Group Cohomology – Exercises

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The ISSSS case. The InterStellar Spectral Sequence Station (ISSSS) has been infiltrated and taken over by a pangalactic group X of con artists. Detective Blorx, an agent of the Cohomological Intelligence Agency, handles the case. He collected the following evidence:

A. The suspects. Only the following groups are sufficiently powerful to be able to infiltrate the ISSSS; here, F_n denotes the free group of rank n and Γ_g denotes the surface group of genus g:

- The Free Group: F_{2018}
- The Freer Group: F_{2019}
- The Bi-Cycle Group: $\mathbb{Z}/4 \times \mathbb{Z}/674 \times F_{2018}$
- GaoS (Group avalaible on Surfaces): Γ_{1010}
- Torus Inc.: $\mathbb{Z}^{2019}$

B. Statements by witnesses.

- The group X was not able to carry invariant means.
- The free products $(F_4 \times F_{674}) * X$ and $(F_4 \times F_{674}) * F_{2018}$ are *not* commensurable.
- Every finite subgroup of X acted freely on some sphere.
- The group X has *no* subgroup that is isomorphic to the fundamental group of an oriented closed connected surface of genus at least 2.

C. Project Euler. Blorx hacked into the servers of the Secret Invariants Service and found the following files on Project Euler:

> Let G be a group of type F, i.e., G admits a classifying space with a finite CW-structure. Then the *Euler characteristic of* G is defined as
>
> $$\chi(G) := \sum_{n \in \mathbb{N}} (-1)^n \cdot \dim_{\mathbb{Q}} H_n(G; \mathbb{Q}).$$

① In this situation, $\chi(G)$ is a well-defined integer and it equals the Euler characteristic of any classifying space for G with finite CW-structure.

② If $H \subset G$ is a subgroup of finite index, then $\chi(H) = [G : H] \cdot \chi(G)$.

③ If H is a group of type F, then $\chi(G \times H) = \chi(G) \cdot \chi(H)$.

④ If H is a group of type F, then $\chi(G * H) = \chi(G) + \chi(H) - 1$.

D. Law of Commensurability. Two groups G and H are *commensurable*, if there exist finite index subgroups $\overline{G} \subset G$ and $\overline{H} \subset H$ with $\overline{G} \cong_{\mathsf{Group}} \overline{H}$.

Please turn over

Exercise ($4 + 8 + 4$ credits)**.** Help Blorx!

1. Establish two of the four claims of Project Euler.
2. Which group infiltrated the ISSSS? Justify your answer!

   Hints. Sphere actions will be discussed in the final lecture.
   How do Project Euler and the Law of Commensurability interact?

3. Help Blorx to get a promotion by working as an informant for him: Pick your favourite field of Mathematics (e.g., algebraic number theory, Riemannian geometry, geometric topology, geometric group theory, operator algebras, functional analysis, graph theory, algebraic geometry, ergodic theory, ...).

   Find (in the literature or on the servers of the Secret Invariants Service) an application of group (co)homology to that field that we did not discuss in the lectures!

Bonus problem (fickle witness)**.** The last witness statement was later withdrawn by the witness and changed to the following:

- The group X is *not* quasi-isometric to F_{2018}.

Does this lead to the same conclusion?
Hints. One can (provided one has access to it) use the secret H2UF-technology.

Optional submission before July 29, 10:00, in the mailbox