ALTERNATION NUMBERS OF TORUS KNOTS WITH SMALL BRAID INDEX

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ABSTRACT. We calculate the alternation number of torus knots with braid index 4 and less. For the lower bound, we use the Upsilon-invariant recently introduced by Ozsváth, Stipsicz, and Szabó. For the upper bound, we use a known bound for braid index 3 and a new bound for braid index 4. Both bounds coincide, so that we obtain a sharp result.

1. INTRODUCTION

Kawauchi introduced the *alternation number* $\mathrm{alt}(K)$ of a knot K—the minimal number of crossing changes needed to turn a diagram of K into the diagram of an alternating knot [14]. Our main result determines the alternation number for all torus knots with braid index 4 or less.

Theorem 1. *If K is a torus knot of braid index 3 or 4, then* $\mathrm{alt}(K) = \lfloor \frac{1}{3} g(K) \rfloor$.
In other words, for all positive integers n, we have

$$\mathrm{alt}(T_{3,3n+1}) = \mathrm{alt}(T_{3,3n+2}) = \mathrm{alt}(T_{4,2n+1}) = n.$$

The proof of Theorem 1 consists of two parts. We use Ozsváth, Stipsicz, and Szabó's Υ-invariant [17] to improve previously known lower bounds for the alternation number. The necessary upper bounds are provided by an explicit geometric construction in the case of braid index 4, and by Kanenobu's bound of [12] for braid index 3.

Let us put Theorem 1 in context. Calculation of the alternation number for general knots is difficult; even for small crossing knots, say 12 or less crossings, for which many invariants are calculated (compare [5]), the alternation number appears to be unknown; compare also Jablan's work which determines many small crossing knots to have alternation number one [11]. Many knot knot invariants, including Floer theoretic invariants, have a particularly simple behavior for alternating knots. Hence it is natural to study the alternation number which can be understood as a 'distance' from alternating knots, and in this sense it is analogous to the better known unknotting number.

We focus on the natural and well-studied torus knots, which are arguably one of the simplest classes of knots and typically non-alternating. Their symmetry and periodic knot diagram representations allow for treatment of infinite families rather than case-by-case treatment. Also, torus knots are L-space knots, that is they are particularly simple from Heegaard-Floer theory point of view and, in particular, the Υ-invariant can be calculated combinatorially; compare Section 2. Let us address

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the alternation number of torus knots by considering them ordered by the braid index:

Torus knots with braid index 2 are alternating; in other words, their alternation number is zero. For torus knots with braid index 3, our result is a slight improvement on previous work of Kanenobu. In [12], he established that

$$\operatorname{alt}(T_{3,3n+1}) = \operatorname{alt}(T_{3,3n+2}) = n$$

for even positive integers n, whereas for odd integers n he is left with the ambiguity that

$$\operatorname{alt}(T_{3,3n+1}), \operatorname{alt}(T_{3,3n+2}) \in \{n-1, n\}.$$

For torus knots of braid index 4, Kanenobu established

$$n \leq \operatorname{alt}(T_{4,2n+1}) \leq \frac{3}{2}n \quad \text{and} \quad n-1 \leq \operatorname{alt}(T_{4,2n+1}) \leq \frac{3}{2}n - \frac{1}{2}$$

for even and odd n, respectively [12]. Therefore, Theorem 1 improves both the previously existing lower and upper bound.

The question whether all positive integers appear as alternation numbers of torus knots has been considered before. This is a natural question given the partial results found by Kanenobu [12]. Theorem 1 answers this question in the positive.

A related knot invariant is the dealternating number $\operatorname{dalt}(K)$ of a knot. This number is the minimal number of crossing changes that one needs for turning a diagram of K into an alternating *diagram*. Clearly,

$$\operatorname{alt}(K) \leq \operatorname{dalt}(K)$$

for all knots K. The dealternating number might appear less appealing at first sight. However, there exists the following interesting connection to quantum topology, due to Asaeda and Przytycki (reproved by Champanerkar-Kofman in [6] with a spanning tree model for Khovanov homology): for all knots K,

$$\operatorname{width}(\operatorname{Kh}(K)) - 2 \leq \operatorname{dalt}(K), \tag{1}$$

where Kh denotes the unreduced Khovanov homology [3], and $\operatorname{width}(\operatorname{Kh}(K))$ denotes the number of δ-diagonals with δ-grading greater or equal the lowest δ-grading on which the Khovanov homology has support and less than or equal the highest δ-grading on which Khovanov homology has support. The inequality (1) can be used to show that the alternation number differs from the dealternating number in general. For instance, any Whitehead double W_K of a (non-trivial) knot K has alternation number 1, while $\operatorname{width}(\operatorname{Kh}(K))$ is in general larger than 3 for Whitehead doubles.

Using Turner's calculation of width(Kh) for torus knots of braid index three [22], Abe and Kishimoto used inequality (1) to calculate the dealternating number for torus knots with braid index 3. However, the width width(Kh) is unknown for torus knots of braid index 4. In fact, by work of Benheddi, one has $n+2 \leq \operatorname{width}(\operatorname{Kh}(T_{4,2n+1}))$, see [4], and, conjecturally, this is an equality.

Question. *Does Theorem 1 also hold for the dealternating number? In other words, are there geometric constructions similar to the ones provided below, that show* $\operatorname{dalt}(T_{4,2n+1}) = \operatorname{alt}(T_{4,2n+1}) = n$*?*

A positive answer would determine $\operatorname{width}(\operatorname{Kh}(T_{4,2n+1}))$ to be $n+2$. This was part of the original motivation for the study conducted in this paper. However, it is impossible to directly use the constructions that we presented in Section 3 below to prove that $\operatorname{dalt}(T_{4,2n+1}) \leq n$; compare Remark 8.

Finally, we would like to point out that our approach of using the Υ-invariant as a lower bound for the alternation number described in Section 2 has inspired successful applications of the Υ-invariant in the study of knot concordances in the context of alternating knots by Friedl, Livingston, and the third author [9].

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2. Lower bounds for the alternation number

In [1], Abe observed that

$$\frac{|s(K)-\sigma(K)|}{2} \leq \operatorname{alt}(K)$$

for all knots K, where s and σ denote Rasmussen's invariant [19] and Trotter's signature [21], respectively. In fact, this lower bound works similarly with other knot invariants:

Proposition 2. *Let ψ_1 and ψ_2 be any real-valued knot invariants such that*

(i) for all alternating knots ψ_1 and ψ_2 are equal and

(ii) if K_+ and K_- are two knots such that K_- is obtained from K_+ by changing a positive crossing to a negative crossing, then

$$\psi_i(K_-) - 1 \leq \psi_i(K_+) \leq \psi_i(K_-)$$

for $i = 1, 2$.

Then for all knots K, we have

$$|\psi_1(K) - \psi_2(K)| \leq \operatorname{alt}(K).$$

Proof. For $i = 0, \ldots, n$, let K_i be a sequence of knots such that for $i = 1, \ldots, n-1$ the knot K_{i+1} results from K_i through a crossing change, and such that K_0 is alternating. Induction on n shows that the difference $|\psi_1(K_n) - \psi_2(K_n)|$ can be at most n. □

For Ozsváth and Szabó's τ-invariant the negative $\psi_1(K) = -\tau(K)$ satisfies (ii) from Proposition 2; see [18]. Similarly, the invariant $\psi_2(K) = \Upsilon_K(1) = \upsilon(K)$ does satisfy (ii), and

$$\psi_1(A) = -\tau(A) = \Upsilon_A(1) = \upsilon(A) = \psi_2(A)$$

for all alternating knots A. Here $\Upsilon_K(t)$ (denoted by $\upsilon(K)$ when $t = 1$) is the real valued knot-invariant (depending piecewise-linearly on a parameter t in $[0, 2]$) introduced by Ozsváth, Stipsicz, and Szabó [17]. Therefore, we get the following.

Corollary 3. *For all knots K, we have*

$$|\tau(K) + \upsilon(K)| \leq \operatorname{alt}(K)\,.$$

We note that other invariants rather than τ can be used and will yield the same lower bounds for the alternation number on torus knots; for example, Rasmussen's s-invariant or any concordance invariant with the properties described in [15, Theorem 1]. The τ-invariant seems to be the canonical choice to work with since Υ is a generalization of it: indeed, one has $-\tau = \lim_{t\to 0} \frac{\Upsilon(t)}{t}$; see [17, Proposition 1.6].

Proposition 4. *For all positive integers n, we have the following bounds for the alternation number.*

$$n \leq \operatorname{alt}(T_{3,3n+1}), \quad n \leq \operatorname{alt}(T_{3,3n+2}), \quad \textit{and} \quad n \leq \operatorname{alt}(T_{4,2n+1}).$$

Proof. This is immediate from calculating $|\tau+\upsilon|$. On positive torus knots τ equals the three-genus:

$$\tau(T_{p,q}) = \frac{(p-1)(q-1)}{2},$$

for all coprime positive integers p and q; see [18, Corollary 1.7]. For torus knots (and more generally L-space knots) Ozsváth, Stipsicz, and Szabó [17, Theorem 1.15] provided a procedure to calculate $\Upsilon(t)$ from the Alexander polynomial. With this procedure one calculates

$$\upsilon(T_{3,3n+1}) = -2n = \upsilon(T_{4,2n+1}) \quad \text{and} \quad \upsilon(T_{3,3n+1}) = -2n-1,$$

for all n; compare [8, Proposition 28], where this elementary calculation is provided. The values for τ and υ combined yield

$$\begin{aligned}|\tau(T_{3,3n+1}) + \upsilon(T_{3,3n+1})| &= 3n - 2n = n\\ |\tau(T_{3,3n+2}) + \upsilon(T_{3,3n+2})| &= 3n + 1 - 2n - 1 = n\\ |\tau(T_{4,2n+1}) + \upsilon(T_{4,2n+1})| &= 3n - 2n = n.\end{aligned}$$

This concludes the proof since $|\tau+\upsilon|$ is a lower bound for the alternation number by Corollary 3. □

3. Upper bounds for the alternation number

For torus knots with braid index 3, upper bounds for the alternation number where calculated by Kanenobu [12]; compare also [7], where this is recovered from a different perspective. Abe and Kishimoto showed that the same upper bounds hold for the dealternating number [2].

Proposition 5 ([12, Theorem 8],[2, Theorem 2.5])**.** *For all positive integers n,*

$$\operatorname{alt}(T_{3,3n+1}) \le \operatorname{dalt}(T_{3,3n+1}) \le n \quad \textit{and} \quad \operatorname{alt}(T_{3,3n+2}) \le \operatorname{dalt}(T_{3,3n+2}) \le n.$$

We provide new upper bounds for torus knots of braid index 4.

Proposition 6. *Let $n \ge 2$ be an integer. There is a diagram of the torus knot $T_{4,2n+1}$ such that n crossing changes yield the knot $T_{2,2n+1} \# T_{2,2n+1}$. In particular,*

$$\operatorname{alt}(T_{4,2n+1}) \le n.$$

Remark 7. Similarly, one can show that for $n \ge 2$ there is a diagram of the torus link $T_{4,2n}$ such that there are n crossing changes which turn this torus link into an alternating link.

Remark 8. It is impossible that the diagram for $T_{2,2n+1} \# T_{2,2n+1}$ provided by Proposition 6 is alternating. Indeed, assume towards a contradiction that there is a diagram D_1 for the torus knot $T_{4,2n+1}$ such that n crossing changes yield an alternating diagram D_2 for the knot $T_{2,2n+1} \# T_{2,2n+1}$. We may assume that D_1 and D_2 are reduced diagrams. Since the minimal crossing number of $T_{4,2n+1}$ is $6n+3$, the diagram D_1, and thus also D_2, has at least $6n+3$ crossings. However, $T_{2,2n+1} \# T_{2,2n+1}$ has an alternating diagram with $4n+2$ crossings, which contradicts Tait's conjecture that two reduced alternating diagrams for the same knot have the same number of crossings [13, 16, 20].

Proof of Proposition 6. We think of the torus knots $T_{4,2n+1}$ as closures of braids. Using braid relations, or equivalently using isotopies of the closure, we see that a 'full twist' can be isotoped according to Figure 1.

Similarly, a braid of three half twists can be isotoped according to Figure 2. We notice a slight asymmetry in the two 'bands' in this case.

We observe that these isotopies are compatible with iterations of full twists respectively multiplication of the braids corresponding to full twists. The result

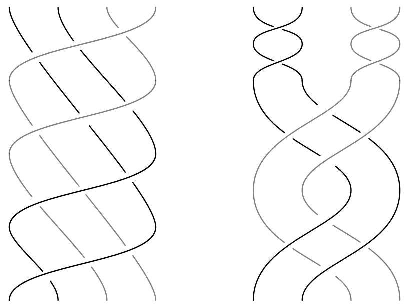

FIGURE 1. These are identical braids corresponding to isotopic diagrams relative to the ends. The left hand side is standard, the right hand side description will be used later on.

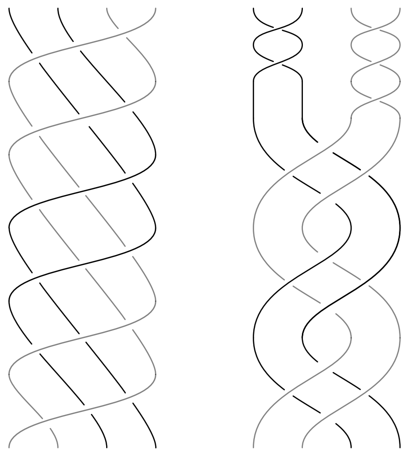

FIGURE 2. Isotopy corresponding to three half twists.

will be two bands which, when seen from the top to the bottom, both first twist, and then cross each other as planar bands.

Now in each full twist, we can find two crossing changes in the region where the bands cross with a geometric significance. Figure 3 below shows how we can arrange for the two outer strands to pass in front of the two inner. Similarly, Figure 4 shows how we can arrange for the two inner strands to pass in front of the two outer strands.

Iterating this, we see that with n crossing changes, we transform the braid corresponding to the torus knot $T_{4,2n+1}$ to the braid on the left hand side of Figure 5 if n is even, and to the braid on the right hand side if n is odd. In the first case, we have used the crossing changes according to Figure 3, in the second case we have used those of Figure 4.

Finally we observe that the braid closure of this is the connected sum

$$T_{2,2n+1} \# T_{2,2n+1}.$$

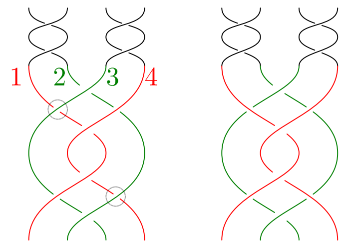

Figure 3. Two crossing changes bringing the outer two strands (number 1 and 4) to the front

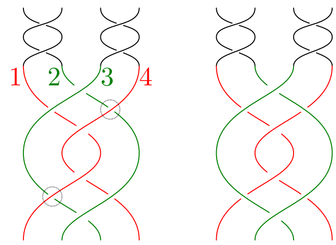

Figure 4. Two crossing changes bringing the inner two strands (number 2 and 3) to the front

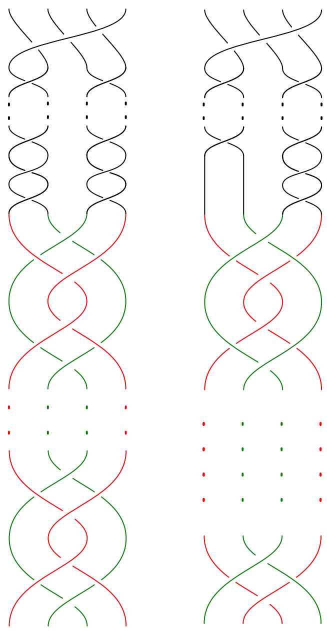

Figure 5. After n crossing changes, we obtain the braid on the left for n even, and the one on the right for n odd, starting from $T_{4,2n+1}$.

To see this, we must distinguish the cases n even and n odd. If n is even, we start with the braid closure of the left hand diagram in Figure 5. We can flip the two inner strands (corresponding to strands 2 and 3 in Figure 3) in the braid closure to the top, passing behind everything else; see Figure 6. Notice that this flipping yields a new crossing between the two flipped strands.

The case where n is odd is entirely analogous. In the braid closure of the right hand braid of Figure 5, we can flip the outer two strands (corresponding to strands

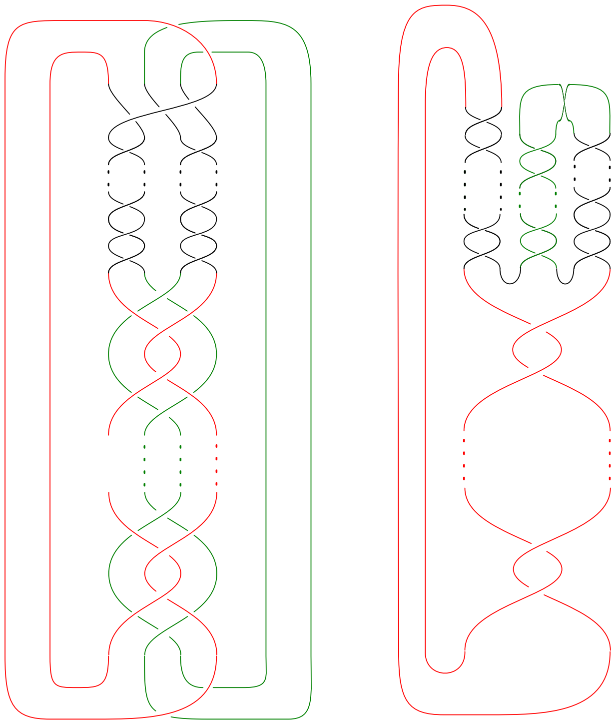

FIGURE 6. An isotopy from the braid closure to the connected sum $T_{2,2n+1} \# T_{2,2n+1}$. The two inner strands pass behind everything else.

1 and 4 in Figure 4) behind everything else. This also resolves the apparent asymmetry in the top of the braid we have started with.

□

4. Proof of the main result

Theorem 1 is an immediate consequence of Propositions 4, 5, and 6. The reformulation that $\mathrm{alt}(K) = \lfloor \frac{1}{3} g(K) \rfloor$ is an easy computation that follows from the formula of the genus of a torus knot, given by

$$g(T_{p,q}) = \frac{(p-1)(q-1)}{2},$$

for $p, q > 1$ coprime integers.

5. Perspectives

It is natural to wonder what the alternation numbers for torus knots of higher braid index are. Even the asymptotic behavior is unclear. To make this precise we set

$$a_p = \limsup_{n \to \infty} \frac{\mathrm{alt}(T_{p,i+np})}{n}$$

for $p \geq 2, 0 \leq i < p$. It follows from [7] that one has

$$|\mathrm{alt}(T_{p,k}) - \mathrm{alt}(T_{p,l})| \leq \frac{p-1}{2} |k - l|,$$

showing that the above limit superior is independent of i. In fact, we suspect that this limit superior is a limit since any geometric constructions that realize the limit superior and is periodic (such as what we found for the case of torus knots of braid index 4) would establish the existence of the limit.

In this setup, Kanenobu's lower bound [12], which he obtained using Abe's lower bound [1] and Gordon, Litherland and Murasugi's signature calculation [10], yields

$$\begin{aligned} \frac{(p-1)^2}{4} &\leq a_p \quad \text{for } p \text{ odd, and} \\ \frac{(p-2)p}{4} &\leq a_p \quad \text{for } p \text{ even.} \end{aligned} \tag{2}$$

In fact, using the $|\tau + \upsilon|$-bound from Section 2, one can recover (2). In particular, using the υ-invariant, one does not get a better asymptotic lower bound than Abe's bound using the signature and the τ-invariant. The same bounds hold for the limit inferior.

Kanenobu's upper bound on the alternation number of torus knots of braid index 3 (compare Proposition 5) shows that (2) is an equality for $p \leq 3$ and our main result Theorem 1 shows that (2) is an equality for $p = 4$ as well. The values a_p for $p \geq 5$ seem out of reach at the moment. However, maybe the geometrically constructed upper bounds generalize such that in the future the following question can be answered in the positive.

Question. *Is* (2) *an equality for all positive integers* p*?*

As a further hint in this direction, we notice that the lower bound in (2), for p even, is equal to the number of 'band crossings' in a full twist for a suitable generalization of Figure 3. For odd $p \geq 5$, we briefly comment on the case $p = 5$, the smallest for which a_p remains unknown. While we are able to determine the alternation number of $T(5,6)$ to be 4 (explicit manipulations allow to make $T(5,6)$ alternating with four crossing changes, which agrees with the lower bound $|\tau(T(5,6)) + \upsilon(T(5,6))| = 10 - 6$ from Corollary 3.), $\mathrm{alt}(T(5,n))$ remains unknown for all $n \geq 7$ and coprime to 5. Even asymptocally, combining the lower bounds (2) and the upper bounds Kanenobu provides in [12, Corollary 28], one only finds $4 \leq a_p \leq 6$.

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