Group Cohomology – Exercises

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Exercise 1 (universal coefficients and Künneth)**.** Let G be a group that satisfies $H_n(G;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(1;\mathbb{Z})$ for all $n \in \mathbb{N}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If H is a group and $n \in \mathbb{N}$, then $H_n(G \times H;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(H;\mathbb{Z})$.
2. If A is a $\mathbb{Z}$-module (with trivial G-action) and $n \in \mathbb{N}$, then $H_n(G;A) \cong_{\mathbb{Z}} H_n(1;A)$.

Exercise 2 (topology of discrete groups)**.** We consider the articles

> G. Baumslag, E. Dyer, A. Heller. The topology of discrete groups, *J. Pure Appl. Algebra*, 16(1), pp. 1–47, 1980.
>
> C.R.F Maunder. A short proof of a theorem of Kan and Thurston. *Bull. London Math. Soc.*, 13(4), pp. 325–327, 1981.

1. What is a *mitotic* group?
2. Sketch the proof that mitotic groups have trivial homology.
3. How does the main theorem of the second paper relate to group homology?
4. How/Where is the first paper used in the second paper?

Exercise 3 (a classifying space for the Heisenberg group)**.** Let $H \subset \mathrm{SL}(3,\mathbb{Z})$ be the integral Heisenberg group (Sheet 2, Exercise 4). Show that there exists a classifying space for H that is a compact 3-manifold.
Hints. Start with the real Heisenberg group.

Exercise 4 (surface groups)**.** For $g \in \mathbb{N}_{\geq 2}$, we consider the group

$$\Gamma_g := \bigl\langle a_1, \dots, a_g, b_1, \dots, b_g \bigm| [a_1, b_1] \cdot [a_2, b_2] \cdot \dots \cdot [a_g, b_g] \bigr\rangle.$$

1. Compute $H_*(\Gamma_g;\mathbb{Z})$, using "the" oriented closed connected surface of genus g as classifying space (and a suitable CW-structure on it).
2. Compute the deficiency of the group Γ_g.

Bonus problem (classifying space of a category)**.**

1. How is the classifying space of a (small) category defined?
2. How can one construct classifying spaces for groups out of classifying spaces of a category?

Hints. Rough sketches suffice.

Bonus problem (lecture notes)**.** Find as many typos/mistakes in the lecture notes as you can!

Submission before July 22, 2019, 10:00, in the mailbox