In this minicourse, we discuss lattices in $\text{SL}_n(\mathbb{R})$.

**Def.** A lattice in $\text{SL}_n(\mathbb{R})$ is a discrete subgroup such that $\text{vol}(\text{SL}_n(\mathbb{R})/\Gamma) < \infty$.

**Example:** $\text{SL}_n(\mathbb{Z})$.

We shall prove the following 3 main results:

1) Borel density thm.

2) (Kazhdan) $\Gamma < \text{SL}_n(\mathbb{R})$, $n \geq 3$, is finitely generated.

3) (Margulis) $\Gamma < \text{SL}_n(\mathbb{R})$, $n \geq 3$, and $N \leq \Gamma$.

Then $\Gamma/N$ is finite.

**Borel density thm.**

**Thm.** $\Gamma$ - lattice in $G = \text{SL}_n(\mathbb{R})$

$\phi: G \to \text{GL}_N(\mathbb{R})$ - polynomial homomorphism.

Then $\forall \sigma \in \mathbb{R}^N$: $\phi(\Gamma) \sigma = \sigma \Rightarrow \phi(G) \sigma = \sigma$.

**Idea:** Recurrence (ergodic theory) $\leftrightarrow$ Transience (algebraic actions)
Lem. 1 (Poincare recurrence)

\( X \) - compact metric space

\( T : X \to X \) - homeomorphism

\( \mu \) - invariant Borel probability measure on \( X \)

Then for \( \mu \)-a.e. \( x \in X \),

\( T^{n_i} x \to x \) along a subsequence \( n_i \to \infty \).

Lem. 2

\( T \in \text{GL}_N(\mathbb{R}) \) - unipotent,

\( T \subset \mathbb{P}^{N-1} \) - projective space,

\( [v] \in \mathbb{P}^{N-1}, \ n_i \to \infty \).

Then

\[ T^{n_i} [v] \to [v] \quad \Rightarrow \quad T v = v. \]

\[ \text{Let} \ T = I + S, \text{ where } S \text{ is nilpotent.} \]

\[ \text{Pick } k \text{ such that } S^k v = 0 \text{ and } S^{k+1} v = 0. \]

\[ T^{n} [v] = [(I+S)^n v] = \left[ \sum_{i=0}^{k} \binom{n}{i} S^i v \right] \to [S^k v]. \]

Since \( S \) is nilpotent, \( [S^k v] = [v] \Rightarrow k = 0. \)

Proof of Thm. Consider the map

\( \pi : G_T \to \mathbb{P}^{N-1} : g \mapsto [f(g)v] \),

and define measure \( \nu \) on \( \mathbb{P}^{N-1} \):

\[ \nu(B) = \mu(\pi^{-1}(B)), \quad B \subset \mathbb{P}^{N-1}, \]

where \( \mu \) is the inv. prob. measure on \( G_T \).
Then \( \nu(P^{N-1}) = 1 \), and \( \nu \) is \( \rho(G) \)-inv.

Take unipotent \( g \in G \), \( g \neq 1 \).

Then \( T = \rho(g) \) is unipotent.

Indeed, if \( Tw = \lambda w \), then
\[
\rho(g^n) w = \rho(g)^n w = \lambda^n w,
\]
so that \( \lambda = 1 \).

By Lem. 1 & Lem. 2, for \( \nu \)-a.e. \( [w] \in P^{N-1} \):
\[
T w = w.
\]

Equivalently, for a.e. \( h \in G \), \( T \rho(h) \omega = \rho(h) \omega \).

Then \( \rho(h-g \cdot h) \cdot \omega = \omega \) for all \( h \in G \),
and \( \rho(G) \cdot \omega = \omega \) because \( SL_n(R) \) has no nontrivial infinite normal subgroups.

**Thm. (Margulis)** \( \Gamma \)-a lattice in \( SL_n(R) \), \( n \geq 3 \).

\( N \triangleleft \Gamma \) - infinite.

Then \(|\Gamma/N| < \infty|\).

**Strategy:** \( \Gamma/N \) is amenable \( \Gamma/N \) has property (T) \( \Rightarrow \) \( \Gamma/N \)-finite.
Amenability

G - topological group (e.g., $G = \text{a closed subgroup of } SL_2(\mathbb{R})$).

**Def.** $V$ - locally convex top vector space
S2 - nonempty, compact, convex
G $\rhd$ S2 - affine continuous action.

The group $G$ is called **amenable** if S2 contains a $G$-fixed point.

**Application.** X - compact metric space
G $\rhd$ X - continuous action.

Then $G \rhd \text{Prob}(X)$ - convex and compact (in weak* topology).

Hence, $\exists$ G-inv. prob. measure on X.

**Prop.** $\mathbb{Z}^d$ and $\mathbb{R}^d$ are amenable.

Consider $\mathbb{Z}^d \rhd S2$.

Let $B_N = [1, N]^d$ and $\omega_N = \frac{1}{|B_N|} \sum_{z \in B_N} z \cdot w$ for $w \in S2$.

| $\frac{|B_N \Delta (z_0 + B_N)|}{|B_N|}$ | $\rightarrow 0$ | $N \rightarrow \infty$ |
|-------------------------------------|-----------------|------------------|
| $B_N$                              | $z_0 + B_N$     |
Then \( z_0 \cdot w_N - w_N = \frac{1}{|B_N|} \left( \sum_{z \in (z_0 + B_N) \setminus B_N} z \cdot w - \sum_{z \in B_N \setminus (z_0 + B_N)} z \cdot w \right) \in \frac{|B_N \setminus (z_0 + B_N)|}{|B_N|} (\pm \Omega) \to 0. \)

By compactness, \( w_{n_i} \to w_\infty \in \mathcal{S}_2, \) and \( z_0 \cdot w = w \) for all \( z_0 \in \mathbb{Z}^d. \)

**Prop.** Suppose that \( G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_e \supseteq G_{e+1} = \{e\} \)
where \( G_i \) is closed and \( G_i/G_{i+1} \) is amenable.

Then \( G \) is amenable.

Consider \( G \subseteq \mathcal{S}_2. \)

Since \( G_e \) is amenable, \( \mathcal{S}_2^{G_e} \neq \emptyset. \)

Clearly, \( \mathcal{S}_2^{G_e} \) is closed (\( \Rightarrow \) compact) and convex.

Consider \( G_{e-1}/G_e \subseteq \mathcal{S}_2^{G_e}. \)

**Prop.** \( SL_n(\mathbb{R}) \) is not amenable.

Consider \( SL_n(\mathbb{R}) \subseteq P^{n-1}. \)

Suppose that \( SL_n(\mathbb{R}) \) is amenable.

Then \( \exists \) invariant prob. measure \( \nu \) on \( P^{n-1}. \)

However, \( \text{supp} (\nu) \subseteq \text{Fix}(g) \) for unipotent \( g. \)

Since \( SL_n(\mathbb{R}) \) is generated by unipotents,
\[ \text{Supp}(\nu) \subset \text{Fix}(G) = \emptyset \]

which is a contradiction.

Now we assume that \( G \) is discrete and countable.

\[ \text{amenability} \quad \iff \quad \text{invariant means} \quad \iff \quad \text{almost invariant vectors} \]

**Def.** A linear map \( M: L^\infty(G) \to \mathbb{C} \) is (left) invariant mean if

1. \( M(1) = 1 \)
2. \( f \geq 0 \Rightarrow M(f) \geq 0 \)
3. \( M(g \cdot f) = M(f) \) for \( g \in G \).
   (here: \( (g \cdot f)(x) = f(gx) \))

Note that

\[ -\|f\|_\infty \leq f \leq \|f\|_\infty \Rightarrow -\|f\|_\infty \leq M(f) \leq \|f\|_\infty \Rightarrow |M(f)| \leq \|f\|_\infty, \]

so that \( M \in L^\infty(G)^* \).

**Thm.** \( G \) is amenable \( \iff \exists \) invariant mean on \( L^\infty(G) \).

\[ \Rightarrow L^\infty(G)^* = \left\{ \begin{array}{c} \text{bounded linear functionals} \\ \text{on } L^\infty(G)^* \text{ with weak* topology} \end{array} \right\} \]

\[ \cup \]

\[ M = \left\{ M : (1) \& (2) \right\} \quad \begin{cases} \text{convex} \\ \text{nonempty} \quad (\text{e.g. Dirac measures}) \end{cases} \quad \text{compact} \quad (\text{Banach- Alaoglu Thm}) \]
This action is continuous in weak* topology. Hence, $M \in G$-fixed point.

Thm. $G$ is amenable $\iff \left[ \forall \text{finite } K \subset G, \exists \psi_n \in L^2(G): \|\psi_n\|=1; \|g \cdot \psi_n - \psi_n\| \to 0 \text{ for } g \in K. \right]$

(almost invariant vector)

Let $M$ be an invariant mean on $L^\infty(G)$. For $\varphi \in L^1(G)$, we define $L\varphi \in L^\infty(G)^*$ by

$$L\varphi(f) = \langle \varphi, f \rangle = \sum_{x \in G} \varphi(x)f(x).$$

For a finite partition $G = \{G_1, \ldots, G_e\}$ of $G$, use define $\psi_\varepsilon(g) = \sum_{i=1}^e M(X_{G_i}) X_{g_i} \in L^1(G)$, where $g_i \in G_i$.

Then $\|\psi_\varepsilon\| = \sum_{i=1}^e M(X_{G_i}) = 1$.

If $\sigma$ is a refinement of the partition $\{A, G \setminus A\}$, then $\langle \psi_\varepsilon, X_A \rangle = M(X_A)$. 

Recall: the weak topology on $L'(G)$:
$$\varphi \to \psi \iff \left( \varphi_n, f \right) \to \left( \psi, f \right) \text{ for all } f \in L^0(G).$$

- If $S \subset L'(G)$ is convex,
  \[ \text{weak-closure}(S) = \text{norm-closure}(S). \]

Approximating functions in $L^\infty(G)$ by linear combinations of characteristic functions, we deduce that
\[ \forall \varphi_n \in L^\infty(G): \exists \varepsilon_n : \left( \varphi_n, f \right) \to M(\mu)^t. \]

Take $g \in G$. Then $\exists \varepsilon_n = \varepsilon_n(f, g)$:
\[ \left( \varphi_n, f \right) \to M(\mu)^t, \quad \left( \varphi_n, g \cdot f \right) \to M(\mu)^t = M(\mu)^t. \]

Then
\[ \left< g \cdot \varphi_n - \varphi_n, f \right> = \left< \varphi_n, \frac{g}{n} \cdot f - f \right> \to 0. \]

This shows that $0 \in \text{weak-closure}\{ g \cdot \varphi_n - \varphi_n \}$. Let $S = \text{convex-closure}\{ g \cdot \varphi_n - \varphi_n \}$. Since $S$ is convex,
\[ \text{norm-closure}(S) = \text{weak-closure}(S) \ni 0. \]

Hence, $\exists \varphi_n = \text{convex-closure}\{ \varphi_n \}$:
\[ \| g \cdot \varphi_n - \varphi_n \|_1 \to 0. \]
\[ \| \varphi_n \|_1 = 1. \]

Given finite $K \subset G$, we apply the same argument to $L'(G)^{\| K \|$ and $\{ g \cdot \varphi_n - \varphi_n : g \in K \}$. We deduce that $\exists \varphi_n : \| \varphi_n \| = 1$:
\[ \| g \cdot \varphi_n - \varphi_n \| \to 0 \text{ for } g \in K. \]
Finally, let $\psi_n = \varphi_n^{1/2}$. Then
\[ \| g \cdot \psi_n - \psi_n \|_2^2 = \sum_x |\psi_n (g^x) - \psi_n (x)|^2 \leq \sum_x |\psi_n (g^x)^2 - \psi_n (x)^2| \]
\[ = \| g \cdot \psi_n - \psi_n \|_1 \rightarrow 0, \]
where we used that $|a-b|^2 \leq |a-b|^2$, $a, b \geq 0$.  

\[ \Leftarrow \text{ is not used below.} \]
Kazhdan property \( T \).

\( G \) - locally compact group

**Def.** \( G \) has property \( T \) if \( \exists \) compact \( K \subset G \): \( \varepsilon > 0 \):

for every continuous unitary representation

\( \pi : G \to U(H), H \)-Hilbert space, \( \pi \) without fixed vectors

\( \forall \psi \in H : \| \psi \| = 1 : \sup_{g \in K} \| \pi(g) \psi - \psi \| \geq \varepsilon. \)

(no almost invariant vectors)

**Thm.** Suppose that \( G \) is discrete/countable.

If \( G \) is amenable and has property \( T \),
then \( G \) is finite.

Consider the regular representation \( \pi : G \to L^2(G) : f \mapsto f(\cdot g^{-1} x) \), \( f \in L^2(G) \).

Since \( G \) is amenable, \( \forall \) finite \( K \subset G \): \( \exists f_n \in L^2(G) : \| f_n \| = 1 \):

\( \max_{g \in K} \| \pi(g) f_n - f_n \| \to 0. \)

Then by property \( T \), \( L^2(G) \) \( \ni \) \( G \)-fixed vector.

Hence, \( 1 \in L^2(G) \) and \( G \) is finite.
Thm. $\text{SL}_n(\mathbb{R})$, $n \geq 3$, has property $T$.

In the proof, we use:

**Spectral Theorem:**

$\{U_t\}$ - one-param. subgroup of unitary operators on $\mathfrak{h}$.

$\dim (\mathfrak{h}) < \infty$:

$$ U_t = \sum_{i=1}^{s} e^{itu_i} P_i, \quad \sum_{i=1}^{s} P_i = \text{id} $$

where $u_i \in \mathbb{R}$ and $P_i$'s are orthogonal projections on the eigenspaces.

$\dim (\mathfrak{h}) = \infty$:

possibly no eigenvectors,

$\exists \ P : \{\text{Borel subsets of } \mathbb{R}\} \rightarrow \{\text{orthogonal projections}\}$

$6$-additive map, $P_\mathbb{R} = \text{id}$.

$$ U_t = \int_{\mathbb{R}} e^{itu} dP(u) $$

**Proof of Thm.**

Consider $\frac{\text{SL}_2(\mathbb{R}) \times \mathbb{R}^2}{G \times A} \rightarrow \text{SL}_n(\mathbb{R})$.

Let $\pi : \text{SL}_n(\mathbb{R}) \rightarrow U(\mathfrak{h}_c)$ be a unitary representation without fixed vectors.
By the spectral theorem for $\pi(g)$,

$$\pi(a) = \int_{\mathbb{R}^2} e^{i\langle a, u \rangle} \, d\pi(u), \quad a \in A,$$

where $\pi$ is a projection-valued measure on $\mathbb{R}^2$.

For $g \in G$ and $u \in A$,

$$\pi(g)\pi(a)\pi(g) = \pi(g^{-1}g) = \pi(g^{-1}(a))$$

$$\int_{\mathbb{R}^2} e^{i\langle a, u \rangle} \, d\pi(g^{-1}(a)u) = \int_{\mathbb{R}^2} e^{i\langle \pi(a)u, u \rangle} \, d\pi(u).$$

Hence,

$$\pi(g)\pi(u)\pi(g) = \pi((tg)(u)).$$

Let $\mathcal{K}$ be a compact generating set of $G$.

Suppose that for some representations $\pi_n$, without fixed vectors, and $v_n \in \mathcal{H}_n$: $\|v_n\| = 1$, $\sup_{g \in \mathcal{K}} \|\pi(g)v_n - v_n\| \to 0$.

Consider the sequence of probability measures on $\mathbb{R}^2$:

$$\mu_n(B) = \langle \pi_n(B)u_n, v_n \rangle$$

for Borel $B \subset \mathbb{R}^2$.

For $g \in \mathcal{K}$,

$$|\mu_n(tgB) - \mu_n(B)| = |\langle \pi_n(B)\pi_n(g)v_n, v_n \rangle - \langle \pi_n(B)u_n, v_n \rangle| \to 0.$$
If $\mu_n(10^j) \neq 0$, then $\pi(10^j) \neq 0$ and $\mathcal{H}$ contains a $\pi(\Lambda)$-fixed vector, but this is impossible by Moore ergodicity thm. (see Furman’s lectures)

Hence, $\mu_n$ are prob. measures on $R^2 \backslash 10^j$ and projecting $R^2 \backslash 10^j \rightarrow \mathcal{P}'$, we obtain a sequence of prob. measures $\mu_n$ on $\mathcal{P}'$.

Let $\mu$ be a weak* limit point $\mu_n$.

Then $\mu$ is $SL_2(R)$-invariant.

This is impossible. Hence, $\exists \varepsilon > 0$.

for all $\pi$’s without invariant vectors and $v$’s with $\|v\| = 1$.

\[\text{Thm. If } \Gamma \text{ is a lattice in } G \text{ and } G \text{ has property } T, \text{ then } \Gamma \text{ has property } T.\]

In the proof we use:

Induced representation:

$\pi: \Gamma \rightarrow U(H)$ - unitary representation of $\Gamma$.

Define $\hat{\mathcal{H}} = \{ f: G \rightarrow H : f(xg) = \pi(x)f(g), \ g \in G, \ x \in \Gamma \}$

\[\|f(\cdot)\| \sum_{x \in \Gamma} \|f(x)\| < \infty\]
\[ \hat{\pi}(g) : \hat{H} \rightarrow \hat{H} : f \mapsto \pi(xg). \]

Then \( \hat{\pi} : G \rightarrow \mathcal{U}(\hat{H}) \) is a unitary representation.

**Proof of Thm:**

For simplicity, let's assume that \( G \) is compact. Then \( \exists \) relatively compact Borel \( C \subset G : \)

\[ G = \bigsqcup_{g \in C} \gamma C \quad (\text{disjoint union}) \]

\[ \gamma g = \pi(g)c(g) \]

Suppose that \( \pi : \Gamma \rightarrow \mathcal{U}(\hat{H}) \) be a representation such that \( \forall \text{finite } \mathcal{S} \subset \Gamma : \exists \psi_0 \in \hat{H} : \| \psi_0 \| = 1 : \)

\[ \max_{\gamma \in \mathcal{S}} \| \pi(\gamma) \psi_0 - \psi_0 \| \rightarrow 0. \]

Consider the induced representation \( \hat{\pi} : G \rightarrow \mathcal{U}(\hat{H}) \)
and \( \hat{f}_n(\gamma) = \pi(\gamma) \psi_0 \in \hat{H}, \| \hat{f}_n \| = 1. \)

Given compact \( K \subset G \), \( \exists \text{finite } \mathcal{S} \subset \Gamma : C.K \subset \mathcal{S}.C \) (since \( \Gamma \) is discrete).

Then for \( g \in K, \) \( \| \hat{\pi}(g) \hat{f}_n - \hat{f}_n \| = \sum_{\gamma \in \mathcal{S}} \| \pi(\gamma) \psi_0 - \psi_0 \| \rightarrow 0. \)

Since \( G \) has property \( T, \) \( \hat{f}_n \) is \( \Gamma \)-inv. vector:
\[ f(g) = \psi_0 \in \hat{H} \text{ for a.e. } g. \]
Since \( f(\gamma g) = \pi(g)f(g), \) \( \psi_0 \) is \( \Gamma \)-inv. Hence, \( \Gamma \) has property \( T. \)
Thm. If $\Gamma$ is discrete and has property $T$, then $\Gamma$ is finitely generated.

Proof. Consider $\mathcal{H} = \mathop{\bigoplus}_{\triangle} L^2(\Gamma/\triangle)$ where $\triangle$ runs over finitely generated subgroups.

For every finite $S \subseteq \Gamma$, $\delta_e \langle s \rangle$ is $S$-invariant.

Since $\Gamma$ has property $T$, $\mathcal{H}$ contains $\Gamma$-inv. vector.

Then $L^2(\Gamma/\triangle) \ni \Gamma$-inv. vector for some $\triangle$, and $\Gamma/\triangle$ is finite.

Cor. If $\Gamma$ is a lattice in $\text{SL}_n(\mathbb{R})$, $n \geq 3$, then $\Gamma$ is finitely generated.
Margulis normal subgroup theorem.

**Thm** Let \( \Gamma \) be a lattice in \( G = \text{SL}(n, \mathbb{R}) \), \( n \geq 3 \), and \( N \trianglelefteq \Gamma \). Then \( |N| < \infty \) or \( |\Gamma/N| < \infty \).

Suppose that \( N \) is infinite normal subgroup in \( \Gamma \). We claim that:

1. \( \Gamma/N \) has property T \( \Rightarrow \) \( \Gamma/N \) is finite.
2. \( \Gamma/N \) is amenable \( \Rightarrow \) \( \Gamma \) has property (T) \( \Rightarrow \) \( \Gamma/N \) is amenable.

It remains to show that \( \Gamma/N \) is amenable.

Let \( V \) - locally convex top. vector space

\( \mathcal{S}_2 \) - nonempty, compact, convex

\( \Gamma \triangleright \mathcal{S}_2 \) - affine continuous action.

We need to show that \( \mathcal{S}_2 \) is \( \Gamma \)-fixed point.

Without loss of generality, \( V \) & \( \mathcal{S}_2 \) are separable.

Consider \( L^\infty(G, \mathcal{S}_2) = \{ f : G \to \mathcal{S}_2 : f(\gamma g) = f(\gamma) \cdot f(g) \} \) for \( \gamma \in \Gamma \), a.e. \( g \in G \)

equipped with weak* topology, namely topology defined by seminorms:
\[ \|f\|_{L^1(G)} = \int_G |f(g)\phi(g)| \, dg \quad \forall \phi \in L^1(G) \]

(\|\cdot\|_2 are the seminorms on \(G\) defining topology).

Then \(L^\infty(G,\mathcal{S})\) is compact.

\(G\) acts continuously on \(L^\infty(G,\mathcal{S})\) by

\[ f \mapsto f(x, \cdot) \]

Let \(B = \{(e, \cdot) \} \subset G\).

Since \(B\) is amenable, \(L^\infty(G,\mathcal{S})\) has \(B\)-fixed point.

Then

\[ \exists \, \tilde{f} : G/B \to \mathcal{S} : \tilde{f}(y \cdot x) = y \cdot f(x) \]

for \(y \in G\), a.e. \(x \in G/B\).

\[\boxed{\text{Margulis Factor Thm.}}\]

If \(f\) is as above, \(\exists\) closed subgroup \(P \supset B\):

\[ \begin{array}{cc}
G/B & \xrightarrow{gB \rightarrow gP} \\
\downarrow f & \downarrow \tilde{f} \xrightarrow{2} \\
\mathcal{S} & \xrightarrow{\tilde{f} \text{-equivariant} \, \text{measurable}} G/P
\end{array} \]

\(\uparrow \text{isomorphism}\)

i.e., every \(P\)-factor is a \(G\)-factor.
Since $N$ acts trivially on $\mathcal{S}$,
$N.gP = gP$ for a.e. $g \in G$, and
$P \supset \langle \tilde{g}^tN\tilde{g} : g \in G \rangle$ - infinite closed normal subgroup of $G$.
Hence, $P = G \Rightarrow f = \text{const}$ a.e.
In particular, $\mathcal{S}$ has a fixed point.

Proof of factor theorem.

Consider $f : G/B \to \mathcal{S}$: $f(\gamma x) = \gamma f(x)$ for $\gamma \in \Gamma$ and a.e. $x \in G/B$.

What is $\mathcal{S}$?
$L^\infty(\mathcal{S}) \subset L^\infty(G/B)$
\text{a } \Gamma\text{-invariant subalgebra}

Classify $\Gamma\text{-inv. subalgebras of } L^\infty(G/B)$?

\begin{align*}
\Gamma\text{-equivariant} & \quad \text{factors} \\
\downarrow \quad 1\text{-to-1} & \\
G/B & \quad \mathcal{S}
\end{align*}

\begin{align*}
\Gamma\text{-invariant} & \quad \text{sub-}\sigma\text{-algebras} \\
\downarrow & \\
\mathcal{A} = \{ \overline{f}(A) : A \in \text{Borel}(\mathcal{S}) \} & \\
f & \quad \mathcal{B} = \text{Borel}(G/B)
\end{align*}
Thm. Every $\Gamma$-invariant sub-$\sigma$-algebra $A$ of $B = \text{Borel}(G)$ is $G$-invariant.

Lem. 1. Every $G$-inv. sub-$\sigma$-algebra $A$ of $\text{Borel}(G)$ is

$$A = \{ \overline{\pi^{-1}(A)} : A \in \text{Borel}(G/P) \} \quad (*)$$

where $\pi : G \to G/P$ and $P$ is a closed subgroup.

$L = L^\infty(G, A)$ - the space of bounded $A$-measurable functions. We equip $L^\infty(G)$ with the topology of convergence in measure, that is, the open sets are of the form:

$$\{ f : m(\{ x \in C : |f(x) - f(y)| < \varepsilon \}) < \delta \}, \quad C \in \text{G-finite measure}$$

Let $L_0 = L \cap L^\infty(G)$. Then we check that:

- $L$ is closed in $L^\infty(G)$,
- $C_c(G) * L \subseteq L_0$,
- $L_0$ is dense in $L$.

We set $P_x = \{ g \in G : f(xg) = f(x) \text{ for all } f \in L_0 \}$.

Since $A$ is $G$-invariant, $P_x$ is independent of $x$.

Since $A$ is $G$-invariant, $P_x$ can be considered as a subalgebra of $C(G/P)$, which separates points. Hence, $L_0 = C(G/P)$ by the Stone-Weierstrass theorem, and (*) holds.
For simplicity, \( G = \mathrm{SL}(3, \mathbb{R}) \).

Up to measure zero,
\[
G/B = U = \left\{ \begin{pmatrix} u_1 & 0 \\ u_3 & u_2 \end{pmatrix} : u_i \in \mathbb{R} \right\}
\]

Intermediate subgroups:
\[
P_1 = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \quad P_2 = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}
\]
\[
U = V \cdot W, \quad V = \begin{pmatrix} 0 & 1 \\ X & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\( \mathcal{G} \)-factors:
\[
\begin{array}{ccc}
\pi_1 & \downarrow & \pi_2 \\
G/B & \to & G/P_1 \\
G/P_1 & \to & G/P_2
\end{array}
\]

\( \mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B} \) — corresponding \( \sigma \)-algebras.

Convergence in measure:
\[
A_n \to A \iff \lim \frac{|(A_n \cap A) \cap B|}{|B|} = 0 \quad \text{for every ball in } U.
\]

\( A \) is closed under convergence in measure.

Example: \( A \subset \mathbb{R}^d \), for a.e. \( a \in A \),
\[
\lim_{r \to \infty} \frac{r(A-a)}{r} = \begin{cases} \mathbb{R}^d & a \in A \\ \emptyset & a \notin A \end{cases}
\]

Indeed, for a.e. \( a \in A \),
\[
|\frac{1}{r}(A-a) \cap B| = \lim_{r \to \infty} \frac{|A \cap a + \frac{1}{r}B|}{|B|} \to |B|
\]
by the Lebesgue density lemma,
and the same holds for \( A^c \).
\[ \text{We set: } a_r = \begin{pmatrix} z^{-1/3} & 0 & 0 \\ 0 & z^{-1/3} & 0 \\ 0 & 0 & z^{1/3} \end{pmatrix}, \quad V = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \]

\[ a_r : G_B \rightarrow G_B : (v, w) \mapsto (z^{-1/3} v, w). \]

As in the example:

**Lem. 2.** For every Borel \( A \subset G_B \sim U \) and a.e. \( v e V \)

\[ a_r (v^* A) \xrightarrow{r \to \infty} V \cdot A_v, \]

where \( A_v = \{ w e W : (v, w) e A \} \).

**Lem. 3.** For a.e. \( v e V \), \( \{ v a_r^{-1} \}_{r \geq 1} = G \).

We know by Moore's ergodicity Thm, that \( \{ a_r \} \) act ergodically on \( r \backslash G \).

This implies that for a.e. \( g e G \),

\[ \{ g a_r^{-1} \}_{r \geq 1} = G. \]

A.e. \( g = v \cdot p \) where \( v e V \) and \( p e P \).

Then \( g a_r^{-1} = v a_r^{-1} (a_r p a_r^{-1}) \), where
\[
\alpha_r \rho a_r^{-1} = \begin{pmatrix}
\rho_1 & \rho_2 \\
\rho_2 & \rho_3 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\rho_1 & \rho_2 \\
\rho_2 & \rho_3 \\
0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
\infty & 0 \\
0 & \rho_3
\end{pmatrix}, \quad r \rightarrow \infty.
\]

Hence, \( \{ \Gamma \nu a_r^{-1} \}_{r=1} = G \iff \{ \Gamma \rho a_r^{-1} \}_{r=1} = G. \)

**Proof of Thm.**

1) Suppose that \( A \subset B_1. \)

Then \( \exists A \in A \) with nontrivial \( A_v \)
(i.e., \( |A_v| \neq 0 \) and \( |A_v| \neq 0 \)),
for \( v \in \) positive measure set in \( V. \)

By Lem. 2 and Lem. 3,
we can pick this \( v \) so that
\( \{ \Gamma \nu a_r^{-1} \}_{r=1} = G, \)

\( a_r (\nu^{-1} A) \rightarrow V \cdot A \nu = \tilde{A}. \)

Note that \( \tilde{A} \) is nontrivial.

Then \( \forall g \in G: g_n = \nu^{-1} \rightarrow g \) for \( n \rightarrow \infty. \)

We have \( g_n a_r \nu^{-1}. A = \tilde{f}_n A \in A. \)

Hence, \( g \tilde{A} \in A \) for all \( g \in G. \)

This shows that \( A \supset \) nontrivial \( G \)-inv. sub-\( \sigma \)-algebra \( = B. \)

Thus, \( B_2 \subset A. \)
2) Similar argument shows that
\[ a \notin B_2 \implies B_1 \subset A. \]

3) Either:
\[ a \in B_1, a \notin B_2 \implies a \in B, B \cap B_2 = \{ \emptyset, \emptyset \} \]
\[ a \in B_1, a \in B_2 \implies B_1 \subset a \implies a = B_1 \]
\[ a \notin B_1, a \in B_2 \implies B_2 \subset a \implies a = B_2 \]
\[ a \notin B_1, a \notin B_2 \implies B_2 \subset a, B_1 \subset a \implies a = B. \]