2.E Exercises

Exercise 2.E.1 (products).

1. Let Γ and Λ be infinite groups of finite type. Show that $b_1^{(2)}(\Gamma \times \Lambda) = 0$.
2. Conclude: If Γ is a group of finite type with $b_1^{(2)}(\Gamma) \neq 0$, then Γ does *not* contain a finite index subgroup that is a product of two infinite groups.

Exercise 2.E.2 (Euler characteristic). Let Γ be a group that admits a finite classifying space. Show that

$$\chi(\Gamma) = \sum_{n \in \mathbb{N}} (-1)^n \cdot b_n^{(2)}(\Gamma).$$

Exercise 2.E.3 (QI?! [51, 63]). Let Γ be a group of finite type and let $r \in \mathbb{N}_{\geq 2}$.

1. Compute all L^2-Betti numbers of $F_r * \Gamma$ in terms of r and the L^2-Betti numbers of Γ.
2. Let $k \in \mathbb{N}_{\geq 2}$. Conclude that the quotient $b_1^{(2)}/b_k^{(2)}$ is *not* a quasi-isometry invariant.
3. Show that the sign of the Euler characteristic is *not* a quasi-isometry invariant.
4. Use these results to prove that there exist groups of finite type that are quasi-isometric but not commensurable.

Hints. If $s \in \mathbb{N}_{\geq 2}$, then it is known that F_r and F_s are bilipschitz equivalent and thus that $F_r * \Gamma$ and $F_s * \Gamma$ are quasi-isometric [51, 63].

Exercise 2.E.4 (deficiency [44]).

1. Let Γ be a group of finite type and let $\langle S \mid R \rangle$ be a finite presentation of Γ. Show that

   $$|S| - |R| \leq 1 - b_0^{(2)}(\Gamma) + b_1^{(2)}(\Gamma) - b_2^{(2)}(\Gamma).$$

   Taking the maximum of all these differences thus shows that the *deficiency* $\operatorname{def}(\Gamma)$ *of* Γ is bounded from above by the right hand side.
2. Let $\Gamma \subset \operatorname{Isom}^+(\mathbb{H}^4)$ be a torsion-free uniform lattice. Show that

   $$\operatorname{def}(\Gamma) \leq 1 - \chi(\Gamma) = 1 - \frac{3}{4 \cdot \pi^2} \cdot \operatorname{vol}(\Gamma \setminus \mathbb{H}^4).$$