3.E Exercises

Exercise 3.E.1 (surface groups, free groups)**.** Prove the approximation theorem for surface groups and free groups by direct computation of the right hand side.

Exercise 3.E.2 (Laplacian)**.** Let Γ be a countable group, let C_* be a chain complex of Hilbert Γ-modules (with boundary operators ∂_*), and let Δ_* be the *Laplacian* of C_*, i.e., for each $n \in \mathbb{N}$, we set

$$\Delta_n := \partial_{n+1} \circ \partial_{n+1}^* + \partial_n^* \circ \partial_n.$$

Show that there exists an isomorphism

$$\ker \Delta_n \longrightarrow \ker \partial_n \;/\; \overline{\operatorname{im} \partial_{n+1}}$$

of Hilbert Γ-modules.
Hints. Consider the orthogonal projection onto $\overline{\operatorname{im} \partial_{n+1}}^{\perp}$.

Exercise 3.E.3 (weak convergence)**.** Give an example of a sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $[0,1]$ (with the Borel σ-algebra) that weakly converges to a probability measure μ on $[0,1]$, but that satisfies

$$\lim_{n\to\infty} \mu_n\bigl(\{0\}\bigr) \neq \mu\bigl(\{0\}\bigr).$$

Exercise 3.E.4 (rank gradients of products)**.** Let Γ and Λ be finitely generated infinite residually finite groups. Compute $\operatorname{rg}(\Gamma \times \Lambda)$.

Exercise 3.E.5 (self-maps)**.** Let M be an oriented closed connected aspherical manifold with residually finite fundamental group Γ. Moreover, we suppose that M admits a self-map $f \colon M \longrightarrow M$ with $|\deg f| \geq 2$.

1. Give examples of this situation.
2. Show that $\operatorname{rg}(\Gamma) = 0$.
3. Show that $b_k^{(2)}(\Gamma) = 0$ for all $k \in \mathbb{N}$.
4. Challenge: Does the vanishing of L^2-Betti numbers of Γ also hold without any residual finiteness or Hopficity condition on Γ ?! (This is an open problem.)

Hints. Covering theory shows that $\operatorname{im} f$ has finite index in Γ. Moreover, it is useful to know that residually finite groups are *Hopfian*, i.e., every self-epimorphism is an automorphism.