1.E Exercises

Exercise 1.E.1 (the "trivial" Hilbert module)**.** For which countable groups Γ is $\mathbb{C}$ (with the trivial Γ-action) a Hilbert Γ-module? Which von Neumann dimension does it have?

Exercise 1.E.2 (Hilbert modules as modules over the von Neumann algebra)**.** Let Γ be a countable group and let V be a Hilbert Γ-module. Show that the left Γ-action on V extends to a left $N\Gamma$-action on V.
Hints. This fact is the reason why often Hilbert Γ-modules are called *Hilbert $N\Gamma$-modules*.

Exercise 1.E.3 (kernels and cokernels)**.** Let Γ be a countable group, let V and W be Hilbert Γ-modules, and let $\varphi\colon V \longrightarrow W$ be a morphism of Hilbert Γ-modules.

1. Show that $\ker \varphi$ (with the induced inner product and Γ-action) is a Hilbert Γ-module
2. Show that $W/\overline{\operatorname{im} \varphi}$ (with the induced inner product and Γ-action) is a Hilbert Γ-module.

   Hints. Orthogonal complement!

Exercise 1.E.4 (restriction formula for the von Neumann dimension)**.** Let Γ be a countable group, let V be a Hilbert Γ-module, and let $\Lambda \subset \Gamma$ be a finite index subgroup. Show that

$$\dim_{N\Lambda} \operatorname{Res}^{\Gamma}_{\Lambda} V = [\Gamma : \Lambda] \cdot \dim_{N\Gamma} V.$$

Exercise 1.E.5 (Atiyah $\Longrightarrow$ Kaplansky)**.** Let Γ be a countable torsion-free group that satisfies the Atiyah conjecture (Outlook 1.2.4). Show that $\mathbb{C}\Gamma$ is a domain.