Classification up to KK-equivalence for circle actions on C*-algebras and some other cases

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SFB Miniworkshop on C*-Algebras, C*-Bundles, and Group Actions

University of Münster
November 2013
Overview

We construct classification functors up to KK-equivalence for

- $\mathbb{T}$-C*-algebras,
- C*-algebras over a unique path space,
Overview

We construct classification functors up to KK-equivalence for

- $\mathbb{T}$-$C^*$-algebras,
- $C^*$-algebras over a unique path space,
- graph $C^*$-algebras over a finite space.
We construct classification functors up to KK-equivalence for

- $\mathbb{T}$-$\mathcal{C}^*$-algebras,
- $\mathcal{C}^*$-algebras over a unique path space,
- graph $\mathcal{C}^*$-algebras over a finite space

by adopting a method from algebraic topology due to Bousfield.
We construct classification functors up to KK-equivalence for
- $\mathbb{T}$-$\text{C}^*$-algebras,
- $\text{C}^*$-algebras over a unique path space,
- graph $\text{C}^*$-algebras over a finite space
by adopting a method from algebraic topology due to Bousfield.
Motivational case: \( KK^\mathbb{T} \)

- Objects: separable \( \mathbb{T}\)-\( C^* \)-algebras
- Morphisms: \( KK^\mathbb{T} \)-classes
- Composition: Kasparov product

Consider the functor \( A \mapsto K^\mathbb{T}_*(A) \cong K_*(A \rtimes \mathbb{T}) \cong KK^\mathbb{T}_*(\mathbb{C}, A) \).

Definition/Theorem (Meyer–Nest)

\( A \in B^\mathbb{T} \iff A \rtimes \mathbb{T} \in B^\mathbb{T} \).

Definition

A lifting of an object \( M \in A \) is an object \( \hat{M} \in B^\mathbb{T} \) together with an isomorphism \( K^\mathbb{T}_*(\hat{M}) \to M \).

Classification + range results \( \leftrightarrow \) uniqueness + existence of liftings

Classification for circle actions on \( C^* \)-algebras

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Motivational case: $\text{KK}^\mathbb{T}$

- Objects: separable $\mathbb{T}$-$\text{C}^*$-algebras
- Morphisms: $\text{KK}^\mathbb{T}$-classes
- Composition: Kasparov product

Consider the functor $A \mapsto K^\mathbb{T}_*(A) \cong K_*(A \rtimes \mathbb{T}) \cong \text{KK}^\mathbb{T}_*(\mathbb{C}, A)$.

We have $K^\mathbb{T}: \text{KK}^\mathbb{T} \to \mathcal{A} := \text{Mod}(\mathbb{Z}[x, x^{-1}])^\mathbb{Z}/2_c$ via the dual action.
Motivational case: $\text{KK}^\mathbb{T}$

- Objects: separable $\mathbb{T}$-$\text{C}^*$-algebras
- Morphisms: $\text{KK}^\mathbb{T}$-classes
- Composition: Kasparov product

Consider the functor $A \mapsto K_*^\mathbb{T}(A) \cong K_*(A \rtimes \mathbb{T}) \cong \text{KK}^\mathbb{T}_*(\mathbb{C}, A)$. We have $K^\mathbb{T}: \text{KK}^\mathbb{T} \to \mathcal{A} := \text{Mod}(\mathbb{Z}[x, x^{-1}])_{\mathbb{Z}/2}^c$ via the dual action.

Definition/Theorem (Meyer–Nest)

$A \in B^\mathbb{T} \iff A \rtimes \mathbb{T} \in B$. 

Classification for circle actions on $\text{C}^*$-algebras 

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Motivational case: $\text{KK}^T$

- Objects: separable $\mathbb{T}$-$\mathbb{C}^*$-algebras
- Morphisms: $\text{KK}^T$-classes
- Composition: Kasparov product

Consider the functor $A \mapsto K^T_*(A) \cong K_*(A \rtimes \mathbb{T}) \cong \text{KK}^T_*(\mathbb{C}, A)$. We have $K^T: \text{KK}^T \rightarrow \mathcal{A} := \text{Mod}(\mathbb{Z}[x, x^{-1}])_{\mathbb{Z}/2}^c$ via the dual action.

**Definition/Theorem (Meyer–Nest)**

$A \in \mathcal{B}^T \iff A \rtimes \mathbb{T} \in \mathcal{B}$.

**Definition**

A **lifting** of an object $M \in \mathcal{A}$ is an object $\hat{M} \in \mathcal{B}^T$ together with an isomorphism $K^T_*(\hat{M}) \rightarrow M$. 

Classification for circle actions on $\mathbb{C}^*$-algebras

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Motivational case: $\text{KK}^\mathbb{T}$

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Definition/Theorem (Meyer–Nest)

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Definition

A lifting of an object $M \in \mathcal{A}$ is an object $\hat{M} \in B^\mathbb{T}$ together with an isomorphism $K_*^\mathbb{T}(\hat{M}) \to M$.

Classification + range results $\iff$ uniqueness + existence of liftings
Motivational case: \( \text{KK}^\mathbb{T} \)

- Objects: separable \( \mathbb{T} \)-\( \text{C}^* \)-algebras
- Morphisms: \( \text{KK}^\mathbb{T} \)-classes
- Composition: Kasparov product

Consider the functor \( A \mapsto \text{K}_*^\mathbb{T}(A) \cong \text{K}_*(A \times \mathbb{T}) \cong \text{KK}_*^\mathbb{T}(\mathbb{C}, A) \).

We have \( \text{K}_*^\mathbb{T} : \text{KK}^\mathbb{T} \rightarrow \mathfrak{A} := \text{Mod}(\mathbb{Z}[x, x^{-1}])_{\mathbb{Z}/2}^\mathfrak{c} \) via the dual action.

**Definition/Theorem (Meyer–Nest)**

\( A \in \mathbb{B}^\mathbb{T} \iff A \times \mathbb{T} \in \mathbb{B} \).

**Definition**

A **lifting** of an object \( M \in \mathfrak{A} \) is an object \( \hat{M} \in \mathbb{B}^\mathbb{T} \) **together with** an isomorphism \( \text{K}_*^\mathbb{T}(\hat{M}) \rightarrow M \).

Classification + range results \( \iff \) uniqueness + existence of liftings
Motivational case: $\mathbf{KK}^\mathbb{T}$ (Zero and Projectives)

**Lemma**

If $A \in \mathcal{B}^\mathbb{T}$ and $K^\mathbb{T}_\ast(A) \cong 0$ then $A \cong 0$.

“$K^\mathbb{T}_\ast$ vanishes on sufficiently few objects.”
Motivational case: $\text{KK}^\mathbb{T}$ (Zero and Projectives)

**Lemma**

If $A \in \mathcal{B}^\mathbb{T}$ and $\text{K}^\mathbb{T}_*(A) \cong 0$ then $A \cong 0$.

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**Corollary**

$0 \in \mathcal{A}$ has a unique lifting.
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**Corollary**

$0 \in \mathcal{A}$ has a unique lifting.

**Theorem (Meyer–Nest)**

There is a fully faithful functor $\mathcal{A} \supset \text{Proj} \xrightarrow{L} KK^\mathbb{T}$ such that

\[ K^\mathbb{T}_*(L(P)) \cong P \]
\[ KK^\mathbb{T}_*(L(P), B) \cong \mathcal{A}(P, K^\mathbb{T}_*(B)) \text{ for every } B \in B^\mathbb{T}. \]
Motivational case: $\text{KK}^T$ (Zero and Projectives)

**Lemma**

If $A \in B^T$ and $K^*_T(A) \cong 0$ then $A \cong 0$.

“$K^*_T$ vanishes on sufficiently few objects.”

**Corollary**

$0 \in \mathcal{A}$ has a unique lifting.

**Theorem (Meyer–Nest)**

There is a fully faithful functor $\mathcal{A} \supset \text{Proj} \xrightarrow{L} \text{KK}^T$ such that

$\triangleright K^*_T(L(P)) \cong P$

$\triangleright \text{KK}^*_T(L(P), B) \cong \mathcal{A}(P, K^*_T(B))$ for every $B \in B^T$.

“$K^*_T$ is universal ($\mathcal{A}$ has sufficiently few morphisms).”

Classification for circle actions on C*-algebras  
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Motivational case: $KK^\mathbb{T}$ (Zero and Projectives)

**Lemma**

If $A \in B^\mathbb{T}$ and $K_*^\mathbb{T}(A) \cong 0$ then $A \cong 0$.

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\[ KK_*^\mathbb{T}(L(P), B) \cong \mathcal{A}(P, K_*^\mathbb{T}(B)) \text{ for every } B \in B^\mathbb{T}. \]

“$K_*^\mathbb{T}$ is universal ($\mathcal{A}$ has sufficiently few morphisms).”

**Corollary**

*Projectives in $\mathcal{A}$ have unique liftings.*
Motivational case: $\text{KK}^\mathbb{T}$ (Zero and Projectives)

Lemma
If $A \in B^\mathbb{T}$ and $K_*^\mathbb{T}(A) \cong 0$ then $A \cong 0$.

“$K_*^\mathbb{T}$ vanishes on sufficiently few objects.”

Corollary
0 ∈ $\mathcal{A}$ has a unique lifting.

Theorem (Meyer–Nest)
There is a fully faithful functor $\mathcal{A} \ni \text{Proj} \xrightarrow{L} \text{KK}^\mathbb{T}$ such that

1. $K_*^\mathbb{T}(L(P)) \cong P$
2. $\text{KK}_*^\mathbb{T}(L(P), B) \cong \mathcal{A}(P, K_*^\mathbb{T}(B))$ for every $B \in B^\mathbb{T}$.

“$K_*^\mathbb{T}$ is universal ($\mathcal{A}$ has sufficiently few morphisms).”

Corollary
Projectives in $\mathcal{A}$ have unique liftings.
Motivational case: $KK^\mathbb{T}$ (Dimensions 1 and 2)

Theorem (Meyer–Nest)

If $A \in \mathcal{B}^\mathbb{T}$ and $K_\ast(A)$ has projective dimension $\leq 1$ then there is a natural short exact sequence

$$\text{Ext}^1_\mathcal{A}(K_\ast(\Sigma A), K_\ast(B)) \hookrightarrow KK^\mathbb{T}(A, B) \twoheadrightarrow \text{Hom}_\mathcal{A}(K_\ast(A), K_\ast(B)).$$

Corollary

If $M \in \mathcal{A}$ has projective dimension $\leq 1$ then it has a unique lifting.
Motivational case: $\text{KK}^\mathbb{T}$ (Dimensions 1 and 2)

**Theorem (Meyer–Nest)**

*If $A \in B^\mathbb{T}$ and $K^\mathbb{T}(A)$ has projective dimension $\leq 1$ then there is a natural short exact sequence*

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**Corollary**

*If $M \in \mathcal{A}$ has projective dimension $\leq 1$ then it has a unique lifting.*

**Theorem**

*If $M \in \mathcal{A}$ has projective dimension $\leq 2$ (automatic in this example) then iso-classes of liftings of $M$ are in bijection with $\text{Ext}_2^\mathbb{A}(\Sigma M, M)$.*
Motivational case: $\mathbf{KK}^T$ (Dimensions 1 and 2)

**Theorem (Meyer–Nest)**

If $A \in B^T$ and $K_*(A)$ has projective dimension $\leq 1$ then there is a natural short exact sequence

$$\text{Ext}_A^1(K_*(\Sigma A), K_*(B)) \hookrightarrow \mathbf{KK}_*(A, B) \rightarrow \text{Hom}_A(K_*(A), K_*(B)).$$

**Corollary**

If $M \in A$ has projective dimension $\leq 1$ then it has a unique lifting.

**Theorem**

If $M \in A$ has projective dimension $\leq 2$ (automatic in this example) then iso-classes of liftings of $M$ are in bijection with $\text{Ext}_A^2(\Sigma M, M)$.

Hence if $M = M_+ \oplus M_-$ then $M_\pm$ have unique liftings $\hat{M}_\pm$.

$\hat{M}_+ \oplus \hat{M}_-$ is the **canonical lifting** of $M$ corresponding to $0 \in \text{Ext}_A^2(\Sigma M, M)$. 

Classification for circle actions on $C^*$-algebras

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Motivational case: $\text{KK}^T$ (Dimensions 1 and 2)

Theorem (Meyer–Nest)

If $A \in B^T$ and $K_\pi^T(A)$ has projective dimension $\leq 1$ then there is a natural short exact sequence

$$
\text{Ext}_{\mathfrak{A}}^1(K_\pi^T(\Sigma A), K_\pi^T(B)) \hookrightarrow \text{KK}^T(A, B) \twoheadrightarrow \text{Hom}_{\mathfrak{A}}(K_\pi^T(A), K_\pi^T(B)).
$$

Corollary

If $M \in \mathfrak{A}$ has projective dimension $\leq 1$ then it has a unique lifting.

Theorem

If $M \in \mathfrak{A}$ has projective dimension $\leq 2$ (automatic in this example) then iso-classes of liftings of $M$ are in bijection with $\text{Ext}_{\mathfrak{A}}^2(\Sigma M, M)$.

Hence if $M = M_+ \oplus M_-$ then $M_\pm$ have unique liftings $\hat{M}_\pm$. $\hat{M}_+ \oplus \hat{M}_-$ is the canonical lifting of $M$ corresponding to $0 \in \text{Ext}_{\mathfrak{A}}^2(\Sigma M, M)$. 

Classification for circle actions on C*-algebras
Motivational case: $\text{KK}^\mathbb{T}$ (Classification)

How to get a classification functor?

**Definition**

$\mathcal{A}\delta$ is the category of pairs $(M, \delta)$ with $M \in \mathcal{A}$, $\delta \in \text{Ext}_{\mathcal{A}}^2(\Sigma M, M)$. A morphism $(M, \delta) \to (M', \delta')$ is $f : M \to M'$ s.t. $\delta' f = f \delta$.

**Theorem**

There is a dense strong classification functor $(\text{K}_\ast^\mathbb{T}, \delta) : \mathcal{B}^\mathbb{T} \to \mathcal{A}\delta$. 

δ̂(M̂) measures the difference of M̂ and M̂ + ⊕ M̂ −. Observed earlier by Aldridge K. Bousfield for $K$-local spectra at an odd prime, Jerome Wolbert, more generally, for certain module spectra.

Example: Cuntz–Krieger algebras

$O_A \cong \text{KK}^\mathbb{T} O_B \iff \text{K}_\ast^\mathbb{T}(O_A) \cong \text{K}_\ast^\mathbb{T}(O_B) \iff A \cong Z_B$. 

Classification for circle actions on $\mathbb{C}^*$-algebras Rasmus Bentmann 6
Motivational case: $\text{KK}^\mathbb{T}$ (Classification)

How to get a classification functor?

**Definition**

$\mathfrak{A}_\delta$ is the category of pairs $(M, \delta)$ with $M \in \mathfrak{A}$, $\delta \in \text{Ext}^2_{\mathfrak{A}}(\Sigma M, M)$. A morphism $(M, \delta) \to (M', \delta')$ is $f : M \to M'$ s.t. $\delta'f = f\delta$.

**Theorem**

*There is a dense strong classification functor $(\text{K}_{\!*}^\mathbb{T}, \delta) : \mathcal{B}^\mathbb{T} \to \mathfrak{A}_\delta$.*

“$\delta(\hat{M})$ measures the difference of $\hat{M}$ and $\hat{M}_+ \oplus \hat{M}_-$. “
Motivational case: \( \mathbf{KK}^\mathbb{T} \) (Classification)

How to get a classification functor?

**Definition**
\( \mathfrak{A} \delta \) is the category of pairs \((M, \delta)\) with \( M \in \mathfrak{A}, \delta \in \text{Ext}^2_{\mathfrak{A}}(\Sigma M, M) \).
A morphisms \((M, \delta) \to (M', \delta')\) is \( f : M \to M' \) s.t. \( \delta' f = f \delta \).

**Theorem**

*There is a dense strong classification functor* \((\mathbf{K}^\mathbb{T}_*, \delta) : \mathcal{B}^\mathbb{T} \to \mathfrak{A} \delta.\)

“\( \delta(\hat{M}) \) measures the difference of \( \hat{M} \) and \( \hat{M}_+ \oplus \hat{M}_- \).”

Observed earlier by

- Aldridge K. Bousfield for K-local spectra at an odd prime,
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Motivational case: \( \text{KK}^\mathbb{T} \) (Classification)

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**Theorem**

*There is a dense strong classification functor \((\text{KT}^\mathbb{T}, \delta): \mathcal{B}^\mathbb{T} \to \mathcal{A}_\delta \).*

“\( \delta(\hat{M}) \) measures the difference of \( \hat{M} \) and \( \hat{M}_+ \oplus \hat{M}_- \).”

Observed earlier by

- Aldridge K. Bousfield for K-local spectra at an odd prime,
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**Example:** Cuntz–Krieger algebras

\[ O_A \simeq_{\text{KK}^\mathbb{T}} O_B \iff \text{KT}_*(O_A) \simeq \text{KT}_*(O_B) \iff A \sim_\mathbb{Z} B. \]
Motivational case: $\text{KK}^\mathbb{T}$ (Classification)

How to get a classification functor?

**Definition**

$\mathcal{A}_\delta$ is the category of pairs $(M, \delta)$ with $M \in \mathcal{A}$, $\delta \in \text{Ext}^2_A(\Sigma M, M)$. A morphism $(M, \delta) \rightarrow (M', \delta')$ is $f: M \rightarrow M'$ s.t. $\delta'f = f\delta$.

**Theorem**

*There is a dense strong classification functor $(K^\mathbb{T}_*, \delta): B^\mathbb{T} \rightarrow \mathcal{A}_\delta$. “$\delta(\hat{M})$ measures the difference of $\hat{M}$ and $\hat{M}_+ \oplus \hat{M}_-.$”*

Observed earlier by

- Aldridge K. Bousfield for K-local spectra at an odd prime,
- Jerome Wolbert, more generally, for certain module spectra.

**Example: Cuntz–Krieger algebras**

$\mathcal{O}_A \simeq_{\text{KK}^\mathbb{T}} \mathcal{O}_B \iff K^\mathbb{T}_*(\mathcal{O}_A) \cong K^\mathbb{T}_*(\mathcal{O}_B) \iff A \simeq_{\mathbb{Z}} B.$
Second application: $KK(X)$

- $X$ finite $T_0$-space
- Objects: separable $C^*$-algebras over $X$
  \((U \subseteq X \text{ open } \leadsto A(U) \triangleleft A)\)
- Morphisms: $KK(X)$-classes
- Composition: Kasparov product
Second application: \( \text{KK}(X) \)

- \( X \) finite \( T_0 \)-space
- Objects: separable \( \text{C}^* \)-algebras over \( X \)
  \( (U \subseteq X \text{ open } \leadsto A(U) \triangleleft A) \)
- Morphisms: \( \text{KK}(X) \)-classes
- Composition: Kasparov product

Definition (Meyer–Nest)

\[ \mathcal{B}(X) \text{ is the localizing subcategory generated by } \{i_x \mathbb{C} \mid x \in X\}. \]
Second application: $\text{KK}(X)$

- $X$ finite $T_0$-space
- Objects: separable $C^*$-algebras over $X$
  $(U \subseteq X \text{ open } \leadsto A(U) \otimes A)$
- Morphisms: $\text{KK}(X)$-classes
- Composition: Kasparov product

**Definition (Meyer–Nest)**

$\mathcal{B}(X)$ is the localizing subcategory generated by $\{i_x \mathbb{C} \mid x \in X\}$.

$\text{KK}_*(X; i_x \mathbb{C}, A) \cong K_*(A(U_x))$
(here $U_x$ is the smallest open neighborhood of $x$)
Second application: $\text{KK}(X)$

- $X$ finite $T_0$-space
- Objects: separable C*-algebras over $X$
  \((U \subseteq X \text{ open } \leadsto A(U) \triangleleft A)\)
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**Definition (Meyer–Nest)**

$\mathcal{B}(X)$ is the localizing subcategory generated by \(\{i_x \mathbb{C} \mid x \in X\}\).

$\text{KK}_*(X; i_x \mathbb{C}, A) \cong K_*(A(U_x))$

(here $U_x$ is the smallest open neighborhood of $x$)

**Definition**

$XK(A) = \left( K_*(A(U_x)) \right)_{x \in X}$
**Second application: \( KK(X) \)**

- \( X \) finite \( T_0 \)-space
- **Objects**: separable \( C^* \)-algebras over \( X \)
  
  \( (U \subseteq X \text{ open } \leadsto A(U) \triangleleft A) \)

- **Morphisms**: \( KK(X) \)-classes

- **Composition**: Kasparov product

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**Definition (Meyer–Nest)**

\( \mathcal{B}(X) \) is the localizing subcategory generated by \( \{ i_x \mathbb{C} \mid x \in X \} \).

\[ KK_\ast(X; i_x \mathbb{C}, A) \cong K_\ast(A(U_x)) \]

(Here \( U_x \) is the smallest open neighborhood of \( x \)).

**Definition**

\[ XK(A) = \left( K_\ast(A(U_x)) \right)_{x \in X} \]

\[ XK : KK(X) \to \text{Mod} (\mathbb{Z}X)_{\mathbb{Z}/2}^c \text{ via maps induced by ideal inclusions.} \]
Second application: $KK(X)$

- $X$ finite $T_0$-space
- Objects: separable C*-algebras over $X$
  
  $(U \subseteq X$ open $\leadsto A(U) \triangleleft A)$
- Morphisms: $KK(X)$-classes
- Composition: Kasparov product

**Definition (Meyer–Nest)**

$B(X)$ is the localizing subcategory generated by $\{i_x\mathbb{C} \mid x \in X\}$.

$KK_*(X; i_x\mathbb{C}, A) \cong K_*(A(U_x))$

(here $U_x$ is the smallest open neighborhood of $x$)

**Definition**

$XK(A) = \left( K_*(A(U_x)) \right)_{x \in X}$

$XK: KK(X) \to \mathcal{Mod}(\mathbb{Z}X)_{\mathbb{Z}/2}$ via maps induced by ideal inclusions.
Second application: KK(\(X\)) over unique path space

Let \(X\) be a unique path space.
For instance \( \bullet \rightarrow \bullet \) but not \( \bullet \rightarrow \bullet \)

\[ \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\end{array} \]

\[ \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\end{array} \]

Theorem

If \(X\) is a unique path space then \( \mathbb{Z}X \) is an integral quiver algebra and hence has cohomological dimension 2.
Second application: \( KK(X) \) over unique path space

Let \( X \) be a unique path space. For instance \( \bullet \rightarrow \bullet \) but not \( \bullet \rightarrow \bullet \).

\begin{center}
\begin{tikzpicture}
    \node (A) at (0,0) [circle,fill,inner sep=1.5pt] {}; 
    \node (B) at (1,1) [circle,fill,inner sep=1.5pt] {}; 
    \node (C) at (1,0) [circle,fill,inner sep=1.5pt] {}; 
    \node (D) at (2,0) [circle,fill,inner sep=1.5pt] {}; 
    \draw (A) -- (B) -- (C) -- (D); 
    \draw (A) -- (D); 
\end{tikzpicture}
\end{center}

Theorem

If \( X \) is a unique path space then \( \mathbb{Z}X \) is an integral quiver algebra and hence has cohomological dimension 2. Hence there is a dense strong classification functor \( (\mathcal{X}K, \delta) : \mathcal{B}(X) \rightarrow \text{Mod}(\mathbb{Z}X)^{\mathbb{Z}/2}_c \delta \).

Corollary (using Kirchberg’s classification)

Strong classification up to \( \star \)-isomorphism of stable Kirchberg \( X \)-algebras in \( \mathcal{B}(X) \) including a description of the range.

(A Kirchberg \( X \)-algebra is a separable nuclear tight \( \mathcal{O}_\infty \)-absorbing \( \mathcal{C}^* \)-algebra over \( X \).)
Second application: $KK(X)$ over unique path space

Let $X$ be a unique path space.
For instance $ \bullet \xrightarrow{} \bullet $ but not $ \bullet \xrightarrow{} \bullet $.

\begin{align*}
\bullet & \xrightarrow{} \bullet \\
\bullet & \xrightarrow{} \bullet
\end{align*}

Theorem

If $X$ is a unique path space then $\mathbb{Z}X$ is an integral quiver algebra and hence has cohomological dimension 2. Hence there is a dense strong classification functor $(\mathcal{XK}, \delta): \mathcal{B}(X) \to \text{Mod}(\mathbb{Z}X)^{\mathbb{Z}/2}_c \delta$.

Corollary (using Kirchberg’s classification)

Strong classification up to $\ast$-isomorphism of stable Kirchberg $X$-algebras in $\mathcal{B}(X)$ including a description of the range.
Second application: \( KK(X) \) over unique path space

Let \( X \) be a unique path space.
For instance \( \bullet \rightarrow \bullet \) but not \( \bullet \rightarrow \bullet \)

\[ \begin{array}{c}
\bullet \rightarrow \bullet \\
\downarrow \\
\bullet \\
\end{array} \]

\[ \begin{array}{c}
\bullet \rightarrow \bullet \\
\downarrow \\
\bullet \\
\end{array} \]

**Theorem**

*If* \( X \) *is a unique path space then* \( \mathbb{Z}X \) *is an integral quiver algebra and hence has cohomological dimension 2. Hence there is a dense strong classification functor* \((XK, \delta) : \mathcal{B}(X) \rightarrow \text{Mod}(\mathbb{Z}X)_{\mathbb{Z}/2}^{\delta}\).*

**Corollary (using Kirchberg’s classification)**

*Strong classification up to *\(-\)isomorphism of stable Kirchberg X-algebras in \( \mathcal{B}(X) \) including a description of the range.*

*(A Kirchberg X-algebra is a separable nuclear tight \( \mathcal{O}_{\infty} \)-absorbing \( \mathbb{C}^* \)-algebra over \( X \).*
Second application: \( \text{KK}(X) \) over unique path space

Let \( X \) be a unique path space. For instance, \( \bullet \rightarrow \bullet \) but not \( \bullet \rightarrow \bullet \)

**Theorem**

*If \( X \) is a unique path space then \( \mathbb{Z}X \) is an integral quiver algebra and hence has cohomological dimension 2. Hence there is a dense strong classification functor \((\mathcal{XK}, \delta): \mathcal{B}(X) \rightarrow \text{Mod}(\mathbb{Z}X)^{\mathbb{Z}/2}_c \delta \).*

**Corollary (using Kirchberg’s classification)**

*Strong classification up to \(*\)-isomorphism of stable Kirchberg \( X \)-algebras in \( \mathcal{B}(X) \) including a description of the range.*

(A Kirchberg \( X \)-algebra is a separable nuclear tight \( \mathcal{O}_\infty \)-absorbing \( C^* \)-algebra over \( X \).)
Second application: $\text{KK}(X)$ for graph algebras

Let $X$ be an arbitrary finite $T_0$-space.

**Theorem**

If $E$ is row-finite and all distinguished ideals of $C^*(E)$ are gauge-invariant then $\text{XK}(C^*(E))$ has projective dimension $\leq 2$. 

**To do:** find range results in this context.
Second application: $\text{KK}(X)$ for graph algebras

Let $X$ be an arbitrary finite $T_0$-space.

**Theorem**

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**Corollary (using Kirchberg’s classification)**

*Strong classification up to $\ast$-isomorphism of stable/unital purely infinite graph $C^*$-algebras with finitely many ideals.*
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**Theorem**

*The element $\delta(C^*(E)) \in \text{Ext}^2_{\mathbb{Z}X}(\text{XK}(\Sigma C^*(E)), \text{XK}(C^*(E)))$ is given by the dual Pimsner–Voiculescu sequence

$\text{XK}_1(C^*(E)) \hookrightarrow \text{XK}_0(C^*(E)^{\mathbb{T}}) \to \text{XK}_0(C^*(E)^{\mathbb{T}}) \to \text{XK}_0(C^*(E)).$*
Second application: $\text{KK}(X)$ for graph algebras

Let $X$ be an arbitrary finite $T_0$-space.

**Theorem**

*If $E$ is row-finite and all distinguished ideals of $C^*(E)$ are gauge-invariant then $\text{XK}(C^*(E))$ has projective dimension $\leq 2$.***

**Corollary (using Kirchberg’s classification)**

*Strong classification up to $^*$-isomorphism of stable/unital purely infinite graph $C^*$-algebras with finitely many ideals.*

**Theorem**

*The element $\delta(C^*(E)) \in \text{Ext}^2_{\mathbb{Z}}(\text{XK}(\Sigma C^*(E)), \text{XK}(C^*(E)))$ is given by the dual Pimsner–Voiculescu sequence*

\[ \text{XK}_1(C^*(E)) \rightarrow \text{XK}_0(C^*(E)\mathbb{T}) \rightarrow \text{XK}_0(C^*(E)\mathbb{T}) \rightarrow \text{XK}_0(C^*(E)). \]

*To do: find range results in this context.*
Second application: $KK(X)$ for graph algebras

Let $X$ be an arbitrary finite $T_0$-space.

**Theorem**

*If $E$ is row-finite and all distinguished ideals of $C^*(E)$ are gauge-invariant then $XK(C^*(E))$ has projective dimension $\leq 2$.***

**Corollary (using Kirchberg’s classification)**

*Strong classification up to $^\ast$-isomorphism of stable/unital purely infinite graph $C^*$-algebras with finitely many ideals.*

**Theorem**

*The element $\delta(C^*(E)) \in \text{Ext}^2_{\mathbb{Z}X}(XK(\Sigma C^*(E)), XK(C^*(E)))$ is given by the dual Pimsner–Voiculescu sequence*

$$XK_1(C^*(E)) \hookrightarrow XK_0(C^*(E)^T) \rightarrow XK_0(C^*(E)^T) \rightarrow XK_0(C^*(E)).$$

To do: find range results in this context.
Thank you for your attention!