4.E Exercises

Exercise 4.E.1 (characterisations of residual finiteness). Let Γ be a finitely generated group. Show that the following are equivalent:

1. The group Γ is residually finite (i.e., it admits a residual chain).
2. For each $g \in G \setminus \{e\}$, there exists a finite group F and a group homomorphism $\varphi\colon \Gamma \longrightarrow F$ with
$$\varphi(g) \neq e.$$
3. The diagonal homomorphism $\Gamma \longrightarrow \widehat{\Gamma} = \varprojlim_{\Lambda \in N(\Gamma)} \Gamma/\Lambda$ into the profinite completion of Γ is injective. Here, $N(\Gamma)$ denotes the set of all finite index normal subgroups of Γ.
4. The diagonal action of Γ on the profinite completion $\widehat{\Gamma}$ is free.

Exercise 4.E.2 ((stable) orbit equivalence via equivalence relations). Reformulate the notion of (stable) orbit equivalence of standard actions in terms of the orbit relations (without using the group actions directly) and prove Corollary 4.2.6.

Exercise 4.E.3 (non-orbit equivalence of groups). Let $m, n \in \mathbb{N}$ and let $r_1, \dots, r_m, s_1, \dots, s_n \in \mathbb{N}_{\geq 2}$. Prove the following: If $m \neq n$, then $\prod_{j=1}^m F_{r_j}$ and $\prod_{j=1}^k F_{s_j}$ are *not* orbit equivalent.
Hints. L^2-Betti numbers . . .

Exercise 4.E.4 (L^2-Betti numbers of amenable groups). Compute the L^2-Betti numbers of amenable groups of finite type, i.e., prove Corollary 4.2.8.

Exercise 4.E.5 (L^2-Betti numbers of topological groups). Compute the L^2-Betti numbers (in the sense of Corollary 4.2.9) of the following topological groups:

1. $\mathbb{R}^{2019}$
2. $\mathrm{PSL}(2, \mathbb{R})$
3. $\mathbb{R}^{2019} \times \mathrm{PSL}(2, \mathbb{R})$
4. $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$
5. the three-dimensional real Heisenberg group
$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}(3, \mathbb{R}).$$