Q}) \to H^*(M; \mathbb{Q})$, equivalently, if $\text{cl}(M) = n$.

So this is a case where the algebraic necessary condition for domination derived from rational cohomology is also sufficient.

Lemma 4.1 combined with (1) tells us that Theorem 2.2 holds for $N = T^n$. Furthermore, we have:

**Proposition 4.2.** If $M_1$ and $M_2$ are manifolds of dimensions $m_1$ and $m_2$ respectively, then $M_1 \times M_2 \geq T^{m_1 + m_2}$ if and only if $M_1 \geq T^{m_1}$ and $M_2 \geq T^{m_2}$.

In particular, Theorem 2.1 also holds for $N = T^n$.

4.2. Infinite products. Gromov has suggested that some non-domination results should extend to infinite products, following his perspective on infinite products and related topics \cite{1, 9, 2}\cite{10, Section 5}.

By increasing the number $d$ of factors in $P^{\times d}$, one would naively end up with a countably infinite product $P^{\times \infty}$, without any extra structure. A better way of looking at infinite products is probably to pick a (discrete, countable) group $\Gamma$, and to look at the space $P^\Gamma = \text{Map}(\Gamma, P)$, equipped with the natural shift action of $\Gamma$. Now in formulating what $P^\Gamma \not\geq N^\Gamma$ might mean, one should only consider $\Gamma$-equivariant continuous maps between these product spaces.

The main issue is of course that for maps between these infinite-dimensional manifolds there is no naive, geometric, notion of degree. Instead, one should make full use of equivariance and define domination via surjectivity in a suitable homology theory, perhaps without necessarily attempting to define a degree.

\footnote{Hopf did not use cohomology, but formulated the conclusion in terms of the Umkehr map on intersection rings.}
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