Group Cohomology – Exercises

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Exercise 1 (~~puzzle~~ convergence)**.** Let R be a ring and let $(E^r, d^r)_{r \in \mathbb{N}_{>1}}$ be a homological spectral sequence that converges to a graded R-module A:

$$E^2_{pq} \Longrightarrow A_{p+q}.$$

Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $E^2_{pq} \cong_R 0$ for all $p, q \in \mathbb{N}$ with $q \neq 2019$, then $A_{2020} \cong_R E^2_{1,2019}$.
2. If $E^2_{pq} \cong_R 0$ for all $p, q \in \mathbb{N}$ for which $p + q$ is odd, then $A_{2019} \cong_R 0$.

Exercise 2 (homology of the Heisenberg group)**.** Let $H \subset \mathrm{SL}(3, \mathbb{Z})$ be the integral Heisenberg group (Sheet 2, Exercise 4). Compute $H_n(H; \mathbb{Z})$ for all $n \in \mathbb{N}$ (where H acts trivially on $\mathbb{Z}$) via the Hochschild-Serre spectral sequence.
Hints. You may use the result on $H_1(H; \mathbb{Z})$ from Exercise 4 on Sheet 2.

Exercise 3 (standard spectral arguments)**.** We consider the article

> M.R. Bridson, P.H. Kropholler. Dimension of elementary amenable groups, *J. Reine Angew. Math.*, 699, p. 217–143, 2015.
> Institut Mittag-Leffler, report no. 38, 2011/2012, spring.

In the paragraph after Theorem I.5 it says that "In both cases of course the inequalities $\leq$ follow at once from standard spectral sequence arguments."

1. Which spectral sequence should be applied?
2. Carry out these "standard spectral sequence arguments."

Hints. Cohomological dimension already appeared in Exercise 4 of Sheet 4.

Exercise 4 (the conjugation action on homology)**.**

1. Let G be a group, let A be a $\mathbb{Z}G$-module, and let $g \in G$. Moreover, let

   $$c(g) := (x \mapsto g \cdot x \cdot g^{-1},\ x \mapsto g \cdot x) \in \mathrm{Mor}_{\mathsf{GroupMod}}\big((G, A), (G, A)\big).$$

   Show that $H_n(c(g)) = \mathrm{id}_{H_n(G;A)}$ for all $n \in \mathbb{N}$.
2. For each $n \in \mathbb{N}$, compute $H_n(\varphi; \mathbb{Z}) \colon H_n(\mathbb{Z}/3; \mathbb{Z}) \longrightarrow H_n(\mathbb{Z}/3; \mathbb{Z})$, where $\varphi \colon \mathbb{Z}/3 \longrightarrow \mathbb{Z}/3$ is the group automorphism $x \mapsto -x$.

Hints. First understand how $H_*(c(g))$ can be described in terms of a projective resolution of $\mathbb{Z}$ and then avoid confusion at all cost.

Bonus problem (centre kills)**.** Let $n, k \in \mathbb{N}_{>0}$ and let K be a field with $\operatorname{char} K \neq 2$. Show that $H_k(\mathrm{GL}(n, K); K^n) \cong_K 0$, where $\mathrm{GL}(n, K)$ acts on K^n by matrix multiplication.
Hints. Let elements of the centre act ...

Submission before July 15, 2019, 10:00, in the mailbox