Ergodic theoretic methods in group homology

A minicourse on L^2-Betti numbers in group theory

Clara Löh

MSRI Summer graduate school (06/2019)

Random and arithmetic structures in topology

Version of July 23, 2019
clara.loeh@mathematik.uni-regensburg.de
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg

Contents

0 Introduction

These are the lecture notes for a five lecture minicourse at the MSRI summer graduate school *Random and arithmetic structures in topology* (organised by Alexander Furman and Tsachik Gelander) in June 2019:

https://www.msri.org/summer_schools/853

This minicourse will be a brief introduction to ergodic theoretic methods in group homology. By now, this is a vast subject. In the present course, we will focus on L^2-Betti numbers.

The underlying fundamental observation is that taking suitable coefficients for group (co)homology allows to connect homological invariants with ergodic theory. This interaction works in both directions:

- On the one hand, (co)homology with coefficients based on dynamical systems leads to orbit/measure equivalence invariants.
- On the other hand, many homological gradient invariants can be accessed through the dynamical system given by the profinite completion.

Overview of this minicourse

- **Lecture 1.** We will start our gentle introduction to L^2-Betti numbers by introducing the von Neumann dimension. In order to keep the technical overhead at a minimum, we will work with an elementary approach to von Neumann dimensions and L^2-Betti numbers.
- **Lecture 2.** The von Neumann dimension allows us to define L^2-Betti numbers. We will then explore basic computational tools and compute L^2-Betti numbers in simple examples.

- **Lecture 3.** On the one hand, L^2-Betti numbers are related to classical Betti numbers through approximation. This residually finite view has applications to homological gradient invariants.
- **Lecture 4.** On the other hand, L^2-Betti numbers are also related to measured group theory. We will compare this dynamical view with the residually finite view.
- **Lecture 5.** Finally, we will briefly discuss L^2-invariants in the context of invariant random subgroups. Moreover, the appendix contains an outlook on simplicial volume and its interaction with ergodic theory.

I tried to keep things as elementary as reasonably possible; this means that a basic background in algebraic topology (fundamental group, covering theory, (co)homology), functional analysis (bounded operators, measure theory), and measured group theory should be sufficient to follow this course.

Convention. The set $\mathbb{N}$ of natural numbers contains 0. All rings are unital and associative (but very often *not* commutative). We write ${}_R\mathsf{Mod}$ for the category of left R-modules and Mod_R for the category of right R-modules.

Some literature

This course will not follow a single source and there are several books that cover the standard topics (all with their own advantages and disadvantages). Therefore, you should individually compose your own favourite selection of material.

The following list is by no means complete and biased by my personal preferences:

Textbooks on L^2-invariants

- H. Kammeyer. *Introduction to ℓ^2-invariants*, lecture notes, 2018.
  https://topology.math.kit.edu/downloads/introduction-to-l2-invariants.pdf
- W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 44, Springer, 2002.

Recommended further reading

- M. Abért, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, I. Samet. On the growth of L^2-invariants for sequences of lattices in Lie groups, *Ann. of Math. (2)*, 185(3), pp. 711–790, 2017.

- M. Abért, N. Nikolov. Rank gradient, cost of groups and the rank versus Heegard genus problem, *J. Eur. Math. Soc.*, 14, 16571677, 2012.
- M.W. Davis. *The Geometry and Topology of Coxeter Groups*, London Mathematical Society Monographs, 32, Princeton University Press, 2008.
- A. Furman. A survey of measured group theory. In *Geometry, Rigidity, and Group Actions* (B. Farb, D. Fisher, eds.), 296–347, The University of Chicago Press, 2011.
- D. Gaboriau. Coût des relations d'équivalence et des groupes, *Invent. Math.*, 139(1), 41–98, 2000.
- D. Gaboriau. Invariants ℓ^2 de relations d'équivalence et de groupes, *Inst. Hautes Études Sci. Publ. Math.*, 95, 93–150, 2002.
- A.S. Kechris, B.D. Miller. *Topics in Orbit Equivalence*, Springer Lecture Notes in Mathematics, vol. 1852, 2004.
- W. Lück. Approximating L^2-invariants by their finite-dimensional analogues, *Geom. Funct. Anal.*, 4(4), pp. 455–481, 1994.
- R. Sauer. Amenable covers, volume and L^2-Betti numbers of aspherical manifolds, *J. reine angew. Math.*, 636, 47–92, 2009.

Acknowledgements. I would like to thank Alex Furman and Tsachik Gelander for organising this summer school. Special thanks go to Gil Goffer, who skillfully taught the discussion session for this course. Moreover, I am grateful to Damien Gaboriau for valuable comments on Chapter 4.

1

The von Neumann dimension

Betti numbers are dimensions of (co)homology groups. In the presence of a group action, we can alternatively also use an equivariant version of dimension; this leads to L^2-Betti numbers.

In this lecture, we will introduce such an equivariant version of dimension, using the group von Neumann algebra. In the second lecture, this dimension will allow us to define L^2-Betti numbers of groups and spaces.

Overview of this chapter.

Running example. the additive group $\mathbb{Z}$, finite (index sub)groups

1.1 From the group ring to the group von Neumann algebra

One of the fundamental invariants in algebraic topology are Betti numbers of spaces and groups, which are numerical invariants extracted from (co)homology (by taking dimensions of homology groups).

We will now pass to an equivariant setting: Let $\Gamma \curvearrowright X$ be a continuous group action on a topological space X. Then the singular chain complex $C_*(X;\mathbb{C})$ and the singular homology $H_*(X;\mathbb{C})$ of X inherit a Γ-action, and thus consist of modules over the group ring $\mathbb{C}\Gamma$.

Unfortunately, the group ring $\mathbb{C}\Gamma$, in general, does not admit an accessible module/dimension theory. We will therefore pass to completions of the group ring: $\ell^2\Gamma$ (for the modules) and the von Neumann algebra $N\Gamma$ (for the morphisms), which lead to an appropriate notion of traces and thus to a Γ-dimension. We will now explain this in more detail.

1.1.1 The group ring

The group ring of a group Γ is an extension of the ring $\mathbb{Z}$ with new units coming from the group Γ:

Definition 1.1.1 (group ring)**.** Let Γ be a group. The *(complex) group ring* of Γ is the $\mathbb{C}$-algebra $\mathbb{C}\Gamma$ (sometimes also denoted by $\mathbb{C}[\Gamma]$ to avoid misunderstandings)

- whose underlying $\mathbb{C}$-vector space is $\bigoplus_{g\in\Gamma}\mathbb{C}$, freely generated by Γ (we denote the basis element corresponding to $g \in \Gamma$ simply by g),
- and whose multiplication is the $\mathbb{C}$-bilinear extension of composition in Γ, i.e.:

$$\begin{aligned} \cdot : \mathbb{C}\Gamma \times \mathbb{C}\Gamma &\longrightarrow \mathbb{C}\Gamma \\ \Bigl(\sum_{g\in\Gamma} a_g \cdot g, \sum_{g\in\Gamma} b_g \cdot g\Bigr) &\longmapsto \sum_{g\in\Gamma}\sum_{h\in\Gamma} a_h \cdot b_{h^{-1}\cdot g} \cdot g \end{aligned}$$

(where all sums are "finite").

Example 1.1.2 (group rings)**.**

- The group ring of "the" trivial group 1 is just $\mathbb{C}[1] \cong_{\mathsf{Ring}} \mathbb{C}$.
- The group ring $\mathbb{C}[\mathbb{Z}]$ of the additive group $\mathbb{Z}$ is isomorphic to $\mathbb{C}[t, t^{-1}]$, the ring of Laurent polynomials over $\mathbb{C}$ (check!).

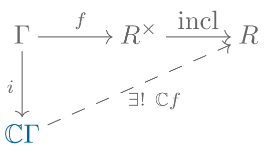
- Let $n \in \mathbb{N}_{>0}$. Then we have $\mathbb{C}[\mathbb{Z}/n] \cong_{\mathsf{Ring}} \mathbb{C}[t]/(t^n - 1)$ (check!).
- In general, group rings are *not* commutative. In fact, a group ring $\mathbb{C}\Gamma$ is commutative if and only if the group Γ is Abelian (check!). Hence, for example, the group ring $\mathbb{C}[F_2]$ of "the" free group F_2 of rank 2 is not commutative.

Caveat 1.1.3 (notation in group rings)**.** When working with elements in group rings, some care is required. For example, the term $4 \cdot 2$ in $\mathbb{C}[\mathbb{Z}]$ might be interpreted in the following different(!) ways:

- the product of 4 times the ring unit and 2 times the ring unit, or
- 4 times the *group* element 2.

We will circumvent this issue in $\mathbb{C}[\mathbb{Z}]$, by using the notation "t" for a generator of the additive group $\mathbb{Z}$ and viewing the infinite cyclic group $\mathbb{Z}$ as multiplicative group. Using this convention, the first interpretation would be written as $4 \cdot 2$ (which equals 8) and the second interpretation would be written as $4 \cdot t^2$. Similarly, also in group rings over other groups, we will try to avoid ambiguous notation.

Proposition 1.1.4 (group ring, universal property)**.** *Let Γ be a group. Then the group ring $\mathbb{C}\Gamma$, together with the canonical inclusion map $i\colon \Gamma \longrightarrow \mathbb{C}\Gamma$ (as standard basis) has the following universal property: For every $\mathbb{C}$-algera R and every group homomorphism $f\colon \Gamma \longrightarrow R^\times$, there exists a unique $\mathbb{C}$-algebra homomorphism $\mathbb{C}f\colon \mathbb{C}\Gamma \longrightarrow R$ with $\mathbb{C}f \circ i = f$.*



Proof. This is a straightforward calculation (check!). □

Outlook 1.1.5 (Kaplansky conjecture)**.** The ring structure of group rings is not well understood in full generality. For example, the following versions of the Kaplansky conjectures are still open: Let Γ be a torsion-free group.

- Then the group ring $\mathbb{C}\Gamma$ is a domain (?!).
- The group ring $\mathbb{C}\Gamma$ does not contain non-trivial idempotents (?!). (I.e., if $x \in \mathbb{C}\Gamma$ with $x^2 = x$, then $x = 1$ or $x = 0$).

However, a positive solution is known for many special cases of groups [23, 75, 26][66, Chapter 10] (such proofs often use input from functional analysis or geometry) and no counterexamples are known.

1.1.2 Hilbert modules

Homology modules are quotient modules. In the presence of an inner product, quotients of the form A/B (by closed subspaces B) can be viewed as *submodules* of A (via orthogonal complements). Therefore, we will pass from the group ring $\mathbb{C}\Gamma$ to the completion $\ell^2\Gamma$:

Definition 1.1.6 ($\ell^2\Gamma$)**.** Let Γ be a group. Then

$$\begin{aligned} \langle \cdot\,, \cdot \rangle \colon \mathbb{C}\Gamma \times \mathbb{C}\Gamma &\longrightarrow \mathbb{C}\Gamma \\ \Bigl(\sum_{g\in G} a_g \cdot g, \sum_{g\in G} b_g\Bigr) &\longmapsto \sum_{g\in G} \overline{a}_g \cdot b_g \end{aligned}$$

is an inner product on $\mathbb{C}\Gamma$. The completion of $\mathbb{C}\Gamma$ with respect to this inner product is denoted by $\ell^2\Gamma$ (which is also a $\mathbb{C}$-algebra with an inner product). More concretely, $\ell^2\Gamma$ is the $\mathbb{C}$-vector space of ℓ^2-summable functions $\Gamma \longrightarrow \mathbb{C}$ with the inner product

$$\begin{aligned} \langle \cdot\,, \cdot \rangle \colon \ell^2\Gamma \times \ell^2\Gamma &\longrightarrow \mathbb{C} \\ \Bigl(\sum_{g\in G} a_g \cdot g, \sum_{g\in G} b_g\Bigr) &\longmapsto \sum_{g\in G} \overline{a}_g \cdot b_g \end{aligned}$$

Example 1.1.7.

- If Γ is a finite group, then $\ell^2\Gamma = \mathbb{C}\Gamma$.
- If $\Gamma = \mathbb{Z} = \langle t \mid \rangle$, then Fourier analysis shows that

$$\begin{aligned} F \colon \ell^2\Gamma &\longrightarrow L^2\bigl([-\pi,\pi],\mathbb{C}\bigr) \\ \sum_{n\in\mathbb{Z}} a_n \cdot t^n &\longmapsto \Bigl(x \mapsto \frac{1}{\sqrt{2\pi}} \cdot \sum_{n\in\mathbb{Z}} a_n \cdot e^{i\cdot n\cdot x}\Bigr) \end{aligned}$$

  is an isomorphism of $\mathbb{C}$-algebras (with inner product).

Remark 1.1.8 (countability and separability)**.** In order to avoid technical complications, in the following, we will always work with countable groups; then, the associated ℓ^2-space will be separable.

Definition 1.1.9 (Hilbert modules)**.** Let Γ be a countable group.

- A *Hilbert Γ-module* is a complex Hilbert space V with a $\mathbb{C}$-linear isometric (left) Γ action such that there exists an $n \in \mathbb{N}$ and an isometric Γ-embedding $V \longrightarrow (\ell^2\Gamma)^n$. Here, we view $\ell^2\Gamma$ as left $\mathbb{C}\Gamma$-module via

$$\begin{aligned} \Gamma \times \ell^2\Gamma &\longrightarrow \ell^2\Gamma \\ (g,f) &\longmapsto \bigl(x \mapsto f(x\cdot g)\bigr). \end{aligned}$$

- Let V and W be Hilbert Γ-modules. A *morphism* $V \longrightarrow W$ *of Hilbert* Γ*-modules* is a Γ-equivariant bounded $\mathbb{C}$-linear map $V \longrightarrow W$.

In a complex Hilbert space, we have the following fundamental equality for (closed) submodules A (check!):

$$\mathbb{C}\text{-dimension of } A = \text{trace of the orthogonal projection onto } A.$$

For Hilbert Γ-modules, we will use this description of the dimension as *definition*. Therefore, we first need to be able to describe orthogonal projections and we need a suitable notion of trace. Both goals can be achieved by means of the group von Neumann algebra.

1.1.3 The group von Neumann algebra

Let Γ be a countable group and let $a \in \mathbb{C}\Gamma$. Then the right multiplication map $M_a \colon \ell^2\Gamma \longrightarrow \ell^2\Gamma$ by a is a (left) Γ-equivariant isometric $\mathbb{C}$-linear map. Similarly, matrices A over $\mathbb{C}\Gamma$ induce morphisms M_A between finitely generated free $\ell^2\Gamma$-modules.

However, morphisms of Hilbert Γ-modules, in general, will not be of this simple form: We will need more general matrix coefficients.

Definition 1.1.10 (group von Neumann algebra)**.** Let Γ be a countable group.

- Let $B(\ell^2\Gamma)$ be the $\mathbb{C}$-algebra of bounded linear operators $\ell^2\Gamma \longrightarrow \ell^2\Gamma$.
- The *group von Neumann algebra of* Γ is the weak closure of $\mathbb{C}G$ (acting by right multiplication on $\ell^2\Gamma$) in $B(\ell^2\Gamma)$.

Remark 1.1.11 (alternative descriptions of the group von Neumann algebra)**.** Let Γ be a a countable group. Then the group von Neumann algebra $N\Gamma$ can equivalently be described as follows [34, Corollary 4.2.2]:

- $N\Gamma$ is the strong closure of $\mathbb{C}\Gamma$ (acting by right multiplication on $\ell^2\Gamma$).
- $N\Gamma$ is the bicommutant of $\mathbb{C}\Gamma$ (acting by right multiplication on $\ell^2\Gamma$).
- $N\Gamma$ is the subalgebra of $B(\ell^2\Gamma)$ consisting of all bounded operators that are *left* $\mathbb{C}\Gamma$-equivariant.

Theorem 1.1.12 (von Neumann trace)**.** *Let* Γ *be a countable group and let*

$$\begin{aligned} \operatorname{tr}_\Gamma \colon N\Gamma &\longrightarrow \mathbb{C} \\ a &\longmapsto \langle e, a(e) \rangle, \end{aligned}$$

where $e \in \mathbb{C}\Gamma \subset \ell^2\Gamma$ *denotes the atomic function at* $e \in \Gamma$. *Then* tr_Γ *satisfies the following properties:*

1. Trace property. *For all* $a, b \in N\Gamma$*, we have* $\operatorname{tr}_\Gamma(a \circ b) = \operatorname{tr}_\Gamma(b \circ a)$*.*

2. Faithfulness. *For all* $a \in N\Gamma$*, we have* $\operatorname{tr}_\Gamma(a^* \circ a) = 0$ *if and only if* $a = 0$*. Here,* a^* *denotes the adjoint operator of* a*.*

3. Positivity. *For all* $a \in N\Gamma$ *with* $a \geq 0$*, we have* $\operatorname{tr}_\Gamma a \geq 0$*. Here,* $a \geq 0$ *if and only if* $\langle x, a(x)\rangle \geq 0$ *for all* $x \in \ell^2\Gamma$*.*

Proof. Ad 1. A straightforward computation shows that the trace property holds on the subalgebra $\mathbb{C}\Gamma$ (check!). By construction, tr_Γ is weakly continuous. Therefore, the trace property also holds on $N\Gamma$.

Ad 2. For the non-trivial implication, let $a \in N\Gamma$ with $\operatorname{tr}_\Gamma(a^* \circ a) = 0$. Then, by definition, we have

$$0 = \operatorname{tr}_\Gamma(a^* \circ a) = \langle e, a^* \circ a(e)\rangle = \langle a(e), a(e)\rangle$$

and thus $a(e) = 0$. Because a is Γ-linear, we also obtain $a(g \cdot e) = g \cdot a(e) = 0$. Continuity of a therefore shows that $a = 0$.

Ad 3. This is clear from the definition of positivity and the trace. □

Example 1.1.13 (some von Neumann traces)**.**

- If Γ is a finite group, then $N\Gamma = \mathbb{C}\Gamma$. The von Neumann trace is

$$\begin{aligned} \operatorname{tr}_\Gamma \colon N\Gamma = \mathbb{C}\Gamma &\longrightarrow \mathbb{C} \\ \sum_{g\in\Gamma} a_g \cdot g &\longmapsto a_e. \end{aligned}$$

- If $\Gamma = \mathbb{Z} = \langle t \mid \rangle$, then we obtain [66, Example 1.4]: The group von Neumann algebra $N\Gamma$ is canonically isomorphic to $L^\infty\bigl([-\pi,\pi],\mathbb{C}\bigr)$ (and the action on $\ell^2\Gamma \cong L^2([-\pi,\pi],\mathbb{C})$ is given by pointwise multiplication); under this isomorphism, the trace tr_Γ on $N\Gamma$ corresponds to the integration map

$$\begin{aligned} L^\infty\bigl([-\pi,\pi],\mathbb{C}\bigr) &\longrightarrow \mathbb{C} \\ f &\longmapsto \frac{1}{2\pi} \cdot \int_{[-\pi,\pi]} f \, d\lambda. \end{aligned}$$

In view of the previous example, the abstract theory of von Neumann algebras is also sometimes referred to as *non-commutative measure theory.*

Remark 1.1.14 (extension of the trace to matrices and morphisms)**.** As in linear algebra, we can extend the trace from the group von Neumann algebra to matrices: Let Γ be a countable group and let $n \in \mathbb{N}$. Then we define the trace

$$\begin{aligned}\operatorname{tr}_\Gamma \colon M_{n\times n}(N\Gamma) &\longrightarrow \mathbb{C}\\ A &\longmapsto \sum_{j=1}^{n} \operatorname{tr}_\Gamma A_{j,j}.\end{aligned}$$

This trace also satisfies the trace property, is faithful, and positive (check!).

Moreover, every bounded (left) Γ-equivariant map $(\ell^2\Gamma)^n \longrightarrow (\ell^2\Gamma)^n$ is represented by a matrix in $M_{n\times n}(N\Gamma)$ (by the last characterisation in Remark 1.1.11). Therefore, every bounded Γ-equivariant map $(\ell^2\Gamma)^n \longrightarrow (\ell^2\Gamma)^n$ has a trace.

1.2 The von Neumann dimension

We can now define the von Neumann dimension of Hilbert modules via the trace of projections:

Proposition and Definition 1.2.1 (von Neumann dimension)**.** *Let Γ be a countable group and let V be a Hilbert Γ-module. Then the* von Neumann Γ-dimension of V *is defined as*

$$\dim_{N\Gamma} V := \operatorname{tr}_\Gamma p,$$

where $n \in \mathbb{N}$, where $i \colon V \longrightarrow (\ell^2\Gamma)^n$ is an isometric Γ-embedding, and $p \colon (\ell^2\Gamma)^n \longrightarrow (\ell^2\Gamma)^n$ is the orthogonal Γ-projection onto $i(V)$. This is well-defined (i.e., independent of the chosen embedding into a finitely generated free $\ell^2\Gamma$-module) and $\dim_{N\Gamma} V \in \mathbb{R}_{\geq 0}$.

Proof. As first step, we note that $i(V)$ is a *closed* subspace of $(\ell^2\Gamma)^n$ (because V is complete and i is isometric). Hence, there indeed exists an orthogonal projection $p \colon (\ell^2\Gamma)^n \longrightarrow V$.

The trace is independent of the embedding: Let $j \colon V \longrightarrow (\ell^2\Gamma)^m$ also be an isometric Γ-embedding and let $q \colon (\ell^2\Gamma)^m \longrightarrow \operatorname{im} j$ be the orthogonal projection. Then we define a partial isometry $u \colon (\ell^2\Gamma)^n \longrightarrow (\ell^2\Gamma)^m$ by taking $j \circ i^{-1}$ on $\operatorname{im} i$ and taking 0 on $(\operatorname{im} i)^\perp$. By construction $j = u \circ i$. Taking adjoints shows that $q = p \circ u^*$ and hence

$$\begin{aligned}\operatorname{tr}_\Gamma q &= \operatorname{tr}_\Gamma(j \circ q) = \operatorname{tr}_\Gamma(u \circ i \circ q) = \operatorname{tr}_\Gamma(u \circ i \circ p \circ u^*)\\ &= \operatorname{tr}_\Gamma(i \circ p \circ u^* \circ u) \qquad \text{(trace property)}\\ &= \operatorname{tr}_\Gamma(i \circ p \circ p) = \operatorname{tr}_\Gamma(i \circ p) = \operatorname{tr}_\Gamma p.\end{aligned}$$

The von Neumann dimension is non-negative: Let $P \in M_{n\times n}(N\Gamma)$ be the matrix representing p. Because p (as an orthogonal projection) is a positive operator (check!), all the diagonal entries $P_{jj} \in N\Gamma$ of P are also positive operators (check!). Therefore, positivity of the von Neumann trace (Theorem 1.1.12) shows that $\dim_{N\Gamma} V = \operatorname{tr}_\Gamma p = \sum_{j=1}^n \operatorname{tr}_\Gamma P_{jj} \geq 0$. □

Example 1.2.2 (von Neumann dimension).

- Let Γ be a finite group and let V be a Hilbert Γ-module. Then
$$\dim_{N\Gamma} V = \frac{1}{|\Gamma|} \cdot \dim_{\mathbb{C}} V,$$
as can be seen from a direct computation (check!) or by applying the restriction formula (Theorem 1.2.3)
- Let $\Gamma = \mathbb{Z} = \langle t \,|\, \rangle$. We will use the description of $\ell^2\Gamma$ and $N\Gamma$ from Example 1.1.13. Let $A \subset [-\pi, \pi]$ be a measurable set. Then $V := \{f \cdot \chi_A \mid f \in L^2([-\pi,\pi], \mathbb{C})\}$ is a Hilbert Γ-module and the matrix
$$(\chi_A) \in M_{1\times 1}(N\Gamma)$$
describes the orthogonal projection onto V. Hence,
$$\dim_{N\Gamma} V = \operatorname{tr}_\Gamma \chi_A = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \chi_A \, d\lambda = \frac{1}{2\pi} \cdot \lambda(A)$$
and thus every number in $[0,1]$ occurs as von Neumann dimension of a Hilbert $N\Gamma$-module (!).

Theorem 1.2.3 (basic properties of the von Neumann dimension). *Let Γ be a countable group.*

1. Normalisation. *We have $\dim_{N\Gamma} \ell^2\Gamma = 1$.*
2. Faithfulness. *For every Hilbert Γ-module V, we have $\dim_{N\Gamma} V = 0$ if and only if $V \cong_\Gamma 0$.*
3. Weak isomorphism invariance. *If $f \colon V \longrightarrow W$ is a morphism of Hilbert Γ-modules with $\ker f = 0$ and $\overline{\operatorname{im} f} = W$, then $\dim_\Gamma V = \dim_\Gamma W$.*
4. Additivity. *Let $0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{\pi} V'' \longrightarrow 0$ be a weakly exact sequence of Hilbert Γ-modules (i.e., i is injective, $\overline{\operatorname{im} i} = \ker \pi$ and $\overline{\operatorname{im} \pi} = V''$). Then*
$$\dim_{N\Gamma} V = \dim_{N\Gamma} V' + \dim_{N\Gamma} V''.$$
5. Multiplicativity. *Let Λ be a countable group, let V be a Hilbert Γ-module, and let W be a Hilbert Λ-module. Then the completed tensor product $V \mathbin{\overline{\otimes}_{\mathbb{C}}} W$ is a Hilbert $\Gamma \times \Lambda$-module and*
$$\dim_{N(\Gamma\times\Lambda)} V \mathbin{\overline{\otimes}_{\mathbb{C}}} W = \dim_{N\Gamma} V \cdot \dim_{N\Lambda} W.$$

6. Restriction. *Let V be a Hilbert Γ-module and let $\Lambda \subset \Gamma$ be a subgroup of finite index. Then*

$$\dim_{N\Lambda} \operatorname{Res}^{\Gamma}_{\Lambda} V = [\Gamma : \Lambda] \cdot \dim_{N\Gamma} V.$$

Proof. *Ad 1.* This is clear from the definition (we can take $\mathrm{id}_{\ell^2\Gamma}$ as embedding and projection).

Ad 2. In view of faithfulness of the von Neumann trace (Theorem 1.1.12), it follows that the von Neumann trace of a projection is 0 if and only if the projection is 0 (check!).

Ad 3. This is a consequence of polar decomposition: Let $f = u \circ p$ be the polar decomposition of f into a partial isometry u and a positive operator p with $\ker u = \ker p$. We now show that u is a Γ-isometry between V and W: As f is injective, we have $\ker u = \ker p = 0$. Moreover, as partial isometry, u has closed image and so $\operatorname{im} u = \overline{\operatorname{im} u} = \overline{\operatorname{im} f} = W$. Hence, u is an isometry. Moreover, the uniqueness of the polar decomposition shows that u is Γ-equivariant. Therefore, $\dim_{N\Gamma} V = \dim_{N\Gamma} W$.

Ad 4. The von Neumann dimension is additive with respect to direct sums (check!). Moreover,

$$\begin{aligned} V &\longrightarrow \overline{\operatorname{im} i} \oplus V'' \\ x &\longmapsto \big(p(x), \pi(x)\big) \end{aligned}$$

is a weak isomorphism of Hilbert Γ-modules (check!), where $p \colon V \longrightarrow \overline{\operatorname{im} i}$ denotes the orthogonal projection. Therefore, weak isomorphism invariance of $\dim_\Gamma$ shows that

$$\dim_\Gamma V = \dim_\Gamma(\overline{\operatorname{im} i} \oplus V'') = \dim_\Gamma \overline{\operatorname{im} i} + \dim_\Gamma V'' = \dim_\Gamma V' + \dim_\Gamma V''.$$

Ad 5. The key observation is that $\ell^2(\Gamma \times \Lambda)$ is isomorphic (as Hilbert $\Gamma \times \Lambda$-module) to $\ell^2\Gamma \mathbin{\overline{\otimes}_{\mathbb{C}}} \ell^2\Lambda$ [66, Theorem 1.12].

Ad 6. This is Exercise 1.E.4. □

Moreover, the von Neumann dimension also satisfies inner and outer regularity [66, Theorem 1.12].

Outlook 1.2.4 (Atiyah conjecture)**.** The Atiyah question/conjecture comes in many flavours (originally formulated in terms of closed Riemannian manifolds). One version is:

> Let Γ be a torsion-free countable group, let $n \in \mathbb{N}$, and let $A \in M_{n\times n}(\mathbb{C}\Gamma)$. Then $\dim_{N\Gamma} \ker M_A \in \mathbb{Z}$ (?!)

This version of the Atiyah conjecture is known to hold for many classes of groups (and no counterexample is known so far); however, more general versions of the Atiyah conjecture are known to be false [7, 46]. One interesting aspect of the Atiyah conjecture is that it implies the Kaplansky zero-divisor conjecture (Exercise 1.E.5).

1.E Exercises

Exercise 1.E.1 (the "trivial" Hilbert module)**.** For which countable groups Γ is $\mathbb{C}$ (with the trivial Γ-action) a Hilbert Γ-module? Which von Neumann dimension does it have?

Exercise 1.E.2 (Hilbert modules as modules over the von Neumann algebra)**.** Let Γ be a countable group and let V be a Hilbert Γ-module. Show that the left Γ-action on V extends to a left $N\Gamma$-action on V.
Hints. This fact is the reason why often Hilbert Γ-modules are called *Hilbert $N\Gamma$-modules*.

Exercise 1.E.3 (kernels and cokernels)**.** Let Γ be a countable group, let V and W be Hilbert Γ-modules, and let $\varphi\colon V \longrightarrow W$ be a morphism of Hilbert Γ-modules.

1. Show that $\ker \varphi$ (with the induced inner product and Γ-action) is a Hilbert Γ-module
2. Show that $W/\overline{\operatorname{im} \varphi}$ (with the induced inner product and Γ-action) is a Hilbert Γ-module.

   Hints. Orthogonal complement!

Exercise 1.E.4 (restriction formula for the von Neumann dimension)**.** Let Γ be a countable group, let V be a Hilbert Γ-module, and let $\Lambda \subset \Gamma$ be a finite index subgroup. Show that

$$\dim_{N\Lambda} \operatorname{Res}^{\Gamma}_{\Lambda} V = [\Gamma : \Lambda] \cdot \dim_{N\Gamma} V.$$

Exercise 1.E.5 (Atiyah $\Longrightarrow$ Kaplansky)**.** Let Γ be a countable torsion-free group that satisfies the Atiyah conjecture (Outlook 1.2.4). Show that $\mathbb{C}\Gamma$ is a domain.

2

L^2-Betti numbers

We will now use the von Neumann dimension to define L^2-Betti numbers of groups and spaces, we will study basic properties of L^2-Betti numbers, and we will compute some simple examples. Furthermore, we will briefly outline generalisations of L^2-Betti numbers beyond our elementary approach.

Overview of this chapter.

Running example. the additive group $\mathbb{Z}$, finite (index sub)groups, free groups

2.1 An elementary definition of L^2-Betti numbers

L^2-Betti numbers are an equivariant version of ordinary Betti numbers. For simplicity, we will only consider L^2-Betti numbers of Γ-CW-complexes of finite type.

2.1.1 Finite type

Definition 2.1.1 (equivariant CW-complex)**.** Let Γ be a group.

- A *free* Γ*-CW-complex* is a CW-complex X together with a free Γ-action such that:
  - the Γ-action permutes the open cells of X and
  - if e is an open cell of X and $g \in \Gamma$ is non-trivial, then $g \cdot e \cap e \neq \emptyset$.
- A *morphism of* Γ*-CW-complexes* is a Γ-equivariant cellular map.

Definition 2.1.2 (finite type)**.**

- A CW-complex is of *finite type* if for each $n \in \mathbb{N}$, there are only finitely many open n-cells.
- Let Γ be a group. A (free) Γ-CW-complex is of *finite type* if for each $n \in \mathbb{N}$, there are only finitely many Γ-orbits of open n-cells.
- A group Γ is of *finite type* if it admits a classifying space of finite type (equivalently, a classifying space whose universal covering with the induced free Γ-CW-structure is a Γ-CW-complex of finite type).

Remark 2.1.3 (more on groups of finite type)**.** Let Γ be a group.

- If Γ is of finite type, then Γ is finitely presented (as fundamental group of a CW-complex of finite type); in particular, Γ is countable.
- If Γ is finitely presented, then Γ is of finite type if and only if $\mathbb{C}$ (with the trivial Γ-action) admits a projective resolution over $\mathbb{C}\Gamma$ that is finitely generated in each degree [17, Chapter VIII].

Example 2.1.4 (groups of finite type)**.**

- The group $\mathbb{Z}$ is of finite type: We can take the circle (with our favourite CW-structure) as classifying space.
- If Γ and Λ are of finite type, then so is $\Gamma \times \Lambda$ (we can take the product of finite type models as model for the classifying space of $\Gamma \times \Lambda$).

- If Γ and Λ are of finite type, then so is $\Gamma * \Lambda$ (we can take the wedge of finite type models as model for the classifying space of $\Gamma * \Lambda$).
- In particular, free Abelian groups of finite rank and free groups of finite rank are of finite type.
- Let $g \in \mathbb{N}_{\geq 2}$ and let Σ_g be "the" oriented closed connected surface of genus g. Then $\pi_1(\Sigma_g)$ is of finite type (because Σ_g is a (finite) model for the classifying space of Σ_g).
- More generally: If M is an oriented closed connected hyperbolic manifold, then $\pi_1(M)$ has finite type (because M is a (finite) model for the classifying space of $\pi_1(M)$).
- If $n \in \mathbb{N}_{\geq 2}$, then $\mathbb{Z}/n$ is of finite type (check!), but there is *no* finite model for the classifying space of $\mathbb{Z}/n$ [17].
- More generally: All finite groups are of finite type (for instance, one can use the simplicial Γ-resolution as blueprint to construct a contractible free Γ-CW-complex and then take its quotient).
- There exist finitely presented groups that are *not* of finite type [10].

2.1.2 L^2-Betti numbers of spaces

Definition 2.1.5 (L^2-Betti numbers of spaces)**.** Let Γ be a countable group and let X be a free Γ-CW-complex of finite type.

- The *cellular L^2-chain complex of X* is the twisted chain complex

$$C_*^{(2)}(\Gamma \curvearrowright X) := \ell^2\Gamma \otimes_{\mathbb{C}\Gamma} C_*(X).$$

  Here, $C_*(X)$ denotes the cellular chain complex of X (with $\mathbb{C}$-coefficients) with the induced Γ-action and $\ell^2\Gamma$ carries the $\ell^2\Gamma$-$\mathbb{C}\Gamma$-bimodule structure given by right and left translation on Γ.

- Let $n \in \mathbb{N}$. The *(reduced) L^2-homology of X in degree n* is defined by

$$H_n^{(2)}(\Gamma \curvearrowright X) := \ker \partial_n^{(2)} \;/\; \overline{\operatorname{im} \partial_{n+1}^{(2)}},$$

  where $\partial_*^{(2)}$ denotes the boundary operator on $C_*^{(2)}(\Gamma \curvearrowright X)$.

- The *n-th L^2-Betti number of X* is defined by

$$b_n^{(2)}(\Gamma \curvearrowright X) := \dim_{N\Gamma} H_n^{(2)}(\Gamma \curvearrowright X),$$

where $\dim_{N\Gamma}$ is the von Neumann dimension (Definition 1.2.1). It should be noted that $H_n^{(2)}(\Gamma \curvearrowright X)$ indeed is a Hilbert Γ-module (this follows from Exercise 1.E.3).

Notation 2.1.6. Moreover, we use the following abbreviation: If X is a CW-complex of finite type with fundamental group Γ and universal covering $\widetilde{X}$ (with the induced free Γ-CW-complex structure), then we write

$$b_n^{(2)}(X) := b_n^{(2)}(\Gamma \curvearrowright \widetilde{X}).$$

It should be noted that in the literature also the notation $b_n^{(2)}(\widetilde{X})$ can be found as abbreviation for $b_n^{(2)}(\Gamma \curvearrowright \widetilde{X})$. However, we prefer the notation $b_n^{(2)}(X)$ as it is less ambiguous (what is $b_n^{(2)}(\mathbb{H}^2)$?!).

Remark 2.1.7 (homotopy invariance)**.** Let Γ be a countable group, let X and Y be free Γ-CW-complexes, and let $n \in \mathbb{N}$. If $f \colon X \longrightarrow Y$ is a (cellular) Γ-homotopy equivalence, then

$$b_n^{(2)}(\Gamma \curvearrowright X) = b_n^{(2)}(\Gamma \curvearrowright Y),$$

because f induces a $\mathbb{C}\Gamma$-chain homotopy equivalence $C_*(X) \simeq_{\mathbb{C}\Gamma} C_*(Y)$ and thus a chain homotopy equivalence $C_*^{(2)}(\Gamma \curvearrowright X) \longrightarrow C_*^{(2)}(\Gamma \curvearrowright Y)$ in the category of chain complexes of Hilbert Γ-modules.

2.1.3 L^2-Betti numbers of groups

Let Γ be a group and let X and Y be models for the classifying space of Γ. Then the universal coverings $\widetilde{X}$ and $\widetilde{Y}$ with the induced Γ-CW-structures are (cellularly) Γ-homotopy equivalent. Therefore, $b_n^{(2)}(X) = b_n^{(2)}(Y)$ for all $n \in \mathbb{N}$ (Remark 2.1.7). Hence, the following notion is well-defined:

Definition 2.1.8 (L^2-Betti numbers of groups)**.** Let Γ be a group of finite type and let $n \in \mathbb{N}$. Then the *n-th L^2-Betti number of* Γ is defined by

$$b_n^{(2)}(\Gamma) := b_n^{(2)}(X),$$

where X is a model for the classifying space of Γ of finite type.

Similarly, we could also define/compute L^2-Betti numbers of groups by tensoring finite type projective resolutions of $\mathbb{C}$ over $\mathbb{C}\Gamma$ with $\ell^2\Gamma$ (check!).

2.2 Basic computations

For simplicity, in the following, we will focus on L^2-Betti numbers of groups. Similar statements also hold for L^2-Betti numbers of spaces [66].

2.2.1 Basic properties

Proposition 2.2.1 (degree 0). *Let Γ be a group of finite type.*

1. *If Γ is finite, then $b_0^{(2)}(\Gamma) = 1/|\Gamma|$.*
2. *If Γ is infinite, then $b_0^{(2)}(\Gamma) = 0$.*

Both cases can conveniently be summarised in the formula

$$b_0^{(2)}(\Gamma) = \frac{1}{|\Gamma|}.$$

Proof. Classical group homology tells us that $b_0^{(2)}(\Gamma) = \dim_{N\Gamma} V$, where

$$V = \ell^2\Gamma \;/\; \overline{\operatorname{Span}_{\mathbb{C}}\{x - g \cdot x \mid x \in \ell^2\Gamma,\ g \in \Gamma\}}.$$

If Γ is *finite*, then $\ell^2\Gamma = \mathbb{C}\Gamma$, whence $V \cong_\Gamma \mathbb{C}$ (with the trivial Γ-action). Therefore (Exercise 1.E.1),

$$b_0^{(2)}(\Gamma) = \dim_{N\Gamma} \mathbb{C} = \frac{1}{|\Gamma|}.$$

If Γ is *infinite*, then it suffices to show that $V \cong_\Gamma 0$: To this end, we only need to show that every bounded $\mathbb{C}$-linear functional $V \longrightarrow \mathbb{C}$ is the zero functional. Equivalently, we need to show that every Γ-invariant bounded $\mathbb{C}$-linear functional $f\colon \ell^2\Gamma \longrightarrow \mathbb{C}$ satisfies $f|_\Gamma = 0$ (check!). As Γ is infinite (and countable), we can enumerate Γ as $(g_n)_{n\in\mathbb{N}}$. The element $x := \sum_{n\in\mathbb{N}} 1/n \cdot g_n$ lies in $\ell^2\Gamma$ and the computation

$$\begin{aligned} f(x) &= \sum_{n\in\mathbb{N}} \frac{1}{n} \cdot f(g_n) && \text{(continuity and linearity of } f\text{)} \\ &= \sum_{n\in\mathbb{N}} \frac{1}{n} \cdot f(e) && \text{(}\Gamma\text{-invariance of } f\text{)} \end{aligned}$$

shows that $f(e) = 0$ (otherwise the series would *not* converge). Hence $f(g) = f(e) = 0$ for all $g \in \Gamma$ (by Γ-invariance), as desired. ☐

Theorem 2.2.2 (inheritance properties of L^2-Betti numbers). *Let Γ be a group of finite type and let $n \in \mathbb{N}$.*

1. Dimension. *If Γ admits a finite model of the classifying space of dimension less than n, then $b_n^{(2)}(\Gamma) = 0$.*
2. Restriction. *If $\Lambda \subset \Gamma$ is a subgroup of finite index, then also Λ is of finite type and*

$$b_n^{(2)}(\Lambda) = [\Gamma : \Lambda] \cdot b_n^{(2)}(\Gamma).$$

3. Künneth formula. *If Λ is a group of finite type, then*

$$b_n^{(2)}(\Gamma \times \Lambda) = \sum_{j=0}^{n} b_j^{(2)}(\Gamma) \cdot b_{n-j}^{(2)}(\Lambda).$$

4. Additivity. *If Λ is a group of finite type, then*

$$b_1^{(2)}(\Gamma * \Lambda) = b_1^{(2)}(\Gamma) + b_1^{(2)}(\Lambda) + 1 - \bigl(b_0^{(2)}(\Gamma) + b_0^{(2)}(\Lambda)\bigr)$$

and, if $n > 1$, then

$$b_n^{(2)}(\Gamma * \Lambda) = b_n^{(2)}(\Gamma) + b_n^{(2)}(\Lambda).$$

5. Poincaré duality. *If Γ admits a classifying space that is an oriented closed connected d-manifold, then*

$$b_n^{(2)}(\Gamma) = b_{d-n}^{(2)}(\Gamma).$$

Sketch of proof. Ad 1. This is clear from the definition.

Ad 2. If X is a finite type model for the classifying space of Γ, then the covering space Y associated with the subgroup $\Lambda \subset \Gamma$ is a model for the classifying space of Λ; moreover, Y is of finite type because the covering degree is $[\Gamma : \Lambda]$, which is finite. Algebraically, on the cellular chain complex of the universal covering space $\widetilde{X} = \widetilde{Y}$ (and whence also on its reduced cohomology with ℓ^2-coefficients), this corresponds to applying the restriction functor $\operatorname{Res}_\Lambda^\Gamma$. Then, we only need to apply the restriction formula for the von Neumann dimension (Theoerm 1.2.3).

Ad 3. Let X and Y be finite type models for the classifying space of Γ and Λ, respectively. Then $X \times Y$ is a model for the classifying space of $\Gamma \times \Lambda$, which is of finite type. One can now use a Künneth argument and the multiplicativity of the von Neumann dimension (Theorem 1.2.3) to prove the claim [66, Theorem 1.35].

Ad 4. Let X and Y be finite type models for the classifying space of Γ and Λ, respectively. Then the wedge $X \vee Y$ is a model for the classifying space of $\Gamma * \Lambda$, which is of finite type. One can now use a (cellular) Mayer-Vietoris argument to prove the additivity formula [66, Theorem 1.35].

Ad 5. The main ingredients are twisted Poincaré duality (applied to the coefficients $\ell^2\Gamma$) and the fact that the L^2-Betti numbers can also be computed in terms of reduced *co*homology [66, Theorem 1.35]. □

Proposition 2.2.3 (Euler characteristic). *Let Γ be a group that admits a* finite *classifying space. Then*

$$\chi(\Gamma) = \sum_{n\in\mathbb{N}} (-1)^n \cdot b_n^{(2)}(\Gamma).$$

Proof. This follows (as in the classical case) from the additivity of the von Neumann dimension (Exercise 2.E.2). □

2.2.2 First examples

Example 2.2.4 (finite groups)**.** Let Γ be a finite group. Then Γ is of finite type (Example 2.1.4) and, for all $n \in \mathbb{N}$, we have

$$b_n^{(2)}(\Gamma) = \frac{1}{|\Gamma|} \cdot \dim_{\mathbb{C}} H_n(\Gamma; \mathbb{C}\Gamma) = \begin{cases} \frac{1}{|\Gamma|} & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

Example 2.2.5 (the additive group $\mathbb{Z}$)**.** There are many ways to see that the L^2-Betti numbers of the additive group $\mathbb{Z}$ are all equal to 0. For instance: Let $n \in \mathbb{N}$. For $k \in \mathbb{N}_{>1}$, we consider the subgroup $k \cdot \mathbb{Z} \subset \mathbb{Z}$ of index k. Then the restriction formula (Theorem 2.2.2) shows that

$$\begin{aligned} b_n^{(2)}(\mathbb{Z}) &= b_n^{(2)}(k \cdot \mathbb{Z}) && (\text{because } k \cdot \mathbb{Z} \cong_{\mathsf{Group}} \mathbb{Z}) \\ &= k \cdot b_n^{(2)}(\mathbb{Z}) && (\text{restriction formula}) \end{aligned}$$

and so $b_n^{(2)}(\mathbb{Z}) = 0$.

Example 2.2.6 (free groups)**.** Let $r \in \mathbb{N}_{\geq 2}$ and let F_r be "the" free group of rank r. Then $X_r := \bigvee^r S^1$ is a model of the classifying space of F_r.

- Because F_r is infinite, we have $b_0^{(2)}(F_r) = 0$ (Proposition 2.2.1).
- Because $\dim X_r = 1$, we have $b_n^{(2)}(F_r) = 0$ for all $r \in \mathbb{N}_{\geq 2}$.
- It thus remains to compute $b_1^{(2)}(F_r)$. Because the Euler characteristic can be calculated via L^2-Betti numbers (Proposition 2.2.3), we obtain

$$b_1^{(2)}(F_r) = -\chi(F_r) + b_0^{(2)}(F_r) = -\chi(X_r) + 0 = r - 1.$$

Example 2.2.7 (surface groups)**.** Let $g \in \mathbb{N}_{\geq 2}$ and let $\Gamma_g := \pi_1(\Sigma_g)$, where Σ_g is "the" oriented closed connected surface of genus g. Then Σ_g is a model of the clasifying space of Σ_g.

- Because Γ_g is infinite, we have $b_0^{(2)}(\Gamma_g) = 0$ (Proposition 2.2.1).
- Because $\dim \Sigma_g = 2$, we obtain from Poincaré duality (Theorem 2.2.2) that $b_2^{(2)}(\Gamma_g) = b_0^{(2)}(\Gamma_g) = 0$ and that $b_n^{(2)}(\Gamma_g) = 0$ for all $n \in \mathbb{N}_{\geq 3}$.

- It thus remains to compute $b_1^{(2)}(\Sigma_g)$. Because the Euler characteristic can be calculated via L^2-Betti numbers (Proposition 2.2.3), we obtain

$$b_1^{(2)}(\Gamma_g) = -\chi(\Gamma_g) + b_0^{(2)}(\Gamma_g) + b_2^{(2)}(\Gamma_g) = -\chi(\Sigma_g) + 0 = 2 \cdot (g-1).$$

Outlook 2.2.8 (hyperbolic manifolds). More generally: Let Γ be the fundamental group of an oriented closed connected hyperbolic manifold M of dimension d.

- If d is odd, then $b_n^{(2)}(\Gamma) = 0$ for all $n \in \mathbb{N}$.
- If d is even, then $b_n^{(2)}(\Gamma) = 0$ for all $n \in \mathbb{N} \setminus \{d/2\}$. Moreover,

$$b_{d/2}^{(2)}(\Gamma) \neq 0.$$

The proof is based on the fact that L^2-Betti numbers can be computed in terms of spaces of L^2-harmonic forms [27][66, Chapter 1.4] and the explicit computation of L^2-harmonic forms of hyperbolic manifolds [28][66, Theorem 1.62]. (A cellular version of harmonic forms is discussed in Exercise 3.E.2.)

Outlook 2.2.9 (Singer conjecture). The Singer conjecture predicts that L^2-Betti numbers of closed aspherical manifolds are concentrated in the middle dimension:

> Let M be an oriented closed connected aspherical manifold of dimension d. Then
> $$\forall_{n \in \mathbb{N} \setminus \{d/2\}} \quad b_n^{(2)}\bigl(\pi_1(M)\bigr) = 0 \quad (?!)$$

2.3 Variations and extensions

- **Analytic definition.** Originally, Atiyah defined L^2-Betti numbers (of closed smooth manifolds) in terms of the heat kernel on the universal covering [6]. Dodziuk proved that these analytic L^2-Betti numbers admit a combinatorial description (in terms of ℓ^2-chain complexes of simplicial/cellular complexes of finite type) [27].
- **Singular definition.** Cheeger, Gromov, Farber, Lück extended the definition of the von Neumann dimension to all modules over the von Neumann algebra [22, 31, 66]; in particular, this allows for a definition of L^2-Betti numbers of spaces in terms of the singular chain complex with twisted coefficients in $\ell^2\pi_1$.
- **Extension to equivalence relations.** Gaboriau extended the definition of L^2-Betti numbers of groups to standard equivalence relations [42, 80, 83]. We will return to this point of view in Chapter 4.

- **Extension to topological groups.** Petersen gave a definition of L^2-Betti numbers of locally compact, second countable, unimodular groups [77].
- **Version for von Neumann algebras.** In a slightly different direction, Connes and Shlyakhtenko introduced a notion of L^2-Betti numbers for tracial von Neumann algebras [24]. However, it is unknown to which extent these L^2-Betti numbers of group von Neumann algebras coincide with the L^2-Betti numbers of groups.

2.E Exercises

Exercise 2.E.1 (products).

1. Let Γ and Λ be infinite groups of finite type. Show that $b_1^{(2)}(\Gamma \times \Lambda) = 0$.
2. Conclude: If Γ is a group of finite type with $b_1^{(2)}(\Gamma) \neq 0$, then Γ does *not* contain a finite index subgroup that is a product of two infinite groups.

Exercise 2.E.2 (Euler characteristic). Let Γ be a group that admits a finite classifying space. Show that

$$\chi(\Gamma) = \sum_{n \in \mathbb{N}} (-1)^n \cdot b_n^{(2)}(\Gamma).$$

Exercise 2.E.3 (QI?! [74, 88]). Let Γ be a group of finite type and let $r \in \mathbb{N}_{\geq 2}$.

1. Compute all L^2-Betti numbers of $F_r * \Gamma$ in terms of r and the L^2-Betti numbers of Γ.
2. Let $k \in \mathbb{N}_{\geq 2}$. Conclude that the quotient $b_1^{(2)}/b_k^{(2)}$ is *not* a quasi-isometry invariant.
3. Show that the sign of the Euler characteristic is *not* a quasi-isometry invariant.
4. Use these results to prove that there exist groups of finite type that are quasi-isometric but not commensurable.

Hints. If $s \in \mathbb{N}_{\geq 2}$, then it is known that F_r and F_s are bilipschitz equivalent and thus that $F_r * \Gamma$ and $F_s * \Gamma$ are quasi-isometric [74, 88].

Exercise 2.E.4 (deficiency [64]).

1. Let Γ be a group of finite type and let $\langle S \,|\, R \rangle$ be a finite presentation of Γ. Show that

   $$|S| - |R| \leq 1 - b_0^{(2)}(\Gamma) + b_1^{(2)}(\Gamma) - b_2^{(2)}(\Gamma).$$

   Taking the maximum of all these differences thus shows that the *deficiency* $\operatorname{def}(\Gamma)$ *of* Γ is bounded from above by the right hand side.
2. Let $\Gamma \subset \operatorname{Isom}^+(\mathbb{H}^4)$ be a torsion-free uniform lattice. Show that

   $$\operatorname{def}(\Gamma) \leq 1 - \chi(\Gamma) = 1 - \frac{3}{4 \cdot \pi^2} \cdot \operatorname{vol}(\Gamma \setminus \mathbb{H}^4).$$

3

The residually finite view: Approximation

The L^2-Betti numbers are related to classical Betti numbers through approximation by the (normalised) Betti numbers of finite index subgroups/finite coverings.

We explain the (spectral) proof of this approximation theorem and briefly discuss the relation with other (homological) gradient invariants.

This resdiually finite view will be complemented by the dynamical view in Chapter 4 and the approximation theorems for lattices in Chapter 5.

Overview of this chapter.

Running example. free Abelian groups, free groups

3.1 The approximation theorem

In the residually finite view, one approximates groups by finite quotients/finite index subgroups and spaces by finite coverings.

Definition 3.1.1 (residual chain, residually finite group). Let Γ be a finitely generated group.

- A *residual chain* for Γ is a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of finite index normal subgroups of Γ with $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ and $\bigcap_{n\in\mathbb{N}} \Gamma_n = \{e\}$.
- The group Γ is *residually finite* if it admits a residual chain.

Example 3.1.2 (residually finite groups).

- All finitely generated linear groups are residually finite [68, 72]. In particular: Fundamental groups of closed hyperbolic manifolds are residually finite. In contrast, it is unknown whether all finitely generated (Gromov-)hyperbolic groups are residually finite(!).
- There exist finitely presented groups that are not residually finite (e.g., each finitely presented infinite simple group will do).

Theorem 3.1.3 (Lück's approximation theorem [65]). *Let X be a connected CW-complex of finite type with residually finite fundamental group Γ, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a residual chain for Γ, and let $k \in \mathbb{N}$. Then*

$$b_k^{(2)}(X) = \lim_{n\to\infty} \frac{b_k(X_n)}{[\Gamma : \Gamma_n]}.$$

Here, X_n denotes the finite covering of X associated with the subgroup $\Gamma_n \subset \Gamma$ and b_k is the ordinary k-th $\mathbb{Q}$- Betti number (which equals the $\mathbb{C}$-Betti number).

Corollary 3.1.4 (approximation theorem, for groups). *Let Γ be a residually finite group of finite type, let $(\Gamma_n)_{n\in\mathbb{N}}$, and let $k \in \mathbb{N}$. Then*

$$b_k^{(2)}(\Gamma) = \lim_{n\to\infty} \frac{b_k(\Gamma_n)}{[\Gamma : \Gamma_n]}.$$

3.2 Proof of the approximation theorem

We will sketch the proof of the approximation theorem (Theorem 3.1.3), which is of a spectral nature; we will roughly follow unpublished notes of Sauer.

3.2.1 Reduction to kernels of self-adjoint operators

As first step, we reduce the approximation theorem to a statement about von Neumann dimensions of kernels of self-adjoint operators:

- On the one hand, if $\Lambda \curvearrowright Y$ is a free Λ-CW-complex (with a countable group Γ), then the (combinatorial) Laplacian Δ_* of the ℓ^2-chain complex of $\Lambda \curvearrowright Y$ is a positive self-adjoint operator on a Hilbert Λ-module that satisfies (Exercise 3.E.2)

$$b_k^{(2)}(\Lambda \curvearrowright Y) = \dim_{N\Lambda} \ker \Delta_k.$$

- On the other hand, the right hand side in the approximation theorem (Theorem 3.1.3), can also be written as von Neumann dimension: For each $n \in \mathbb{N}$, the finite group Γ/Γ_n is of finite type, we have (Example 1.2.2)

$$\frac{b_k(X_n)}{[\Gamma : \Gamma_n]} = b_k^{(2)}(\Gamma/\Gamma_n \curvearrowright X_n),$$

  and the boundary operator [Laplacian] on $C_*^{(2)}(\Gamma/\Gamma_n \curvearrowright X_n)$ is the reduction of the boundary operator [Laplacian] on $C_*^{(2)}(\Gamma \curvearrowright X)$ modulo Γ_n.

Therefore, Theorem 3.1.3 follows from the following (slightly more algebraically looking) version (because the cellular Laplacian is defined over the integral group ring):

Theorem 3.2.1 (approximation theorem for kernels). *Let Γ be a finitely generated residually finite group with residual chain $(\Gamma_n)_{n\in\mathbb{N}}$, let $m \in \mathbb{N}$, and let $A \in M_{m\times m}(\mathbb{Z}\Gamma)$ be self-adjoint and positive. Then*

$$\begin{aligned} &\dim_{N\Gamma} \ker\bigl(M_A \colon (\ell^2\Gamma)^m \to (\ell^2\Gamma)^m\bigr) \\ &= \lim_{n\to\infty} \dim_{N(\Gamma/\Gamma_n)} \ker\bigl(M_{A_n} \colon (\ell^2(\Gamma/\Gamma_n))^m \to (\ell^2(\Gamma/\Gamma_n))^m\bigr) \end{aligned}$$

where $A_n \in M_{m\times m}(\mathbb{Z}[\Gamma/\Gamma_n])$ denotes the reduction of A modulo Γ_n.

We will now prove Theorem 3.2.1.

3.2.2 Reformulation via spectral measures

We reformulate the claim of Theorem 3.2.1 in terms of spectral measures: Let μ_A be the spectral measure on $\mathbb{R}$ (with the Borel σ-algebra) of the self-adjoint operator A. This measure has the following properties [13, Chapter 6][52, Chapter 4.2]:

- The measure μ_A is supported on the compact set $[0, a]$, where $a := \|A\|$.
- If $f\colon [0, \infty] \longrightarrow \mathbb{R}$ is a measurable bounded function, then the bounded linear operator $f(M_A)$ defined by functional calculus satisifies

$$\int_{\mathbb{R}} f \, d\mu_A = \operatorname{tr}_\Gamma f(M_A).$$

- In the same way, for each $n \in \mathbb{N}$, also the spectral measure μ_{A_n} of the reduction A_n of A is supported on $[0, a]$ (because $\|A_n\| \leq \|A\|$).

Therefore, we obtain

$$\begin{aligned} \dim_{N\Gamma} \ker M_A &= \operatorname{tr}_\Gamma(\text{orthogonal projection onto } \ker M_A) \\ &= \operatorname{tr}_\Gamma\bigl(\chi_{\{0\}}(M_A)\bigr) \\ &= \mu_A\bigl(\{0\}\bigr) \end{aligned}$$

and, for all $n \in \mathbb{N}$,

$$\dim_{N(\Gamma/\Gamma_n)} \ker M_{A_n} = \mu_{A_n}\bigl(\{0\}\bigr).$$

Hence, the claim of the theorem is equivalent to the following property of the spectral measures: $\mu_A(\{0\}) = \lim_{n\to\infty} \mu_{A_n}(\{0\})$. We will now prove this statement on spectral measures.

3.2.3 Weak convergence of spectral measures

We first establish weak convergence of the spectral measures:

Lemma 3.2.2 (weak convergence of spectral measures)**.** *In this situation, the sequence $(\mu_{A_n})_{n\in\mathbb{N}}$ of measures on $\mathbb{R}$ weakly converges to μ_A, i.e., for all continuous functions $f\colon [0, a] \longrightarrow \mathbb{R}$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f \, d\mu_{A_n} = \int_{\mathbb{R}} f \, d\mu_A.$$

Proof. The measures μ_{A_n} with $n \in \mathbb{N}$ and the measure μ_A are all supported on a common compact set. Therefore, by the Weierstraß approximation theorem, it suffices to take test functions of the form $f = (x \mapsto x^d)$ with $d \in \mathbb{N}$.

Thus, let $d \in \mathbb{N}$ and $f := (x \mapsto x^d)$. We then have

$$\int_{\mathbb{R}} f \, d\mu_A = \operatorname{tr}_\Gamma f(M_A) = \operatorname{tr}_\Gamma(A^d)$$

and $\int_{\mathbb{R}} f \, d\mu_{A_n} = \operatorname{tr}_{\Gamma/\Gamma_n} A_n^d$.

Let $F \subset \Gamma$ be the support of A^d (i.e., all elements of Γ that occur with non-zero coefficient in A). Because F is finite and $(\Gamma_n)_{n\in\mathbb{N}}$ is a residual chain, there exists an $N \in \mathbb{N}$ with

$$\forall_{n\in\mathbb{N}_{\geq N}} \quad F \cap \Gamma_n \subset \{e\}.$$

Then (by definition of the trace; check!)

$$\operatorname{tr}_{\Gamma/\Gamma_n} A_n^d = \operatorname{tr}_\Gamma A^d$$

for all $n \in \mathbb{N}_{\geq N}$. This shows weak convergence. □

3.2.4 Convergence at 0

In general, weak convergence does *not* imply convergence of the measures on $\{0\}$ (Exercise 3.E.3). But by the portmanteau theorem [11], we at least obtain the following inequalities from Lemma 3.2.2:

$$\limsup_{n\to\infty} \mu_{A_n}\bigl(\{0\}\bigr) \leq \mu\bigl(\{0\}\bigr)$$
$$\forall_{\varepsilon\in\mathbb{R}_{>0}} \quad \liminf_{n\to\infty} \mu_{A_n}\bigl((-\varepsilon,\varepsilon)\bigr) \geq \mu\bigl((-\varepsilon,\varepsilon)\bigr).$$

The first inequality already gives $\limsup_{n\to\infty} 1/[\Gamma_n : \Gamma] b_k(X_n) \leq b_k^{(2)}(X)$, the *Kazhdan inequality*.

In order to show the missing lower bound $\liminf_{n\to\infty} \mu_{A_n}(\{0\}) \geq \mu(\{0\})$, we will use integrality of the coefficients of A. More precisely, we will show:

Lemma 3.2.3. *In this situation, for all $n \in \mathbb{N}$ and all $\varepsilon \in (0,1)$, we have*

$$\mu_{A_n}\bigl((0,\varepsilon)\bigr) \leq \frac{m \cdot \ln(C)}{\bigl|\ln(\varepsilon)\bigr|},$$

where $C := \max\bigl(\|A\|, 1\bigr)$ (which does not *depend on n).*

Proof. Let $n \in \mathbb{N}$ and let $d := [\Gamma : \Gamma_n]$. Computing $\mu_{A_n}((0,\varepsilon))$ amounts to counting eigenvalues. We can view A_n as matrix in $M_{m\times m}(\mathbb{Z}[\Gamma/\Gamma_n]) \subset M_{d\cdot m\times d\cdot m}(\mathbb{Z})$. In this view, A_n is symmetric and positive semi-definite (check!); let

$$0 = \lambda_1 \leq \cdots \leq \lambda_z = 0 < \lambda_{z+1} \leq \cdots \leq \lambda_{d\cdot m}$$

be the eigenvalues of A_n (listed with multiplicities). Then the characteristic polynomial of A_n is of the form $T^z \cdot q$ with $q \in \mathbb{Z}[T]$. In particular, $q(0) \neq 0$ and thus (because of integrality!)

$$\lambda_{z+1} \cdot \cdots \cdot \lambda_{d\cdot m} = \bigl|q(0)\bigr| \geq 1.$$

For $\varepsilon \in (0,1)$, let $M(\varepsilon)$ be the number of eigenvalues of A_n in $(0,\varepsilon)$. Then

$$1 \leq \lambda_{z+1} \cdot \dots \cdot \lambda_{d \cdot m} \leq \varepsilon^{M(\varepsilon)} \cdot \|M_{A_n}\|^{d \cdot m - z - M(\varepsilon)} \leq \varepsilon^{M(\varepsilon)} \cdot C^{d \cdot m},$$

and so $M(\varepsilon) \leq d \cdot m \cdot \ln C / |\ln \varepsilon|$. Therefore, we obtain

$$\begin{aligned} \mu_{A_n}\bigl((0,\varepsilon)\bigr) &= \dim_{N(\Gamma/\Gamma_n)} \text{all eigenspaces of } A_n \text{ for eigenvalues in } (0,\varepsilon) \\ &= \frac{M(\varepsilon)}{d} \\ &\leq \frac{m \cdot \ln C}{\bigl|\ln(\varepsilon)\bigr|}. \end{aligned}$$

□

We can now complete the proof of Theorem 3.2.1 as follows: We have

$$\begin{aligned} \liminf_{n\to\infty} \mu_{A_n}\bigl(\{0\}\bigr) &= \liminf_{n\to\infty} \bigl(\mu_{A_n}\bigl([0,\varepsilon]\bigr) - \mu_{A_n}\bigl((0,\varepsilon)\bigr)\bigr) \\ &\geq \liminf_{n\to\infty} \mu_{A_n}\bigl((-\varepsilon,\varepsilon)\bigr) - \frac{m \cdot \ln C}{\bigl|\ln(\varepsilon)\bigr|} && \text{(Lemma 3.2.3)} \\ &\geq \mu_A\bigl((-\varepsilon,\varepsilon)\bigr) - \frac{m \cdot \ln C}{\bigl|\ln(\varepsilon)\bigr|} && \text{(portmanteau theorem)} \\ &\geq \mu_A\bigl(\{0\}\bigr) - \frac{m \cdot \ln C}{\bigl|\ln(\varepsilon)\bigr|}. \end{aligned}$$

Taking $\varepsilon \to 0$ yields the desired estimate $\liminf_{n\to\infty} \mu_{A_n}(\{0\}) \geq \mu_A(\{0\})$. This finishes the proof of Theorem 3.2.1 and whence also of the approximation theorem (Theorem 3.1.3).

3.3 Homological gradient invariants

If I is a numerical invariant of (finitely generated residually finite) groups, then one can consider the associated *gradient invariants* $\widehat{I}$: If Γ is a finitely generated residually finite group and Γ_* is a residual chain in Γ, then

$$\widehat{I}(\Gamma, \Gamma_*) := \lim_{n\to\infty} \frac{I(\Gamma_n)}{[\Gamma : \Gamma_n]}.$$

This raises the following questions:

- Does the limit exist?
- Does $\widehat{I}(\Gamma, \Gamma_*)$ depend on the residual chain Γ_* of Γ ?
- Does $\widehat{I}$ have a different interpretation?

3.3.1 Betti number gradients

For the gradient invariant associated to the ordinary Betti numbers, the approximation theorem (Theorem 3.1.3) gives a satisfying answer for finitely presented residually finite groups (of finite type).

Caveat 3.3.1. There exist finitely *generated* residually finite groups Γ with a residual chain $(\Gamma_n)_{n\in\mathbb{N}}$ such that the limit $\lim_{n\to\infty} b_1(\Gamma_n)/[\Gamma : \Gamma_n]$ does *not* exist [30].

In Chapter 5, we will briefly discuss the situation of convergence of ordinary Betti numbers when moving from residual chains to BS-convergent sequences. Generalisations of the more classical version of the approximation theorem are surveyed in the literature [67, 55].

For $\mathbb{F}_p$-Betti number gradients the situation is much less understood. There are known positive examples of convergence/independence, but good candidates for alternative interpretations of the limits are rare.

3.3.2 Rank gradient

A non-commutative version of the first Betti number gradient is the rank gradient:

Definition 3.3.2 (rank gradient [57])**.** Let Γ be a finitely generated infinite residually finite group.

- For a finitely generated group Λ, we write $d(\Lambda)$ for the minimal size of a generating set of Λ.
- If Γ_* is a residual chain, then we define the *rank gradient of* Γ *with respect to* Γ_* by
$$\operatorname{rg}(\Gamma, \Gamma_*) := \lim_{n\to\infty} \frac{d(\Gamma_n) - 1}{[\Gamma : \Gamma_n]}.$$
- Moreover, the *(absolute) rank gradient* of Γ is defined as
$$\operatorname{rg}\Gamma := \inf_{\Lambda\in F(\Gamma)} \frac{d(\Lambda) - 1}{[\Gamma : \Lambda]},$$
where $F(\Gamma)$ denotes the set of all finite index subgroups of Γ.

Remark 3.3.3. If Γ is a finitely generated group and $\Lambda \subset \Gamma$ is a finite index subgroup, then the rank estimate of the Nielsen-Schreier theorem shows that

$$d(\Lambda) - 1 \leq [\Gamma : \Lambda] \cdot \bigl(d(\Gamma) - 1\bigr).$$

Hence, the limit in the definition of the rank gradient indeed exists (and is equal to the infimum).

Remark 3.3.4 (rank gradient via normal subgroups). Let Γ be a finitely generated infinite group. Then

$$\operatorname{rg}\Gamma = \inf_{\Lambda \in F(\Gamma)} \frac{d(\Lambda) - 1}{[\Gamma : \Lambda]} = \inf_{\Lambda \in N(\Gamma)} \frac{d(\Lambda) - 1}{[\Gamma : \Lambda]},$$

where $N(\Gamma)$ denotes the set of all finite index normal subgroups of Γ (because every finite index subgroup of Γ contains a finite index subgroup that is normal in Γ).

Corollary 3.3.5 (rank gradient estimate for the first L^2-Betti number). *Let Γ be a finitely generated infinite residually finite group (of finite type). Then*

$$b_1^{(2)}(\Gamma) \leq \operatorname{rg}\Gamma.$$

Proof. If Λ is a finitely generated group, then there exists a classifying space of Λ with $d(\Lambda)$ one-dimensional cells. Therefore, $b_1(\Lambda) \leq d(\Lambda)$. Applying the approximation theorem (Theorem 3.1.3) to a residual chain Γ_* of Γ shows that

$$b_1^{(2)}(\Gamma) = \lim_{n\to\infty} \frac{b_1(\Gamma_n)}{[\Gamma : \Gamma_n]} \leq \lim_{n\to\infty} \frac{d(\Gamma_n)}{[\Gamma : \Gamma_n]} = \lim_{n\to\infty} \frac{d(\Gamma_n) - 1}{[\Gamma : \Gamma_n]} = \operatorname{rg}(\Gamma, \Gamma_*).$$

Taking the infimum over all residual chains, we obtain with Remark 3.3.4 that

$$b_1^{(2)}(\Gamma) \leq \inf_{\Lambda \in N(\Gamma)} \frac{d(\Lambda) - 1}{[\Gamma : \Lambda]} = \operatorname{rg}\Gamma.$$

Alternatively, one can also show directly that $b_1^{(2)}(\Lambda) \leq d(\Lambda)$ for all finitely generated groups and then use multiplicativity of $b_1^{(2)}$ under finite coverings (Theorem 2.2.2). □

However, it is an open problem to determine whether the rank gradient depends on the residual chain; in all known cases, the inequality in Corollary 3.3.5 is an equality (!) and the absolute rank gradient can be computed by every residual chain. This is related to the fixed price problem (Outlook 4.3.6).

The Betti number-rank estimate can be improved to estimates for the minimal size of normal generating sets in terms of the first L^2-Betti number; in particular, this gives lower bounds on the girth of certain Cayley graphs [86, Theorem 5.1].

3.3.3 More gradients

Further examples of gradient invariants are:

- Homology log-torsion gradients (which conjecturally might be related to L^2-torsion?!) [67].
- Simplicial volume gradients (Appendix A).

3.E Exercises

Exercise 3.E.1 (surface groups, free groups)**.** Prove the approximation theorem for surface groups and free groups by direct computation of the right hand side.

Exercise 3.E.2 (Laplacian)**.** Let Γ be a countable group, let C_* be a chain complex of Hilbert Γ-modules (with boundary operators ∂_*), and let Δ_* be the *Laplacian* of C_*, i.e., for each $n \in \mathbb{N}$, we set

$$\Delta_n := \partial_{n+1} \circ \partial_{n+1}^* + \partial_n^* \circ \partial_n.$$

Show that there exists an isomorphism

$$\ker \Delta_n \longrightarrow \ker \partial_n \bigm/ \overline{\operatorname{im} \partial_{n+1}}$$

of Hilbert Γ-modules.
Hints. Consider the orthogonal projection onto $\overline{\operatorname{im} \partial_{n+1}}^{\perp}$.

Exercise 3.E.3 (weak convergence)**.** Give an example of a sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $[0,1]$ (with the Borel σ-algebra) that weakly converges to a probability measure μ on $[0,1]$, but that satisfies

$$\lim_{n\to\infty} \mu_n\bigl(\{0\}\bigr) \neq \mu\bigl(\{0\}\bigr).$$

Exercise 3.E.4 (rank gradients of products)**.** Let Γ and Λ be finitely generated infinite residually finite groups. Compute $\operatorname{rg}(\Gamma \times \Lambda)$.

Exercise 3.E.5 (self-maps)**.** Let M be an oriented closed connected aspherical manifold with residually finite fundamental group Γ. Moreover, we suppose that M admits a self-map $f \colon M \longrightarrow M$ with $|\deg f| \geq 2$.

1. Give examples of this situation.
2. Show that $\operatorname{rg}(\Gamma) = 0$.
3. Show that $b_k^{(2)}(\Gamma) = 0$ for all $k \in \mathbb{N}$.
4. Challenge: Does the vanishing of L^2-Betti numbers of Γ also hold without any residual finiteness or Hopficity condition on Γ ?! (This is an open problem.)

Hints. Covering theory shows that $\operatorname{im} f$ has finite index in Γ. Moreover, it is useful to know that residually finite groups are *Hopfian*, i.e., every self-epimorphism is an automorphism.

4

The dynamical view: Measured group theory

The theory of von Neumann algebras can be view as a model of non-commutative measure theory. Therefore, it is plausible that L^2-Betti numbers can be computed in terms of probability measure preserving actions.

- On the one hand, this leads to an additional way of computing L^2-Betti numbers of groups.
- On the other hand, in this way, L^2-Betti numbers provide orbit equivalence invariants.

We will first recall basic terminology from measured group theory. Then we will discuss L^2-Betti numbers of standard equivalence relations and cost, and their interactions.

Overview of this chapter.

Running example. amenable groups, free groups

4.1 Measured group theory

Measured group theory is the theory of dynamical systems, i.e., of (probability) measure preserving actions of groups. We briefly introduce some of the terminology. More information can be found in the literature [54, 40, 43, 44].

4.1.1 Standard actions

Definition 4.1.1 (standard action)**.** Let Γ be a countable group.

- A *standard action of* Γ is an action of Γ on a standard Borel probability space by measure preserving Borel automorphisms.
- A standard action $\Gamma \curvearrowright (X, \mu)$ is *essentially free* if μ-almost every point has trivial stabiliser group.
- A standard action $\Gamma \curvearrowright (X, \mu)$ is *ergodic* if every measurable subset $A \subset X$ with $\Gamma \cdot A = A$ satisfies $\mu(A) \in \{0, 1\}$.

A *standard Borel space* is a measure space that is isomorphic to Polish space with its Borel σ-algebra. It turns out that standard Borel spaces form a convenient category for measure theory [53].

Example 4.1.2 (Bernoulli shift)**.** Let Γ be a countable group. Then the shift action of Γ on the product space $\prod_\Gamma \{0, 1\}$ (with the product σ-algebra and the product of the uniform distribution on $\{0, 1\}$) is a standard action of Γ.

Moreover, if Γ is infinite, this action is essentially free and ergodic [83, Lemma 3.37].

Example 4.1.3 (finite quotients)**.** Let Γ be a countable group and let $\Lambda \subset \Gamma$ be a finite index subgroup. Then the translation action of Γ on the coset space Γ/Λ (with the discrete σ-algebra and the uniform distribution) is a standard action. It is ergodic, but apart from pathological cases, not essentially free.

Example 4.1.4 (profinite completion)**.** Let Γ be a finitely generated group. We then consider the *profinite completion of* Γ, defined by

$$\widehat{\Gamma} := \varprojlim_{\Lambda \in N(\Gamma)} \Gamma/\Lambda,$$

where $N(\Gamma)$ denotes the set of all finite index normal subgroups of Γ. Then $\widehat{\Gamma}$ is a group with the induced composition and the diagonal map $\Gamma \longrightarrow \widehat{\Gamma}$ is a group homomorphism, which leads to a Γ-action on $\widehat{\Gamma}$ (by translation of each component). The group Γ is residually finite if and only if this action on $\widehat{\Gamma}$ is free (Exercise 4.E.1).

Moreover, we can equip $\widehat{\Gamma}$ with the inverse limit of the discrete σ-algebras and the inverse limit of the uniform probability measures on the finite factors. This action $\Gamma \curvearrowright \widehat{\Gamma}$ then is a standard action.

4.1.2 Measure/orbit equivalence

We will now compare measure preserving actions of groups via couplings and their orbit structure, respectively. Measure equivalence is a measure-theoretic version of quasi-isometry (the connection being given by Gromov's topological criterion for quasi-isometry [48, 0.2.C_2'])

Definition 4.1.5 (measure equivalence [48, 0.5.E])**.** Let Γ and Λ be countable infinite groups.

- An *ME coupling* between the groups Γ and Λ is a standard Borel measure space (Ω, μ) of infinite measure together with a measure preserving action of $\Gamma \times \Lambda$ by Borel automorphisms so that both actions $\Gamma \curvearrowright (\Omega, \mu)$ and $\Lambda \curvearrowright (\Omega, \mu)$ admit fundamental domains Y and X, respectively, of finite measure. The *index* of such an ME coupling is the quotient $\mu(X)/\mu(Y)$.
- The groups Γ and Λ are *measure equivalent* if there exists an ME coupling between them; in this case, we write $\Gamma \sim_{\mathrm{ME}} \Lambda$.

Example 4.1.6 (lattices are measure equivalent)**.** Let G be a locally compact second countable group (with infinite Haar measure μ) and let $\Gamma, \Lambda \subset G$ be lattices in G. Then the action

$$\begin{aligned}(\Gamma \times \Lambda) \times G &\longrightarrow G\\ \big((\gamma, \lambda), x\big) &\longmapsto \gamma \cdot x \cdot \lambda^{-1}\end{aligned}$$

shows that the ambient group G yields an ME coupling between Γ and Λ.

In particular, countable infinite commensurable groups are measure equivalent. For instance, $F_n \sim_{\mathrm{ME}} F_m$ for all $n, m \in \mathbb{N}_{\geq 2}$.

Measure equivalence indeed defines an equvialence relation on the class of all countable infinite groups [40, p. 300].

Definition 4.1.7 ((stable) orbit equivalence)**.** Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be standard actions.

- The actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are *orbit equivalent* if there exists a measure preserving Borel isomorphism $f\colon X' \longrightarrow Y'$ between co-null subsets $X' \subset X$ and $Y' \subset Y$ with

$$\forall_{x \in X'} \quad f(\Gamma \cdot x \cap X') = \Lambda \cdot f(x) \cap Y'.$$

In this case, we write $\Gamma \curvearrowright X \sim_{\mathrm{OE}} \Lambda \curvearrowright Y$.

- The actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are *stably orbit equivalent* if there exists a Borel isomorphism $f \colon X' \longrightarrow Y'$ between measurable subsets $X' \subset X$ and $Y' \subset Y$ with $\mu(X') > 0$ and $\mu(Y')$ that satisfies $1/\mu(X') \cdot f_* \mu|_{X'} = 1/\nu(Y') \cdot \nu|_{Y'}$ with

$$\forall_{x \in X'} \quad f(\Gamma \cdot x \cap X') = \Lambda \cdot f(x) \cap Y'.$$

The *index* of such a stable orbit equivalence f is $\mu(Y')/\mu(X')$ and we write $\Gamma \curvearrowright X \sim_{\mathrm{SOE}} \Lambda \curvearrowright Y$.

Moreover, we call Γ and Λ [stably] orbit equivalent if they admit [stably] orbit equivalent standard actions.

Theorem 4.1.8 (measure equivalence [40, Theorem 2.5])**.** *Two countable groups are measure equivalent if and only if they admit essentially free standard actions that are stably orbit equvialent.*

Example 4.1.9 (lattices are stably orbit equivalent)**.** In view of Theorem 4.1.8 and Example 4.1.6, we obtain: Lattices in locally compact second countable topological groups with infinite Haar measure are stably orbit equivalent.

In particular, $F_n \sim_{\mathrm{SOE}} F_m$ for all $n, m \in \mathbb{N}_{\geq 2}$.

In order to get a better understanding of measure/orbit equivalence of groups, we need suitable invariants. A first example is the property of being amenable [76, 79].

Theorem 4.1.10 (dynamical characterisation of amenable groups [73, 54])**.**

1. *A countable infinite group is amenable if and only if it is measure equivalent to $\mathbb{Z}$.*
2. *Any two ergodic standard actions of any two countable infinite amenable groups are orbit equivalent.*

Further examples of suitable invariants are (signs of) L^2-Betti numbers and cost, as we will outline now.

4.2 L^2-Betti numbers of equivalence relations

Orbit equivalence is a notion that does not directly involve a group (action), but only the orbit equivalence relations of standard actions. It is therefore natural to widen the context by studying equivalence relation in this measured setting. Gaboriau discovered that one can define L^2-Betti numbers of such equivalence relations and how these relate to L^2-Betti numbers of groups [42].

4.2.1 Standard equivalence relations

Definition 4.2.1 (standard equivalence relation).

- A *standard equivalence relation* is an equivalence relation $\mathcal{R} \subset X \times X$ on a standard Borel space X with the following properties:
  - The subset $\mathcal{R} \subset X \times X$ is measurable.
  - Each $\mathcal{R}$-equivalence class is countable.
- If $\mathcal{R}$ is a standard equivalence relation on X and $A \subset X$ is a measurable subset, then we define the *restriction to* A by

$$\mathcal{R}|_A := \bigl\{(x,y) \bigm| x,y \in A,\ (x,y) \in \mathcal{R}\bigr\} \subset A \times A.$$

- A *measured standard equivalence relation* is a standard equivalence relation $\mathcal{R}$ on a standard Borel probability space (X,μ) with the following property: Every partial $\mathcal{R}$-automorphism of X is μ-preserving. A *partial* $\mathcal{R}$*-automorphism* is a Borel automorphism $A \longrightarrow B$ between measurable subsets $A, B \subset X$ whose graph is contained in $\mathcal{R}$.

Example 4.2.2 (orbit relations). Let $\Gamma \curvearrowright X$ be a standard action. Then the *orbit relation*

$$\mathcal{R}_{\Gamma \curvearrowright X} := \bigl\{(x, \gamma \cdot x) \bigm| x \in X,\ \gamma \in \Gamma\bigr\} \subset X \times X$$

is a (measured) standard equivalence relation. Conversely, every standard equivalence relation arises in this way [38].

4.2.2 Construction and fundamental properties

We now introduce L^2-Betti numbers of measured standard equivalence relations, following Sauer's "algebraic" approach [80] (and will also add the references for proofs in this language); the original construction is due to Gaboriau and has a more simplicial flavour [42]. For the definition of L^2-Betti numbers of a group Γ (of finite type), we needed the following ingredients:

- Base ring: The field $\mathbb{C}$.
- Extension by the group: The group ring $\mathbb{C}\Gamma$.
- Completion of scalars: The group von Neumann algebra $N\Gamma$.
- The trace/dimension of $N\Gamma$.
- Modules with a projectivity condition: Hilbert Γ-modules.
- A suitable model of the classifying space of Γ.

If $\mathcal{R}$ is a measured standard equivalence relation on (X, μ), one can use the following replacements:

- Base ring: The function space $L^\infty(X) := L^\infty(X, \mathbb{C})$.
- Extension by the equivalence relation: The ring

$$\mathbb{C}\mathcal{R} := \Bigl\{ f \in L^\infty(\mathcal{R}, \nu) \Bigm| \sup_{x \in X} \bigl|\{y \mid f(x,y) \neq 0\}\bigr| < \infty, \sup_{y \in X} \bigl|\{x \mid f(x,y) \neq 0\}\bigr| < \infty \Bigr\}$$

  with the "convolution" product $(f \cdot g)(x,y) := \sum_{z \in [x]_\mathcal{R}} f(x,z) \cdot g(z,y)$. Here, we use the following measure on $\mathcal{R}$:

$$\begin{aligned} \nu \colon \text{Borel } \sigma\text{-algebra on } \mathcal{R} &\longrightarrow \mathbb{R}_{\geq 0} \\ A &\longmapsto \int_X \bigl|A \cap (\{x\} \times X)\bigr| \, d\mu(x,y). \end{aligned}$$

  Moreover, in the case of $\mathcal{R}_{\Gamma \curvearrowright X}$ we also have the algebraic crossed product $L^\infty(X) \rtimes G$ (where $g \in G$ acts by $f \mapsto (x \mapsto f(g^{-1} \cdot x))$ on $L^\infty(X)$).
- Completion of scalars: The von Neumann algebra $N\mathcal{R}$ is the weak closure of $\mathbb{C}\mathcal{R}$ in $B(L^2(\mathcal{R}, \nu))$ (with respect to the right convolution action).
- This von Neumann algebra also admits a trace

$$\begin{aligned} \operatorname{tr}_\mathcal{R} \colon N\mathcal{R} &\longrightarrow \mathbb{C} \\ a &\longmapsto \bigl\langle \chi_\Delta, a(\chi_\Delta) \bigr\rangle \end{aligned}$$

  and a corresponding dimension function $\dim_{N\mathcal{R}}$ (see below).
- If P is a finitely generated projective $N\mathcal{R}$-module, we set

$$\dim_{N\mathcal{R}} P := \operatorname{tr}_\mathcal{R} p = \sum_{j=1}^n \operatorname{tr}_\mathcal{R} A_{j,j} \in \mathbb{R}_{\geq 0},$$

  where $p \colon (N\mathcal{R})^n \longrightarrow (N\mathcal{R})^n$ is a projection with $P \cong_{N\mathcal{R}} \operatorname{im} p$ and associated matrix $A \in M_{n \times n}(N\mathcal{R})$. More generally, for an $N\mathcal{R}$-module V, one sets

$$\dim_{N\mathcal{R}} V := \sup \bigl\{ \dim_{N\mathcal{R}} P \bigm| P \text{ is a finitely generated projective } N\mathcal{R}\text{-submodule of } V \bigr\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

  (This only makes sense because the module theory of $N\mathcal{R}$ is well-behaved! More precisely: The ring $N\mathcal{R}$ is semi-hereditary.)
- Instead of a classifying space (as in Gaboriau's work), we use the algebraic description of group homology as a Tor-functor.

Definition 4.2.3 (L^2-Betti numbers of equivalence relations)**.** Let $\mathcal{R}$ be a measured standard equivalence relation on (X, μ) and let $n \in \mathbb{N}$. Then the *n-th L^2-Betti number of* $\mathcal{R}$ is defined by

$$b_n^{(2)}(\mathcal{R}) := \dim_{N\mathcal{R}} \operatorname{Tor}_n^{\mathbb{C}\mathcal{R}}\bigl(N\mathcal{R}, L^\infty(X)\bigr) \in \mathbb{R}_{\geq 0}.$$

Theorem 4.2.4 (restriction formula [42, Corollaire 5.5][80])**.** *Let $\mathcal{R}$ be a measured standard equivalence relation on (X, μ) and let $A \subset X$ be a measurable subset with $\mu(A) > 0$. Then, for all $n \in \mathbb{N}$,*

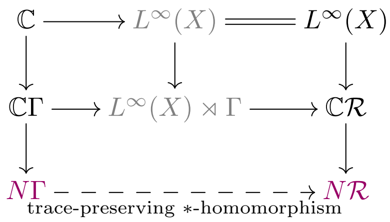
$$b_n^{(2)}(\mathcal{R}|_A) = \frac{1}{\mu(A)} \cdot b_n^{(2)}(\mathcal{R}).$$

The key observation is that L^2-Betti numbers of orbit relations coincide with the L^2-Betti numbers of the group:

Theorem 4.2.5 (L^2-Betti numbers of groups vs. equivalence relations [42, Corollaire 3.16])**.** *Let Γ be a group of finite type and let $\Gamma \curvearrowright X$ be an essentially free standard action of Γ. Then, for all $n \in \mathbb{N}$,*

$$b_n^{(2)}(\Gamma) = b_n^{(2)}(\mathcal{R}_{\Gamma \curvearrowright X}).$$

Sketch of proof [80]. We have the following commutative diagram of rings (where all unmarked arrows denote canonical inclusions and $\mathcal{R} := \mathcal{R}_{\Gamma \curvearrowright X}$):



One then obtains [80] (via suitable tools from homological algebra)

$$\begin{aligned}
b_n^{(2)}(\Gamma) &= \dim_{N\Gamma} \text{reduced } n\text{-th } \ell^2\Gamma\text{-homology of } B\Gamma && \text{(by definition)}\\
&= \dim_{N\Gamma} \text{algebraic } n\text{-th } \ell^2\Gamma\text{-homology of } B\Gamma && \text{([66, Theorem 6.24])}\\
&= \dim_{N\Gamma} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(N\Gamma, \mathbb{C}) && \text{(by definition of Tor)}\\
&= \dim_{N\mathcal{R}} \bigl(N\mathcal{R} \otimes_{N\Gamma} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(N\Gamma, \mathbb{C})\bigr) && \text{(trace-pres. $*$-homs of finite vN-algs are dim-pres.)}\\
&= \dim_{N\mathcal{R}} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(N\mathcal{R}, \mathbb{C}) && \text{(trace-pres. $*$-homs of finite vN-algs are faithfully flat)}\\
&= \dim_{N\mathcal{R}} \operatorname{Tor}_n^{L^\infty(X) \rtimes \Gamma}\bigl(N\mathcal{R}, (L^\infty(X) \rtimes G) \otimes_{\mathbb{C}\Gamma} \mathbb{C}\bigr) && \text{($L^\infty(X) \rtimes \Gamma$ is flat as right $\mathbb{C}\Gamma$-module)}\\
&= \dim_{N\mathcal{R}} \operatorname{Tor}_n^{\mathbb{C}\mathcal{R}}\bigl(N\mathcal{R}, L^\infty(X)\bigr) && \text{($L^\infty(X) \rtimes \Gamma \hookrightarrow \mathbb{C}\mathcal{R}$ is a $\dim_{L^\infty(X)}$-iso)}\\
&= b_n^{(2)}(\mathcal{R}),
\end{aligned}$$

as claimed. □

4.2.3 Applications to orbit equivalence

Corollary 4.2.6 (OE/ME-invariants [42, Corollaire 5.6]). *Let Γ and Λ be infinite groups of finite type and let $n \in \mathbb{N}$.*

1. *If $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent (essentially) free standard actions, then $b_n^{(2)}(\Gamma) = b_m^{(2)}(\Lambda)$.*
2. *If Γ and Λ are measure equivalent (with index c), then*

$$b_n^{(2)}(\Gamma) = c \cdot b_n^{(2)}(\Lambda).$$

   In particular, $b_n^{(2)}(\Gamma)$ and $b_n^{(2)}(\Lambda)$ have the same sign.

Proof. This follows from Theorem 4.2.5 and 4.2.4 (Exercise 4.E.2). □

Corollary 4.2.7 (non-orbit equivalence of free groups). *Let $n, m \in \mathbb{N}$ and let F_n and F_m be free groups of rank n and m, respectively. Then F_n and F_m admit orbit equivalent standard actions if and only if $n = m$.*

Proof. If F_n and F_m admit orbit equivalent standard actions, then $b_1^{(2)}(F_n) = b_1^{(2)}(F_m)$ (Corollary 4.2.6). Because of $b_1^{(2)}(F_n) = n - 1$ and $b_1^{(2)}(F_m) = m - 1$ (Example 2.2.6), we obtain $n = m$. □

4.2.4 Applications to L^2-Betti numbers of groups

Corollary 4.2.8 (L^2-Betti numbers of amenable groups). *Let Γ be an amenable group of finite type and let $n \in \mathbb{N}$. Then*

$$b_n^{(2)}(\Gamma) = \begin{cases} \frac{1}{|G|} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

In particular: If Γ admits a finite classifying space, then $\chi(\Gamma) = 0$.

Proof. This can be deduced from our previous computations, Theorem 4.1.10, and Corollary 4.2.6 (Exercise 4.E.4). (The original proof of Cheeger and Gromov used a Følner-type argument [22].) □

Corollary 4.2.9 (proportionality principle for L^2-Betti numbers [42, Corollaire 0.2, Théorème 6.3]). *Let Γ, Λ be lattices in a locally compact second countable topological group G (with a given Haar measure) and let $n \in \mathbb{N}$. Then*

$$\frac{b_n^{(2)}(\Gamma)}{\operatorname{vol}(\Gamma \setminus G)} = \frac{b_n^{(2)}(\Lambda)}{\operatorname{vol}(\Lambda \setminus G)}.$$

This common quotient is denoted by $b_n^{(2)}(G)$.

Proof. Lattices in the same group are measure equivalent (Example 4.1.6) and the index is the ratio of covolumes. We then apply Corollary 4.2.6.

(In the locally symmetric case, this kind of proportionality can also be obtained analytically via the heat kernel.) □

It should be noted that L^2-Betti numbers for locally compact second countable groups can also be defined directly [77] and that the values depend on the choice of a Haar measure. Usually, in applications, such L^2-Betti numbers appear together with the covolume of a lattice (and so the dependence on the Haar measure is irrelevant).

4.3 Cost of groups

Cost of measured equivalence relations is a measure theoretic version of "minimal size of a generating set". More precisely:

Definition 4.3.1 (graphing, cost [59])**.** Let $\mathcal{R}$ be a measured equivalence relation on (X, μ).

- A *graphing of* $\mathcal{R}$ is a family $\Phi = (\varphi_i)_{i \in I}$ of partial $\mathcal{R}$-automorphisms of (X, μ) such that
$$\langle \Phi \rangle = \mathcal{R},$$
where $\langle \Phi \rangle$ denotes the minimal (with respect to inclusion) equivalence relation on X that contains the graphs of all φ_i with $i \in I$.

- The *cost* of a graphing Φ of $\mathcal{R}$ is
$$\operatorname{cost} \Phi := \sum_{\varphi \in \Phi} \mu(\text{domain of } \varphi) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

- The *cost of* $\mathcal{R}$ is
$$\operatorname{cost} \mathcal{R} := \inf\bigl\{\operatorname{cost} \Phi \bigm| \Phi \text{ is a graphing of } \mathcal{R}\bigr\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Definition 4.3.2 (cost of a group [41])**.** Let Γ be a countable group. Then the *cost of* Γ is defined as
$$\operatorname{cost} \Gamma := \inf\bigl\{\operatorname{cost} \mathcal{R}_{\Gamma \curvearrowright X} \bigm| \Gamma \curvearrowright X \text{ is a standard action}\bigr\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Remark 4.3.3 (the (trivial) rank estimate)**.** If Γ is a group, then $\operatorname{cost}(\Gamma) \leq d(\Gamma)$: In each orbit equivalence relation, group elements yield (total) automorphisms and the automorphisms associated with a generating set clearly also generate the orbit equivalence relation.

The residually finite view and the dynamical view are united through the profinite completion:

Theorem 4.3.4 (cost of the profinite completion [5]). *Let* Γ *be a finitely generated residually finite infinite group. Then*

$$\operatorname{rg}\Gamma = \operatorname{cost}(\Gamma \curvearrowright \widehat{\Gamma}) - 1.$$

Moreover, we have the dynamical version of the rank gradient estimate (Corollary 3.3.5):

Theorem 4.3.5 (cost estimate [42, Corollaire 3.23]). *If* Γ *is a countable infinite group (of finite type), then*

$$b_1^{(2)}(\Gamma) \leq \operatorname{cost}\Gamma - 1.$$

In particular, $\operatorname{cost}\Gamma \geq 1$.

By construction, cost of countable groups is an orbit equivalence invariant. However, in general, cost of groups is hard to compute and the dependence on the underlying dynamical system remains a mystery:

Outlook 4.3.6 (fixed price problem). The fixed price problem asks for the (in)dependence of cost on the action:

> Let Γ be a countable group and let $\Gamma \curvearrowright X$, $\Gamma \curvearrowright Y$ be essentially free standard actions. Do we then have
>
> $$\operatorname{cost}(\Gamma \curvearrowright X) = \operatorname{cost}(\Gamma \curvearrowright Y) \quad (?!)$$

This problem is wide open. In fact, for finitely presented residually finite groups Γ, in all known examples, one has $b_1^{(2)}(\Gamma) = \operatorname{cost}\mathcal{R}_{\Gamma \curvearrowright X} - 1 = \operatorname{rg}(\Gamma, \Gamma_*)$ for all essentially free ergodic standard actions $\Gamma \curvearrowright X$ and all residual chains Γ_* of Γ; this includes infinite amenable groups, free groups, surface groups ... [41].

Abért and Nikolov proved that the following (bold) conjectures exclude each other [5]:

- The [stable] rank vs. Heegaard genus conjecture for orientable compact hyperbolic 3-manifolds (the Heegaard genus of M equals $d(\pi_1(M))$).
- The fixed price conjecture.

4.E Exercises

Exercise 4.E.1 (characterisations of residual finiteness)**.** Let Γ be a finitely generated group. Show that the following are equivalent:

1. The group Γ is residually finite (i.e., it admits a residual chain).
2. For each $g \in G \setminus \{e\}$, there exists a finite group F and a group homomorphism $\varphi \colon \Gamma \longrightarrow F$ with
$$\varphi(g) \neq e.$$
3. The diagonal homomorphism $\Gamma \longrightarrow \widehat{\Gamma} = \varprojlim_{\Lambda \in N(\Gamma)} \Gamma/\Lambda$ into the profinite completion of Γ is injective. Here, $N(\Gamma)$ denotes the set of all finite index normal subgroups of Γ.
4. The diagonal action of Γ on the profinite completion $\widehat{\Gamma}$ is free.

Exercise 4.E.2 ((stable) orbit equivalence via equivalence relations)**.** Reformulate the notion of (stable) orbit equivalence of standard actions in terms of the orbit relations (without using the group actions directly) and prove Corollary 4.2.6.

Exercise 4.E.3 (non-orbit equivalence of groups)**.** Let $m, n \in \mathbb{N}$ and let $r_1, \dots, r_m, s_1, \dots, s_n \in \mathbb{N}_{\geq 2}$. Prove the following: If $m \neq n$, then $\prod_{j=1}^{m} F_{r_j}$ and $\prod_{j=1}^{k} F_{s_j}$ are *not* orbit equivalent.
Hints. L^2-Betti numbers ...

Exercise 4.E.4 (L^2-Betti numbers of amenable groups)**.** Compute the L^2-Betti numbers of amenable groups of finite type, i.e., prove Corollary 4.2.8.

Exercise 4.E.5 (L^2-Betti numbers of topological groups)**.** Compute the L^2-Betti numbers (in the sense of Corollary 4.2.9) of the following topological groups:

1. $\mathbb{R}^{2019}$
2. $\mathrm{PSL}(2, \mathbb{R})$
3. $\mathbb{R}^{2019} \times \mathrm{PSL}(2, \mathbb{R})$
4. $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$
5. the three-dimensional real Heisenberg group
$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}(3, \mathbb{R}).$$

5

Invariant random subgroups

We will now consider an approximation theorem for covolume-normalised Betti numbers for uniform lattices in semi-simple Lie groups. We will first explain the statement of the theorem and two instructive examples. Finally, we will sketch how invariant random subgroups help to handle such homology gradients and briefly outline the structure of the proof of the theorem.

Overview of this chapter.

Running example. lattices in $\mathrm{SL}(n, \mathbb{R})$ and in $\mathrm{SO}(n, 1)$, respectively

5.1 Generalised approximation for lattices

In the classical approximation theorem (Theorem 3.1.3), we looked at the limit behaviour of normalised ordinary Betti numbers of the form

$$\frac{b_k(\Gamma_n)}{[\Gamma : \Gamma_n]} \longrightarrow \ ?$$

If $(\Gamma_n)_{n\in\mathbb{N}}$ is a residual chain in an ambient group Γ of finite type, then this sequence indeed converges and the limit is $b_k^{(2)}(\Gamma)$ (Theorem 3.1.3).

We will now consider a version of the approximation theorem, where the conditions on the sequence $(\Gamma_n)_{n\in\mathbb{N}}$ are substantially relaxed, provided that these groups lie as lattices in a joint ambient group G. In view of Corollary 4.2.9, one might expect that the limit in this case is $b_k^{(2)}(G)$:

Remark 5.1.1 (classical approximation for lattices)**.** Let G be a second countable locally compact topological group (with a given Haar measure), let $\Gamma \subset G$ be a lattice of finite type, and let $(\Gamma_n)_{n\in\mathbb{N}}$ be a residual chain in Γ (provided it exists). Then, for all $k \in \mathbb{N}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{b_k(\Gamma_n)}{\operatorname{vol}(\Gamma_n \setminus G)} &= \lim_{n\to\infty} \frac{b_k(\Gamma_n)}{[\Gamma : \Gamma_n]} \cdot \frac{[\Gamma : \Gamma_n]}{\operatorname{vol}(\Gamma_n \setminus G)} \\
&= \frac{b_k^{(2)}(\Gamma)}{\operatorname{vol}(\Gamma \setminus G)} \qquad \text{(approximation theorem; Theorem 3.1.3)} \\
&= b_k^{(2)}(G). \qquad \text{(by definition of } b_k^{(2)}(G)\text{; Corollary 4.2.9)}
\end{aligned}$$

5.1.1 Statement of the approximation theorem

We now state (a selection of the) approximation results of the "seven samurai" Abért, Bergeron, Biringer, Gelander, Nikolov, Raimbault, Samet [3, 4].

Setup 5.1.2. Let G be a connected centre-free semi-simple Lie group without compact factors (and a chosen Haar measure), let $K \subset G$ be a maximal compact subgroup, and let $X := G/K$ be the associated symmetric space.

Theorem 5.1.3 (BS-approximation for lattices [3, Corollary 1.4])**.** *In the situation of Setup 5.1.2, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a uniformly discrete sequence of uniform lattices in G such that $(\Gamma_n \setminus X)_{n\in\mathbb{N}}$ BS-converges to X, and let $k \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{b_k(\Gamma_n)}{\operatorname{vol}(\Gamma_n \setminus G)} = b_k^{(2)}(G).$$

Theorem 5.1.4 (a sufficient condition for BS-convergence [3, Theorem 1.5]). *In the situation of Setup 5.1.2, let G have property (T) and* $\operatorname{rk}_{\mathbb{R}} G \geq 2$. *Moreover, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of pairwise non-conjugate irreducible lattices in G. Then $(\Gamma_n \setminus X)_{n\in\mathbb{N}}$ BS-converges to X.*

Combining these two theorems gives the following approximation result for lattices:

Corollary 5.1.5 (an approximation theorem for uniformly discrete lattices [3, Corollary 1.6]). *In the situation of Setup 5.1.2, let G have property (T) and* $\operatorname{rk}_{\mathbb{R}} G \geq 2$. *Moreover, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a uniformly discrete sequence of pairwise non-conjugate irreducible lattices in G. Then*

$$\lim_{n\to\infty} \frac{b_k(\Gamma_n)}{\operatorname{vol}(\Gamma_n \setminus G)} = b_k^{(2)}(G).$$

5.1.2 Terminology

Remark 5.1.6 (BS-convergence). In the situation of Setup 5.1.2, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of lattices in G. Then the X-orbifolds $(\Gamma_n \setminus X)_{n\in\mathbb{N}}$ *BS-converge* (*Benjamini-Schramm-converge*) to X if for every $R \in \mathbb{R}_{>0}$, the probability that the R-ball around a random point in $\Gamma_n \setminus X$ is isometric to the R-ball in X tends to 1, i.e., if

$$\forall_{R\in\mathbb{R}_{>0}} \quad \lim_{n\to\infty} \frac{\operatorname{vol}(R\text{-thin part of } \Gamma_n \setminus X)}{\operatorname{vol}(\Gamma_n \setminus X)} = 0.$$

The *R-thin part* of a Riemannian manifold M is $\{x \in M \mid \operatorname{injrad}_M(x) < R\}$.

Remark 5.1.7 (uniform discreteness). In the situation of Setup 5.1.2, a family $(\Gamma_i)_{i\in I}$ of lattices in G is *uniformly discrete*, if there exists an open neighbourhood $U \subset G$ of $e \in G$ with the property that

$$\forall_{i\in I} \quad \forall_{g\in G} \quad g \cdot \Gamma_i \cdot g^{-1} \cap U = \{e\}.$$

Example 5.1.8. In the situation of Setup 5.1.2, let $\Gamma \subset G$ be a uniform lattice and let $(\Gamma_n)_{n\in\mathbb{N}}$ be a family of finite index subgroups of Γ. Then (Exercise 5.E.1):

- The family $(\Gamma_n)_{n\in\mathbb{N}}$ is uniformly discrete in G.
- If $(\Gamma_n)_{n\in\mathbb{N}}$ is a residual chain of Γ, then $(\Gamma_n \setminus X)_{n\in\mathbb{N}}$ BS-converges to X.

Remark 5.1.9 (property (T)). A locally compact second countable topological group G has *(Kazhdan's) property (T)*, if the trivial representation of G is an isolated point in the unitary dual of G (with respect to the Fell topology); equivalently, such groups can also be characterised in terms of (almost) invariant vectors in unitary representations [8].

- If $n \in \mathbb{N}_{\geq 3}$, then $\mathrm{SL}(n, \mathbb{R})$ has property (T). More generally, every simple Lie group G with $\mathrm{rk}_{\mathbb{R}} G \geq 2$ has property (T).
- The group $\mathrm{SL}(2, \mathbb{R})$ does *not* have property (T).
- Property (T) is an antagonist of amenability: If a group both is amenable and has property (T), then it is compact. This fact is beautifully exploited in the proof of the Margulis normal subgroup theorem [69].

In order to use Theorem 5.1.3 in concrete situations, it is useful to know the values on the right hand side, which have been computed through locally symmetric spaces by analytic means.

Remark 5.1.10 (L^2-Betti numbers of locally symmetric spaces/semi-simple Lie groups)**.** In the situation of Setup 5.1.2, the L^2-Betti numbers of G can be computed as follows [66, Theorem 5.12] (this goes back to Borel [14]): Let $\operatorname{frk} G := \mathrm{rk}_{\mathbb{C}}(G) - \mathrm{rk}_{\mathbb{C}}(K)$ be the *fundamental rank of G*. Then:

- If $\operatorname{frk} G \neq 0$, then $b_k^{(2)}(G) = 0$ for all $k \in \mathbb{N}$.
- If $\operatorname{frk} G = 0$, then, for all $k \in \mathbb{N}$

$$b_k^{(2)}(G) = \begin{cases} 0 & \text{if } 2 \cdot k \neq \dim X \\ \neq 0 & \text{if } 2 \cdot k = \dim X. \end{cases}$$

In particular, all (closed) locally symmetric spaces satisfy the Singer conjecture (Outlook 2.2.9).

5.2 Two instructive examples

As in the original paper [3], we will now discuss two instructive examples related to Theorem 5.1.3.

5.2.1 Lattices in $\mathrm{SL}(n, \mathbb{R})$

Example 5.2.1 ([3, Example 1.7])**.** Let $n \in \mathbb{N}_{\geq 3}$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a cocompact lattice, and let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of distinct, finite index subgroups of Γ. Moreover, let $k \in \mathbb{N}$. Then

$$\lim_{n \to \infty} \frac{b_k(\Gamma_n)}{[\Gamma : \Gamma_n]} = 0.$$

To apply Corollary 5.1.5, we verify that the hypotheses are satisfied:

- Because $n \geq 3$, we know that $\operatorname{rk}_{\mathbb{R}} \mathrm{SL}(n, \mathbb{R}) \geq 2$ and that $\mathrm{SL}(n, \mathbb{R})$ has property (T) (Remark 5.1.9).

- As $\mathrm{SL}(n, \mathbb{R})$ is sufficiently irreducible, also its lattices are irreducible.

- The family $(\Gamma_n)_{n \in \mathbb{N}}$ is uniformly discrete (Example 5.1.8).

- Moreover, the sequence $(\Gamma_n)_{n \in \mathbb{N}}$ can, for each conjugacy class, contain at most finitely members (this can be seen from the covolumes).

Hence, we obtain (from Theorem 5.1.3 and the argument in Remark 5.1.1)

$$\lim_{n \to \infty} \frac{b_k(\Gamma_n)}{[\Gamma : \Gamma_n]} = b_k^{(2)}\bigl(\mathrm{SL}(n, \mathbb{R})\bigr) \cdot \operatorname{vol}\bigl(\Gamma \setminus \mathrm{SL}(n, \mathbb{R})\bigr).$$

Moreover, the fundamental rank of $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$ is non-zero. Hence, the right-hand side is 0 (Remark 5.1.10).

5.2.2 Why doesn't it work in rank 1 ?!

Example 5.2.2 (generalised approximation fails in rank 1 [3])**.** There exist closed connected hyperbolic manifolds M with $d := \dim M \geq 3$ and the following property (Exercise 5.E.2): There exists a surjective group homomorphism $\pi \colon \pi_1(M) \longrightarrow F_2$.

We now consider the cocompact lattice $\Gamma := \pi_1(M)$ in $\mathrm{SO}(n, 1)$ and the following sequence of subgroups: For each $n \in \mathbb{N}_{\geq 1}$, let $\Lambda_n \subset F_2$ be a subgroup of index n (these exist), and let

$$\Gamma_n := \pi^{-1}(\Lambda_n) \subset \Gamma := \pi_1(M).$$

Then $[\Gamma : \Gamma_n] = [F_2 : \Lambda_n] = n$.

In this siutation, we have (which shows that the conclusion of Theorem 5.1.3 does *not* hold in this rank 1-situation):

- For each $n \in \mathbb{N}_{\geq 1}$, we have

$$\begin{aligned} b_1(\Gamma_n) &= \dim_{\mathbb{C}} H_1(\Gamma_n; \mathbb{C}) = \operatorname{rk}_{\mathbb{Z}} H_1(\Gamma_n; \mathbb{Z}) = \operatorname{rk}_{\mathbb{Z}} (\Gamma_n)_{\mathsf{Ab}} \\ &\geq \operatorname{rk}_{\mathbb{Z}} (\Lambda_n)_{\mathsf{Ab}} = \text{rank of the free group } \Lambda_n \\ &= n \cdot (2 - 1) + 1 \qquad \text{(Nielsen-Schreier)} \end{aligned}$$

  and so $\liminf_{n \to \infty} \frac{b_1(\Gamma_n)}{[\Gamma : \Gamma_n]} = \liminf_{n \to \infty} \frac{n+1}{n} \geq 1$.

- In contrast, $b_1^{(2)}(\mathrm{SO}(n, 1)) = 0$ (Remark 5.1.10 or Outlook 2.2.8).

5.3 Convergence via invariant random subgroups

To prove the sufficient condition for BS-convergence (Theorem 5.1.4) as well as the convergence of Betti numbers in the presence of BS-convergence (Theorem 5.1.3), one can make use of invariant random subgroups.

5.3.1 Invariant random subgroups

We first quickly recall basic terminology:

Definition 5.3.1 (invariant random subgroup)**.** Let G be a locally compact second countable topological group.

- We write $\mathrm{Sub}(G)$ for the set of all closed subgroups of G, endowed with the subspace topology of the space of closed subsets of G (with the Chabauty topology [21]).
- An *invariant random subgroup* is a Borel probability measure on $\mathrm{Sub}(G)$ that is invariant under the conjugation action of G on $\mathrm{Sub}(G)$.
- Let $\mathrm{IRS}(G)$ be the space of all invariant random subgroups on G (with the weak topology).

Example 5.3.2 (invariant random subgroups)**.** Let G be a locally compact second countable group.

- If N is a normal subgroup of G, then the Dirac measure δ_N is an invariant random subgroup on G.
- If $\Gamma \subset G$ is a lattice, then the push-forward of the normalised Haar measure on G/Γ via

$$\begin{aligned} G/\Gamma &\longrightarrow \mathrm{Sub}(G) \\ g \cdot \Gamma &\longmapsto g \cdot \Gamma \cdot g^{-1} \end{aligned}$$

  is an invariant random subgroup, which we will denote by μ_Γ. In particular, each sequence of lattices gives rise to a sequence of corresponding invariant random subgroups.

Moreover, many exotic examples of invariant random subgroups exist [4, 15, 2]. However, we will focus on the benign situation.

5.3.2 Benjamini-Schramm convergence

In our setting, Benjamini-Schramm convergence can be translated into weak convergence of the corresponding invariant random subgroups:

Theorem 5.3.3 (BS-convergence and IRS-convergence [3, Corollary 3.8]). *In the situation of Setup 5.1.2, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of lattices in G. Then the following are equivalent:*

1. *The sequence $(\Gamma_n \setminus X)_{n\in\mathbb{N}}$ BS-converges to X.*
2. *The sequence $(\mu_{\Gamma_n})_{n\in\mathbb{N}}$ of invariant random subgroups on G weakly converges to δ_1.*

Theorem 5.1.4 is then a consequence of the following fact:

Theorem 5.3.4 (the IRS accumulation point [3, Theorem 4.4]). *Let G be a connected centre-free semi-simple Lie group with property (T) and $\operatorname{rk}_{\mathbb{R}} G \geq 2$. Then the set*

$$\{\mu_\Gamma \mid \Gamma \textit{ is an irreducible lattice in } G\}$$

has exactly one accumulation point in $\operatorname{IRS}(G)$, namely δ_1.

The proof of Theorem 5.3.4 relies on the following input:

- If G is a connected centre-free semi-simple Lie group such that each simple factor has real rank at least 2, then every ergodic invariant random subgroup is either
  - δ_N for a normal subgroup N of G
  - μ_Λ for a lattice Λ in a normal subgroup M of G
  - or a product of two measures of the previous two types, where N and M commmute.

  The proof [3, Theorem 4.2] is based on the Nevö-Stück-Zimmer theorem.

- If G is a connected centre-free semi-simple Lie group with $\operatorname{rk}_{\mathbb{R}} G \geq 2$ and property (T), then every non-atomic irreducible invariant random subgroup of G is of the form μ_Γ for some irreducible lattice Γ in G.

  The proof [3, Theorem 4.1] is also based on the Nevö-Stück-Zimmer theorem.

- The Glasner-Weiss theorem on ergodicity of weak limits of ergodic G-invariant Borel probability measures (this uses property (T)).

- Wang's finiteness theorem

5.3.3 Reduction to Plancherel measures

As in the proof of the classical approximation theorem, the proof of Theorem 5.1.3 is based on a convergence of measures, the corresponding Plancherel measures:

In the situation of Theorem 5.1.3, we introduce the following notation:

- Let ν be the Plancherel measure on the unitary dual $\widehat{G}$ of G.
- For $n \in \mathbb{N}$, let ν_n be the relative Plancherel measure on $\widehat{G}$ of Γ_n in G, i.e.,
$$\nu_n := \frac{1}{\operatorname{vol}(\Gamma_n \setminus G)} \cdot \sum_{\pi \in \widehat{G}} m(\pi, \Gamma_n) \cdot \delta_\pi,$$
where $m(\pi, \Gamma_n)$ denotes the multiplicity of π in the right regular representation $L^2(\Gamma_n \setminus G)$ of G.

Similarly to the case of the classical approximation theorem (or the approximation theorem of DeGeorge-Wallach [25]), one can now express L^2-Betti numbers and covolume-normalised Betti numbers in terms of such measures, using eigenspace representations of the geometric Laplace operator on (locally) symmetric spaces [3, Section 6.23] (this geometric approach leads to the same L^2-Betti numbers for the ambient group G [77, 78]).

Therefore, it suffices to prove a corresponding convergence result for the Plancherel measures on singletons.

5.3.4 Convergence of Plancherel measures

Theorem 5.3.5 (convergence of Plancherel measures [3, Theorem 6.7])**.** *In the situation of Setup 5.1.2, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a uniformly discrete seuqence of lattices in G such that $(\Gamma_n \backslash X)_{n\in\mathbb{N}}$ BS-converges to X. Then, for every relatively compact ν-regular open subset $A \subset \widehat{G}$, we have*
$$\lim_{n\to\infty} \nu_n(A) = \nu(A).$$
The same conclusion also holds for relatively compact ν-regular open subsets of the tempered unitary dual of G.

The proof [3, Theorem 6.7] has the following ingredients:

- Because the lattices BS-converge, the corresponding invariant random subgroups weakly converge to δ_1 (Theorem 5.3.3).

- In combination with the Harish-Chandra formula and uniform discreteness, this weak convergence can be used to show that certain integration functionals (over the invariant random subgroups) converge to each other.
- One then applies Sauvageot's density principle to conclude that the Plancherel measures converge as stated.

Corollary 5.3.6 (pointwise convergence of Plancherel measures [3, Corollary 6.9])**.** *In the situation of Setup 5.1.2, let $(\Gamma_n)_{n\in\mathbb{N}}$ be a uniformly discrete sequence of lattices in G such that $(\Gamma_n\backslash X)_{n\in\mathbb{N}}$ BS-converges to X. Then, for every $\pi \in \widehat{G}$, we have*

$$\lim_{n\to\infty} \nu_n\bigl(\{\pi\}\bigr) = \nu\bigl(\{\pi\}\bigr).$$

Proof. We distinguish two cases:

- If $\nu(\{\pi\}) = 0$, we take a sequence $(A_k)_{k\in\mathbb{N}}$ of relatively compact ν-regular open subsets of $\widehat{G}$ with $\{\pi\} = \bigcap_{k\in\mathbb{N}} A_k$. Then the convergence in Theorem 5.3.5 shows that

$$0 \le \lim_{n\to\infty} \nu_n\bigl(\{\pi\}\bigr) \le \limsup_{n\to\infty} \nu_n\biggl(\bigcap_{k\in\mathbb{N}} A_k\biggr) \le \nu\biggl(\bigcap_{k\in\mathbb{N}} A_k\biggr) = \nu\bigl(\{\pi\}\bigr) = 0.$$

- If $\nu(\{\pi\}) \neq 0$, then π is a discrete series representation of G and thus an isolated point in the tempered unitary dual of G. Therefore, we can apply Theorem 5.3.5 directly to $\{\pi\}$. □

Alternatively, one can also give a more direct proof of this corollary [3, Section 6.10] This finishes the proof outline of Theorem 5.1.3.

Recently, the BS-convergence results for Betti number gradients have been extended and generalised in many ways [4, 39, 29, 1, 12, 15, 71, 45, 2, 84, 56].

5.E Exercises

Exercise 5.E.1 (BS-convergence, uniform discreteness). In the situation of Setup 5.1.2, let $\Gamma \subset G$ be a uniform lattice and let $(\Gamma_n)_{n\in\mathbb{N}}$ be a family of finite index subgroups of Γ. Show the following:

1. There exists an open neighbourhood U of e in G with
$$\forall_{g\in G} \quad g\cdot\Gamma\cdot g^{-1}\cap U = \{e\}.$$
2. The family $(\Gamma_n)_{n\in\mathbb{N}}$ is uniformly discrete in G.
3. If $(\Gamma_n)_{n\in\mathbb{N}}$ is a residual chain of Γ, then $(\Gamma_n\setminus X)_{n\in\mathbb{N}}$ BS-converges to X.
Hints. Given $R\in\mathbb{R}_{>0}$, what happens with the R-thin part of $\Gamma_n\setminus X$ for large n ?

Exercise 5.E.2 (fundamental groups of hyperbolic manifolds). Why are there closed hyperbolic manifolds of dimension at least 3, whose fundamental group surjects onto the free group of rank 2 ?
Hints. You will need some (non-trivial) tool to do this.

Exercise 5.E.3 (lattices in rank 1). In Example 5.2.2 (generalised approximation fails in rank 1), (why) is it important to work in dimension at least 3 ?

Exercise 5.E.4 (convergence of measures?). Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on $[0,1]$ and let μ be a Borel probability measure on $[0,1]$ with the property that
$$\lim_{n\to\infty}\mu_n(U) = \mu(U)$$
holds for all open subsets $U\subset[0,1]$. Moreover, let $(U_k)_{k\in\mathbb{N}}$ be a nested decreasing sequence of open subsets of $[0,1]$ and let $V := \bigcap_{k\in\mathbb{N}} U_k$.

1. Do we always have $\limsup_{n\to\infty}\mu_n(V)\le\mu(V)$?
2. Do we always have $\liminf_{n\to\infty}\mu_n(V)\ge\mu(V)$?

A

Appendix The lost lecture: Simplicial volume

Simplicial volume is a numerical topological invariant of manifolds, measuring the "size" of manifolds in terms of the "number" of singular simplices. Simplicial volume is also related to Riemannian volume and geometric structures on manifolds and therefore is a suitable invariant for certain rigidity phenomena.

We quickly survey basic properties of simplicial volume and its similarities/differences with L^2-Betti numbers and related invariants. In particular, we will discuss the residually finite and the dynamical view on simplicial volume.

Overview of this chapter.

A.1 Simplicial volume

Simplicial volume was first introduced by Gromov [70] in his proof of Mostow rigidity. Simplicial volume is the ℓ^1-semi-norm of the $\mathbb{R}$-fundamental class, defined in terms of *singular* chains:

Definition A.1.1 (simplicial volume)**.** Let M be an oriented closed connected n-manifold. The *simplicial volume of* M is defined by

$$\|M\| := \inf\biggl\{\sum_{j=1}^k |a_j| \biggm| \sum_{j=1}^k a_j\cdot\sigma_j \in C_n^{\mathrm{sing}}(M;\mathbb{R}) \text{ is an } \mathbb{R}\text{-fundamental cycle of } M\biggr\} \in \mathbb{R}_{\geq 0}.$$

The main tools to compute simplicial volume are concrete geometric constructions on the singular chain complex and bounded cohomology. By now, simplicial volume has been computed in a rich class of examples [47, 89, 58, 18, 20, 19, 60, 50]. For simplicity, we will only list a few properties that fit in the context of L^2-Betti numbers and related invariants:

- **Multiplicativity.** If M is an oriented closed connected manifold and $N \longrightarrow M$ is a d-sheeted covering, then [47] (Exercise A.E.1)
$$\|N\| = d \cdot \|M\|.$$
- **Hyperbolicity.** If M is an oriented closed connected hyperbolic n-manifold, then [47, 87, 9]
$$\|M\| = \frac{\operatorname{vol} M}{\text{volume of the ideal regular } n\text{-simplex in } \mathbb{H}^n}.$$
- **Amenability.** If M is an oriented closed connected manifold of non-zero dimension with amenable fundamental group, then [47, 51]
$$\|M\| = 0.$$
- **Proportionality principle.** If M and N are oriented closed connected Riemannian manifolds with isometric Riemannian universal covering, then [47, 87, 85]
$$\frac{\|M\|}{\operatorname{vol}(M)} = \frac{\|N\|}{\operatorname{vol}(N)}.$$

In view of these striking similarities with L^2-Betti numbers, Gromov asked the following question [48, p. 232].

Questions A.1.2. Let M be an oriented closed connected aspherical manifold with $\|M\| = 0$.

- Does this imply that $\chi(M) = 0$?!
- Does this imply that $b_n^{(2)}(M) = 0$ for all $n \in \mathbb{N}$?!

In general, these questions are wide open. Betti number estimates become available when one passes to a more integral setting (see below).

A.2 The residually finite view

Taking integral instead of real coefficients leads to integral simplicial volume. In the residually finite view, we then stabilise along finite-sheeted coverings:

Definition A.2.1 (stable integral simplicial volume)**.** Let M be an oriented closed connected n-manifold.

- The *integral simplicial volume of M* is defined by

$$\|M\|_{\mathbb{Z}} := \inf\left\{ \sum_{j=1}^{k} |a_j| \;\middle|\; \sum_{j=1}^{k} a_j \cdot \sigma_j \in C_n^{\text{sing}}(M;\mathbb{Z}) \text{ is a } \mathbb{Z}\text{-fundamental cycle of } M \right\} \in \mathbb{N}.$$

- The *stable integral simplicial volume of M* is defined by

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf_{(p\colon N \to M) \in F(M)} \frac{\|N\|_{\mathbb{Z}}}{|\deg(p)|} \in \mathbb{R}_{\geq 0},$$

  where $F(M)$ denotes the class of all finite-sheeted coverings of M.

Stable integral simplicial volume has the following properties:

- **Multiplicativity.** If M is an oriented closed connected manifold and $N \longrightarrow M$ is a d-sheeted covering, then a straightforward transfer argument (Exercise A.E.1) shows that

$$\|N\|_{\mathbb{Z}}^{\infty} = d \cdot \|M\|_{\mathbb{Z}}^{\infty}.$$

- **Hyperbolicity.** Let M be an oriented closed connected hyperbolic n-manifold.
  - If $n = 2$, then $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|$ (by direct computation [47]).
  - If $n = 3$, then also $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|$ (by indirect computation [37]; indeed, the only known proof requires passage through the dynamical view and non-trivial results from ergodic theory!).

  – If $n \geq 4$, then there the ration $\|M\|_{\mathbb{Z}}^{\infty} / \|M\|$ is uniformly bounded away from 1 [35] (!). In particular, approximation fails in general for simplicial volume.

- **Amenability.** Let M be an oriented closed connected aspherical manifold of non-zero dimension with amenable residually finite fundamental group. Then [37]
$$\|M\|_{\mathbb{Z}}^{\infty} = 0.$$
- **Betti number estimate.** If M is an oriented closed connected manifold, then the explicit description of Poincaré duality on the singular chain complex shows that, for all $n \in \mathbb{N}$, we have [49]
$$b_n(M) \leq \|M\|_{\mathbb{Z}}.$$
Stabilisation therefore leads to an L^2-Betti number bound. In addition, als log-torsion-homology [gradient] bounds in terms of [stable] integral simplicial volume are known [82].
- **Rank gradient estimate.** If M is an oriented closed connected manifold of with residually finite (infinite) fundamental group, then [61]
$$\|M\|_{\mathbb{Z}}^{\infty} \leq \operatorname{rg}\bigl(\pi_1(M)\bigr).$$

As in the case of the rank gradient, it is not clear how/whether the choice of specific residual chains affects the limit of the normalised integral simplicial volumes.

A.3 The dynamical view

In the dynamical view, we use twisted coefficients, based on standard actions of the fundamental group.

Definition A.3.1 (integral foliated simplicial volume [83])**.** Let M be an oriented closed connected n-manifold with fundamental group Γ.

- Let $\alpha = (\Gamma \curvearrowright (X, \mu))$ be a standard action. Then the *α-parametrised simplicial volume of M* is defined by
$$|M|^{\alpha} := \inf\biggl\{\sum_{j=1}^{n} \int_X |f_j| \, d\mu \biggm| \sum_{j=1}^{n} f_j \otimes \sigma_j \in C_n(M; L^\infty(X, \mu; \mathbb{Z})) \text{ is a fund. cycle of } M\biggr\} \in \mathbb{R}_{\geq 0}.$$
- The *integral foliated simplicial volume of M* is then defined by
$$|M| := \inf_{\alpha \in S(\Gamma)} |M|^{\alpha},$$
where $S(\Gamma)$ denotes the class of all standard actions of Γ.

Remark A.3.2 (comparing/combining the different views)**.** These simplicial volumes are related as follows:

- For all oriented closed connected manifolds M, we have [63]
$$\|M\| \leq |M| \leq \|M\|_{\mathbb{Z}}^{\infty}$$
and (if the fundamental group Γ of M is residually finite) [37]
$$|M|^{\Gamma \curvearrowright \widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty}.$$
- This relation between the stable integral simplicial volume and the parametrised simplicial volume has recently proved useful to compute stable integral simplicial volume in cases where direct approaches failed (such as the case of hyperbolic 3-manifolds [37], manifolds with S^1-actions [32], or higher-dimensional graph-manifolds [33]).

Integral foliated simplicial volume has the following properties:

- **Multiplicativity.** If M is an oriented closed connected manifold and $N \longrightarrow M$ is a d-sheeted covering, then a transfer argument [63] shows that
$$|N| = d \cdot |M| .$$
- **Hyperbolicity.** Let M be an oriented closed connected hyperbolic n-manifold.
  - If $n = 2$, then $|M| = \|M\|$ (by direct computation [63]).
  - If $n = 3$, then $|M| = \|M\|$ (using an argument involving measure equivalence [63]).
  - If $n \geq 4$, then $|M| / \|M\|$ is uniformly bounded away from 1 [37].
- **Amenability.** If M is an oriented closed connected aspherical manifold of non-zero dimension with amenable fundamental group, then [37]
$$|M| = 0.$$
- **L^2-Betti number estimate.** If M is an oriented closed connected manifold, then a parametrised Poincaré duality argument shows that, for all $n \in \mathbb{N}$, we have [83]
$$b_n^{(2)}(M) \leq |M| .$$
- **Cost estimate.** If M is an oriented closed connected manifold, then [62]
$$\operatorname{cost}\bigl(\pi_1(M)\bigr) - 1 \leq |M| .$$

As in the case of cost, it is not clear how/whether the choice of specific essentially free standard actions affects the corresponding parametrised simplicial volume [37].

Furthermore, the various simplicial volumes and L^2-Betti numbers also show similar behaviour with respect to:

- S^1-actions [66, 47, 89, 32]
- graph manifolds [66, 33]
- minimal volume estimates [81, 16]
- (certain) mapping tori [66, 20]
- ...

While simplicial volume and integral foliated simplicial volume are different for general aspherical manifolds (e.g., for hyperbolic manifolds in high dimensions), they might still have the same vanishing behaviour. In the context of Question A.1.2 it is therefore natural ask the following:

Questions A.3.3. Let M be an oriented closed connected aspherical manifold with $\|M\| = 0$. Does this imply that $\mathbf{|}M\mathbf{|} = 0$?

A.E Exercises

Exercise A.E.1 (multiplicativity of simplicial volumes). Let M be an oriented closed connected manifold and let $N \longrightarrow M$ be a d-sheeted (finite) covering of M.

1. Show that $\|M\| \leq 1/d \cdot \|N\|$ (using push-forwards of fundamental cycles of N).
2. Show that $\|N\| \leq d \cdot \|M\|$ and $\|N\|_{\mathbb{Z}} \leq d \cdot \|M\|_{\mathbb{Z}}$ (using the transfer of fundamental cycles of M).
3. Conclude that $\|N\|_{\mathbb{Z}}^{\infty} = d \cdot \|M\|_{\mathbb{Z}}^{\infty}$.

Exercise A.E.2 (simplicial volume of spheres and tori). Use self-maps of non-trivial degree to show that spheres and tori in non-zero dimension have simplicial volume equal to 0. What about stable integral simplicial volume?

Exercise A.E.3 (simplicial volumes of surfaces). Let $g \in \mathbb{N}_{\geq 2}$ and let Σ_g be "the" oriented closed connected surface of genus g.

1. Use explicit (singular) triangulations of surfaces and covering theory to prove that
$$\|\Sigma_g\| \leq \|\Sigma_g\|_{\mathbb{Z}}^{\infty} \leq 4 \cdot (g-1) = 2\big|\chi(\Sigma_g)\big|.$$
2. Use integration of (smooth, geodesically straightened) singular simplices and hyperbolic geometry to prove that
$$2 \cdot \big|\chi(\Sigma_g)\big| \leq \|M\|.$$

Exercise A.E.4 (vanishing stable integral simplicial volume?!).

1. Does there exist an oriented closed connected manifold M satisfying $\pi_1(M) \cong F_2$ and $\|M\|_{\mathbb{Z}}^{\infty} = 0$?!
2. Does there exist an oriented closed connected manifold M satisfying $\pi_1(M) \cong F_2 \times F_2$ and $\|M\|_{\mathbb{Z}}^{\infty} = 0$?!

Hints. (L^2-)Betti numbers might help ...

Exercise A.E.5 (simplicial volume, L^2-Betti numbers, and the Singer conjecture). Let M be an oriented closed connected aspherical manifold. How are Question A.1.2 and the Singer conjecture (Outlook 2.2.9) related?

Bibliography

[1] M. Abért, I. Biringer. Unimodular measures on the space of all Riemannian manifolds, preprint, 2016. arXiv:1606.03360 [math.GT] Cited on page: 55

[2] M. Abért, N. Bergeron, I. Biringer, T. Gelander. Convergence of normalized Betti numbers in nonpositive curvature, preprint, 2018. arXiv:arXiv:1811.02520 [math.GT] Cited on page: 52, 55

[3] M. Abért, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, I. Samet. On the growth of L^2-invariants for sequences of lattices in Lie groups, *Ann. of Math. (2)*, 185(3), pp. 711–790, 2017. Cited on page: 48, 49, 50, 51, 53, 54, 55

[4] M. Abért, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, I. Samet. On the growth of L^2-invariants of locally symmetric spaces, II: exotic invariant random subgroups in rank one, preprint, 2016. arXiv:1612.09510v1 [math.GT] Cited on page: 48, 52, 55

[5] M. Abért, N. Nikolov. Rank gradient, cost of groups and the rank versus Heegard genus problem, *J. Eur. Math. Soc.*, 14, 16571677, 2012. Cited on page: 44

[6] M.F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras, *Astérisque*, 32–33, pp. 43–72, 1976. Cited on page: 22

[7] T. Austin. Rational group ring elements with kernels having irrational dimension, *Proc. Lond. Math. Soc.*, 107(6), pp. 1424–1448, 2013. Cited on page: 13

[8] B. Bekka, P. de la Harpe, A. Valette. *Kazhdans Property (T)*, New Mathematical Monographs, 11, Cambridge University Press, 2008. Cited on page: 49

[9] R. Benedetti, C. Petronio. *Lectures on Hyperbolic Geometry*, Universitext, Springer, 1992. Cited on page: 58

[10] M. Bestvina, N. Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3), pp. 445–470, 1997. Cited on page: 17

[11] P. Billingsley. *Convergence of Probability Measures*, second edition, Wiley-Interscience, 1999. Cited on page: 29

[12] I. Biringer, J. Raimbault, Ends of unimodular random manifolds, *Proc. Amer. Math. Soc.*, 145(9), pp. 4021–4029, 2017. Cited on page: 55

[13] M.Sh. Birman, M.Z. Solomjak. *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Mathematics and its Applications, 5, Springer, 1987. Cited on page: 27

[14] A. Borel. The L^2-cohomology of negatively curved Riemannian symmetric spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10, pp. 95–105, 1985. Cited on page: 50

[15] L. Bowen. Cheeger constants and L^2-Betti numbers, *Duke Math. J.*, 164(2), pp. 569–615, 2015. Cited on page: 52, 55

[16] S. Braun. *Simplicial Volume and Macroscopic Scalar Curvature*, PhD thesis, KIT, 2018.
https://publikationen.bibliothek.kit.edu/1000086838 Cited on page: 62

[17] K.S. Brown. *Cohomology of Groups*, Graduate Texts in Mathematics, 82, Springer, 1982. Cited on page: 16, 17

[18] M. Bucher-Karlsson. The simplicial volume of closed manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$, *J. Topol.*, 1(3), pp. 584–602, 2008. Cited on page: 58

[19] M. Bucher, C. Connell, J. Lafont. Vanishing simplicial volume for certain affine manifolds, *Proc. Amer. Math. Soc.*, 146, pp. 1287–1294, 2018. Cited on page: 58

[20] M. Bucher, C. Neofytidis. The simplicial volume of mapping tori of 3-manifolds, preprint, 2018. arXiv:1812.10726 [math.GT] Cited on page: 58, 62

[21] C. Chabauty. Limite densembles et géométrie des nombres, *Bull. Soc. Math. France*, 78, pp. 143–151, 1950. Cited on page: 52

[22] J. Cheeger, M. Gromov. L^2-cohomology and group cohomology, *Topology*, 25(2), pp. 189–215, 1986. Cited on page: 22, 42

[23] J.M. Cohen. Zero divisors in group rings, *Comm. Algebra*, 2, pp. 1–14, 1974. Cited on page: 7

[24] A. Connes, D. Shlyakhtenko. L^2-homology for von Neumann algebras, *J. Reine Angew. Math.*, 586, pp. 125–168, 2005. Cited on page: 23

[25] D.L. de George, N.R. Wallach. Limit formulas for multiplicities in $L^2(\Gamma \setminus G)$, *Ann. of Math. (2)*, 107(1), pp. 133–150, 1978. Cited on page: 54

[26] T. Delzant. Sur lanneau dun groupe hyperbolique, *C. R. Acad. Sci. Paris Ser. I Math.*, 324(4), pp. 381–384, 1997. Cited on page: 7

[27] J. Dodziuk. de Rham-Hodge theory for L^2-cohomology of infinite coverings, *Topology*, 16(2), pp. 157–165, 1977. Cited on page: 22

[28] J. Dodziuk. L^2 harmonic forms on rotationally symmetric Riemannian manifolds, *Proc. Amer. Math. Soc.*, 77(3), pp. 395–400, 1979. Cited on page: 22

[29] G. Elek. Betti numbers are testable, *Fete of Combinatorics and Computer Science*, pp. 139–149, Springer, 2010. Cited on page: 55

[30] M. Ershov, W. Lück. The first L^2-Betti number and approximation in arbitrary characteristic, *Documenta Math.*, 19, pp. 313–331, 2014. Cited on page: 31

[31] M. Farber. von Neumann categories and extended L^2-cohomology, *K-Theory*, 15(4), pp. 347–405, 1998. Cited on page: 22

[32] D. Fauser. Integral foliated simplicial volume and S^1-actions, preprint, 2017. arXiv:1704.08538 [math.GT], Cited on page: 61, 62

[33] D. Fauser, S. Friedl, C. Löh. Integral approximation of simplicial volume of graph manifolds, to appear in *Bull. Lond. Math. Soc.*, 2019, DOI 10.1112/blms.12266 Cited on page: 61, 62

[34] P.A. Fillmore. *A user's guide to operator algebras*, Canadian Mathematical Society Series of Monographs and Advanced Texts, 14, Wiley, 1996. Cited on page: 9

[35] S. Francaviglia, R. Frigerio, B. Martelli. Stable complexity and simplicial volume of manifolds, *J. Topol.*, 5(4), pp. 977–1010, 2012. Cited on page: 60

[36] R. Frigerio. *Bounded Cohomology of Discrete Groups*, Mathematical Surveys and Monographs, 227, AMS, 2017. Cited on page:

[37] R. Frigerio, C. Löh, C. Pagliantini, R. Sauer. Integral foliated simplicial volume of aspherical manifolds, *Israel J. Math.*, 216(2), pp. 707–751, 2016. Cited on page: 59, 60, 61, 62

[38] J. Feldman, C.C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I, *Trans. Amer. Math. Soc.*, 234(2), pp. 289–324, 1977. Cited on page: 39

[39] T. Finis, E. Lapid, W. Müller. Limit multiplicities for principal congruence subgroups of GL(n) and SL(n), *J. Inst. Math. Jussieu*, 14(3), pp. 589–638, 2015. Cited on page: 55

[40] A. Furman. A survey of measured group theory. In *Geometry, Rigidity, and Group Actions* (B. Farb, D. Fisher, eds.), 296–347, The University of Chicago Press, 2011. Cited on page: 36, 37, 38

[41] D. Gaboriau. Coût des relations d'équivalence et des groupes, *Invent. Math.*, 139(1), 41–98, 2000. Cited on page: 43, 44

[42] D. Gaboriau. Invariants l^2 de relations d'équivalence et de groupes, *Publ. Math. Inst. Hautes Études Sci.*, 95, pp. 93–150, 2002. Cited on page: 22, 38, 39, 41, 42, 44

[43] D. Gaboriau. On orbit equivalence of measure preserving actions, *Rigidity in dynamics and geometry (Cambridge, 2000)*, pp. 167–186, Springer, 2002. Cited on page: 36

[44] D. Gaboriau. Orbit equivalence and measured group theory, *Proceedings of the International Congress of Mathematicians. Volume III*, pp. 1501–1527, Hindustan Book Agency, 2010. Cited on page: 36

[45] T. Gelander, A. Levit. Invariant random subgroups over non-archimedean local fields, *Math. Ann.*, 372(3/4), pp. 1503–1544, 2018. Cited on page: 55

[46] Ł. Grabowski. On Turing dynamical systems and the Atiyah problem, *Invent. Math.*, 198(1), pp. 27–69, 2014. Cited on page: 13

[47] M. Gromov. Volume and bounded cohomology. *Publ. Math. IHES*, 56, pp. 5–99, 1982. Cited on page: 58, 59, 62

[48] M. Gromov. Asymptotic invariants of infinite groups, Geometric group theory, Vol. 2 (Sussex 1991). *London Math. Soc. Lectures Notes Ser.*, 182, Cambridge Univ. Press, Cambridge, pp. 1–295, 1993. Cited on page: 37, 58

[49] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* With appendices by M. Katz, P. Pansu, and S. Semmes, translated by S. M. Bates. Progress in Mathematics, 152, Birkhäuser, 1999. Cited on page: 60

[50] N. Heuer, C. Löh. The spectrum of simplicial volume, preprint, 2019. arXiv:1904.04539 [math.GT] Cited on page: 58

[51] .V. Ivanov. Foundations of the theory of bounded cohomology, *J. Soviet Math.*, 37, pp. 1090–1114, 1987. Cited on page: 58

[52] H. Kammeyer. *Introduction to ℓ^2-invariants*, lecture notes, 2018. https://topology.math.kit.edu/downloads/introduction-to-l2-invariants.pdf Cited on page: 27

[53] A. Kechris. *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, 156. Springer, 1995. Cited on page: 36

[54] A.S. Kechris, B.D. Miller. *Topics in Orbit Equivalence*, Springer Lecture Notes in Mathematics, vol. 1852, 2004. Cited on page: 36, 38

[55] S. Kionke. The growth of Betti numbers and approximation theorems, Borel seminar, 2017. arXiv:1709.00769 [math.AT] Cited on page: 31

[56] S. Kionke, M. Schrödl-Baumann. Equivariant Benjamini-Schramm convergence of simplicial complexes and ℓ^2-multiplicities, preprint, 2019. arXiv:1905.05658 [math.AT] Cited on page: 55

[57] M. Lackenby. Expanders, rank and graphs of groups, *Israel J. Math.*, 146, pp. 357–370, 2005. Cited on page: 31

[58] J.-F. Lafont, B. Schmidt. Simplicial volume of closed locally symmetric spaces of non-compact type, *Acta Math.*, 197(1), pp. 129–143, 2006. Cited on page: 58

[59] G. Levitt. On the cost of generating an equivalence relation, *Ergodic Theory Dynam. Systems*, 15(6), pp. 1173–1181, 1995. Cited on page: 43

[60] C. Löh. Simplicial Volume, *Bull. Man. Atl.*, pp. 7–18, 2011 Cited on page: 58

[61] C. Löh. Rank gradient vs. stable integral simplicial volume, *Period. Math. Hung.*, 76, pp. 88–94, 2018. Cited on page: 60

[62] C. Löh. Cost vs. integral foliated simplicial volume, *Groups Geom. Dyn.*, to appear. arXiv: 1809.09660 [math.GT] Cited on page: 61

[63] C. Löh, C. Pagliantini. Integral foliated simplicial volume of hyperbolic 3-manifolds, *Groups Geom. Dyn.*, 10(3), pp. 825–865, 2016. Cited on page: 61

[64] J. Lott. Deficiencies of lattice subgroups of Lie groups, *Bull. London Math. Soc.*, 31(2), pp. 191–195, 1999. Cited on page: 24

[65] W. Lück. Approximating L^2-invariants by their finite-dimensional analogues, *Geom. Funct. Anal.*, 4(4), pp. 455–481, 1994. Cited on page: 26

[66] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 44, Springer, 2002. Cited on page: 7, 10, 13, 18, 20, 22, 41, 50, 62

[67] W. Lück. Approximating L^2-invariants by their classical counterparts, *EMS Surveys in Math. Sci.*, 3(2), pp. 259–344, 2016. Cited on page: 31, 32

[68] A. Malcev. On isomorphic matrix representations of infinite groups, *Rec. Math. [Mat. Sbornik] N.S.*, 8(50), 405–422, 1940. Cited on page: 26

[69] G.A. Margulis. *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 17, Springer, 1991. Cited on page: 50

[70] H.J. Munkholm. Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston). In *Topology Symposium, Siegen 1979*, Volume 788 of *Lecture Notes in Mathematics*, pp. 109–124. Springer, 1980. Cited on page: 58

[71] H. Namazi, P. Pankka, J. Souto. Distributional limits of Riemannian manifolds and graphs with sublinear genus growth, *Geom. Funct. Anal.*, 24(1), pp. 322–359, 2014. Cited on page: 55

[72] B. Nica. Linear groups – Malcev's theorem and Selberg's lemma, preprint, 2013. arXiv:1306.2385 [math.GR] Cited on page: 26

[73] D.S. Ornstein, B. Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma, *Bull. Amer. Math. Soc.*, 2(1), pp. 161–164, 1980. Cited on page: 38

[74] P. Papasoglu. Homogeneous trees are bi-Lipschitz equivalent, *Geom. Dedicata*, 54, 301–306, 1995. Cited on page: 24

[75] D.S. Passman. *Group rings, crossed products and Galois theory.* CBMS Regional Conference Series in Mathematics, 64. AMS, 1986. Cited on page: 7

[76] A.L.T. Paterson. *Amenability*, Mathematical Surveys and Monographs, 29, AMS, 1988. Cited on page: 38

[77] H.D. Petersen. L^2-Betti numbers of locally compact groups, *C. R. Math. Acad. Sci. Paris*, 351(9–10), pp. 339–342, 2013. Cited on page: 23, 43, 54

[78] H.D. Petersen, A. Valette. L^2-Betti numbers and Plancharel measures, *J. of Funct. Analysis*, 266(5), pp. 3156–3169, 2014. Cited on page: 54

[79] V. Runde. *Amenability*, Springer Lecture Notes in Mathematics, 1774, Springer, 2002. Cited on page: 38

[80] R. Sauer. L^2-Betti numbers of discrete measured groupoids, *Internat. J. Algebra Comput.*, 15(5–6), pp. 1169–1188, 2005. Cited on page: 22, 39, 41

[81] R. Sauer. Amenable covers, volume and L^2-Betti numbers of aspherical manifolds, *J. reine angew. Math.*, 636, 47–92, 2009. Cited on page: 62

[82] R. Sauer. Volume and homology growth of aspherical manifolds, *Geom. Topol.*, 20, 1035–1059, 2016. Cited on page: 60

[83] M. Schmidt. *L^2-Betti Numbers of $\mathcal{R}$-spaces and the Integral Foliated Simplicial Volume.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2005. http://nbn-resolving.de/urn:nbn:de:hbz:6-05699458563 Cited on page: 22, 36, 60, 61

[84] M. Schrödl-Baumann. ℓ^2-Betti numbers of random rooted simplicial complexes, *Manuscripta math.*, to appear, *10.1007/s00229-019-01131-y* Cited on page: 55

[85] C. Strohm (= C. Löh). *The Proportionality Principle of Simplicial Volume.* Diplomarbeit, WWU Münster, 2004. arXiv:math.AT/0504106 Cited on page: 58

[86] A. Thom. A note on normal generation and generation of groups, *Communications in Mathematics*, 23(1), pp. 1-11, 2015. Cited on page: 32

[87] W. P. Thurston. *The Geometry and Topology of 3-Manifolds*, mimeographed notes, 1979. Available online at http://www.msri.org/publications/books/gt3m. Cited on page: 58

[88] K. Whyte. Amenability, bi-Lipschitz equivalence, and the von Neumann conjecture, *Duke Math. J.*, 99(1), 93–112, 1999. Cited on page: 24

[89] K. Yano. Gromov invariant and S^1-actions, *J. Fac. Sci. U. Tokyo, Sec. 1A Math.*, 29(3), pp. 493–501, 1982. Cited on page: 58, 62

Symbols

Index

G

H

I

K

L

M

O

P

Q

R

S

T

U

V

W