A NEW ALGORITHM FOR 3-SPHERE RECOGNITION

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Abstract. We prove the existence of a new algorithm for 3-sphere recognition based on Gröbner basis methods applied to the variety of $SL(2,\mathbb{C})$-representation of the fundamental group. An essential input is a recent result of the second author, stating that any integer homology 3-sphere different from the 3-sphere admits an irreducible representation of its fundamental group in $SL(2,\mathbb{C})$. This result, and hence our algorithm, build on the geometrisation theorem of 3-manifolds.

Introduction

Rubinstein [17] has introduced an algorithm that recognises the 3-sphere, starting from a triangulation of the 3-manifold. His approach is based on normal surface theory, and it was simplified later by Thompson [19]. These algorithms have been established before the Poincaré conjecture and the geometrisation conjecture in dimension 3 were known to hold. Since then, the problem of 3-sphere recognition is equivalent to the recognition of the trivial group amongst fundamental groups of 3-manifolds.

For instance, it is known that the following algorithm detects if a finite presentation $\langle\, S \mid R \,\rangle$ of the fundamental group π of a 3-manifold represents the trivial group:

(1) Use two computers, Computer 1 and Computer 2.
(2) On Computer 1, check successively whether there is a non-trivial homomorphism from π to the symmetric group S_n, for $n = 2, 3, \dots$.
(3) On Computer 2, run the Todd-Coxeter algorithm applied to the trivial subgroup $\{1\} \subseteq \langle\, S \mid R \,\rangle$, see [14, 4].

If the presentation represents a non-trivial group, then the program on Computer 1 will eventually stop because 3-manifold groups are residually finite [9]. If the presentation represents the trivial group, the program on Computer 2 will eventually stop.

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Notice that the algorithm does stop because we suppose we are *guaranteed* that $\langle\, S \,|\, R \,\rangle$ is the presentation of the fundamental group of a 3-manifold.

We suggest a new, somewhat simpler, and presumably more practical algorithm below.

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1. Review of Krull dimension, Hilbert polynomials, and Gröbner bases

By a (complex) affine algebraic variety we understand a subset of $\mathbb{C}^N$, for some $N \in \mathbb{N}$, which is the zero set of finitely many polynomials in the ring $R = \mathbb{C}[x_1, \dots, x_N]$. An affine algebraic variety is called irreducible if it is not the union of two non-empty strictly smaller varieties which are closed in the Zariski topology.

Definition 1.1. *The Krull dimension of an affine algebraic variety V is defined to be the maximal number d such that there are irreducible subvarieties $V_0, \dots, V_d$ which form a strictly increasing chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V$.*

We refer to [5, Chapter 9, §3] for the notions of Hilbert series and Hilbert polynomials of an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_N]$. The following result is classical and can be found, for instance, in [11, Section 5.6].

Proposition 1.2. *Let V be an affine algebraic variety over $\mathbb{C}$, determined by the ideal $I \subseteq R$. Then the Krull dimension of V is equal to the degree of the Hilbert polynomial of the ideal $I \subseteq R$,*

$$\dim(V) = \deg(HP_{R/I}).$$

We fix some graded order on the monomials of R. For instance, this can be the graded lexicographical order. Then every element of R has a well-determined leading term. Following standard notation, we denote by $\langle \mathrm{LT}(I) \rangle$ the ideal generated by the leading terms of the elements in I, see for instance [5]. Recall that a Gröbner basis for I associated to the chosen order is a finite subset of I whose leading terms generate $\langle \mathrm{LT}(I) \rangle$.

Proposition 1.3. *The following algorithm computes the Krull-dimension of an affine algebraic variety $V(I)$ determined by an ideal $I \subseteq R$.*

(1) Compute a Gröbner basis for I with respect to the chosen order.
(2) Consider the subsets $S \subseteq \{x_1, \ldots, x_n\}$ such that x^s, defined to be the product of the elements of S, does not lie in the monomial ideal $\langle \mathrm{LT}(I) \rangle$. Let m denote the maximal cardinality of all these subsets S.
(3) $\dim(V(I)) = \dim(V(\langle \mathrm{LT}(I) \rangle)) = m$.

Proof. We refer to [5, Chapter 9, §1, Proposition 3] for the proof that the second step determines the dimension of a variety associated to a monomial ideal such as $\langle \mathrm{LT}(I) \rangle$. The following key observation is attributed to Macaulay, see [5, Chapter 9, §3, Proposition 4]: The Hilbert series (and hence the Hilbert polynomial) of the monomial ideal $\langle \mathrm{LT}(I) \rangle$ is equal to the Hilbert series of the monomial I, and hence we have equality of the associated Hilbert polynomials,

$$HP_{R/I} = HP_{R/\langle \mathrm{LT}(I) \rangle} .$$

Therefore $\dim(V(I)) = m$ by Proposition 1.2. □

2. The representation variety

Let π be a finitely generated group, and let $\langle s_1, \ldots, s_n \,|\, r_1, \ldots, r_m \rangle$ be a presentation of π. A $SL(2, \mathbb{C})$-representation of π is a homomorphism $\rho \colon \pi \to SL(2, \mathbb{C})$.

Definition 2.1. *The* $\mathrm{SL}(2, \mathbb{C})$*-representation variety is*

$$R(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}(2, \mathbb{C})) \subseteq \mathrm{SL}(2, \mathbb{C})^n \subseteq M(2, \mathbb{C})^n \cong \mathbb{C}^{4n} .$$

The representation variety $R(\pi)$ is contained in $SL(2, \mathbb{C})^n$ via the inclusion $\rho \mapsto \big(\rho(s_1), \ldots, \rho(s_n)\big)$, and it is the set of solutions of a finite system of polynomial equations in the matrix coefficients (in fact, $4m + n$ many), hence it is an affine algebraic variety.

3. A new algorithm for detecting the trivial group among 3-manifold groups

The following result has recently been established by the second author [20].

Theorem 3.1. *Let Y be an integer homology 3-sphere different from the 3-sphere. Then there is an irreducible representation $\rho\colon \pi_1(Y) \to SL(2,\mathbb{C})$.*

With this at hand, we are able to prove the following

Theorem 3.2. *Let $\pi = \langle s_1, \dots, s_n \,|\, r_1, \dots, r_m \rangle$ be a presentation of the fundamental group of a 3-manifold with n generators. Then the following algorithm decides whether or not π is the trivial group.*

(1) Abelianise π. If the abelianisation is non-trivial, π isn't the trivial group.

(2) If the abelianisation π_{ab} of π is trivial, fix a graded monomial order in $\mathbb{C}[x_1, \dots, x_{4n}]$, and compute a Gröbner basis for the affine algebraic variety

$$R(\pi) = Hom(\pi, SL(2,\mathbb{C})) \subseteq \mathbb{C}^{4n}\,.$$

(3) From the Gröbner basis, determine if the Krull dimension $\dim(R(\pi))$ of $R(\pi)$ is equal to 0 or bigger than 0, following the algorithm in Proposition 1.3 above.

(4) If $\dim(R(\pi)) \neq 0$, then π is not the trivial group.
If $\dim(R(\pi)) = 0$, then π is the trivial group.

Remark 3.3. *Theorem 3.2 is in contrast to the following general fact: Whether or not a finite presentation represents the trivial group is undecidable, see* [2, 1, 16] *and for a survey* [13].

Remark 3.4. *In the preceding result, the presentation is not required to be geometrical (for instance, obtained from a Morse decomposition of a 3-manifold, or a triangulation.) However, we do require that the presentation is that of the fundamental group of a 3-manifold. In general, it is undecidable whether or not a given group is the fundamental group of a 3-manifold, see for instance the work of Groves, Manning, and Wilton* [8]*, and the work Aschenbrenner, Friedl, Wilton for a survey* [3].

The proof of this Theorem 3.2 will make use of the following lemma.

Lemma 3.5. *Let π be a finitely generated group. If the representation variety $V(\pi)$ contains an irreducible representation, then $\dim V(\pi) \geq 3$.*

Proof. Let $\rho\colon \pi \to SL(2,\mathbb{C})$ be an irreducible representation. The group $SL(2,\mathbb{C})$ acts by conjugation on the representation variety $R(\pi)$. More precisely, for $A \in SL(2,\mathbb{C})$ we define $(A.\rho)(\gamma) = A\rho(\gamma)A^{-1}$ for all $\gamma \in \pi$, and we let $O(\rho) = \{A.\rho \mid A \in SL(2,\mathbb{C})\}$ denote the orbit of ρ. Notice that the stabiliser of an irreducible representation is the centre of $SL(2,\mathbb{C})$.

Now, Theorem 1.27 of [12] implies that $O(\rho) \subset R(\pi)$ is a closed algebraic subset, and Lemma 3.7 of [15] implies that $\dim O(\rho) = 3$ since ρ is irreducible. Hence $3 = \dim O(\rho) \leq \dim R(\pi)$ by definition of the notion of Krull dimension. □

Proof of Theorem 3.2. If π has trivial abelianisation and is not the fundamental group of the 3-sphere, then there is an irreducible representation $\rho : \pi \to SL(2, \mathbb{C})$ by Theorem 3.1. By Lemma 3.5, we conclude that $R(\pi) = \mathrm{Hom}(\pi, SL(2, \mathbb{C}))$ has Krull dimension at least 3. Hence if the Gröbner basis computation yields $\dim(V) = 0$, then π must be the trivial group. □

4. A new algorithm for 3-sphere recognition

We think of a 3-manifold Y as being given by a Heegaard diagram. From this we can read off a presentation of the fundamental group. If g is the genus of the Heegaard diagram, and if k is the number of intersections in the Heegaard diagram (counted absolutely, and not up to sign), we obtain a presentation of the fundamental group $\pi_1(Y)$ of length $g + k$.

Corollary 4.1. *The combination of*

(1) the standard algorithm to pass from a Heegaard diagram of a 3-manifold Y to a presentation $\pi = \langle\, S \,|\, R \,\rangle$ of its fundamental group together with
(2) the algorithm of Theorem 3.2

is an algorithm that detects the 3-sphere.

The input data of this algorithm is given by a Heegaard diagram, and not by a triangulation, as it is the case in the Rubinstein-Thompson algorithm. This may turn out more practical in concrete cases. In fact, we only need a presentation of the fundamental group.

Furthermore, any triangulation comes with a canonical Heegaard diagram, and a theorem of Reidemeister and Singer states that any two Heegaard diagrams of the same 3-manifold are stably equivalent (see [6]). However, it still seems unknown how much the Heegaard genus of a 3-manifold can differ from the Heegaard genus of a diagram coming from a minimal triangulation, and how many stabilisations/destabilisations one needs to pass from one to the other. From this point of view, it may be that our algorithm uses essentially smaller input data than the previously mentioned one.

5. Complexity questions

For the notion of complexity classes such as NP we refer to [7].

5.1. **Our algorithm.** The question whether finite systems of polynomial equations define algebraic varieties of dimension greater or equal to d is NP-hard for any $d \geq 0$ by a result of Koiran, see [10, Proposition 1.1].

In our situation, we know a few more facts about the varieties $R(\pi)$ in question. For instance, these are always determined by polynomial equations with integer coefficients. Furthermore, $R(\pi)$ always contains the trivial representation, and therefore this variety is always non-empty.

It is unclear to us whether these facts decrease the complexity of our algorithm, but given Koiran's result, we rather expect the algorithm not to be of polynomial length in terms of the input size, and hence not to lie in the complexity class P. This also seems consistent with some numerical evidence we have obtained.

5.2. **NP algorithms.** Schleimer has shown that the 3-sphere recognition problem lies in the complexity class NP, i.e. there is a non-deterministic algorithm that detects the 3-sphere in polynomial time [18]. Kuperberg has shown that the unknot detection problem lies in the complexity class coNP, provided the generalised Riemann hypothesis (GRH) holds. Based on Kuperberg's approach, the second author has shown in [20] that the 3-sphere recognition problem lies in the complexity class coNP modulo GRH.

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