5.E Exercises

Exercise 5.E.1 (BS-convergence, uniform discreteness). In the situation of Setup 5.1.2, let $\Gamma \subset G$ be a uniform lattice and let $(\Gamma_n)_{n\in\mathbb{N}}$ be a family of finite index subgroups of Γ. Show the following:

1. There exists an open neighbourhood U of e in G with
$$\forall_{g\in G} \quad g\cdot\Gamma\cdot g^{-1}\cap U = \{e\}.$$
2. The family $(\Gamma_n)_{n\in\mathbb{N}}$ is uniformly discrete in G.
3. If $(\Gamma_n)_{n\in\mathbb{N}}$ is a residual chain of Γ, then $(\Gamma_n\setminus X)_{n\in\mathbb{N}}$ BS-converges to X.
Hints. Given $R\in\mathbb{R}_{>0}$, what happens with the R-thin part of $\Gamma_n \setminus X$ for large n ?

Exercise 5.E.2 (fundamental groups of hyperbolic manifolds). Why are there closed hyperbolic manifolds of dimension at least 3, whose fundamental group surjects onto the free group of rank 2 ?

Exercise 5.E.3 (lattices in rank 1). In Example 5.2.2 (generalised approximation fails in rank 1), (why) is it important to work in dimension at least 3 ?

Exercise 5.E.4 (convergence of measures?). Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on $[0,1]$ and let μ be a Borel probability measure on $[0,1]$ with the property that
$$\lim_{n\to\infty}\mu_n(U) = \mu(U)$$
holds for all open subsets $U \subset [0,1]$. Moreover, let $(U_k)_{k\in\mathbb{N}}$ be a nested decreasing sequence of open subsets of $[0,1]$ and let $U := \bigcap_{k\in\mathbb{N}} U_k$.

1. Do we always have $\limsup_{n\to\infty}\mu_n(U)\leq\mu(U)$?
2. Do we always have $\liminf_{n\to\infty}\mu_n(U)\geq\mu(U)$?