A NOTE ON LOGARITHMIC TRANSFORMATIONS ON THE HOPF SURFACE

RAPHAEL ZENTNER

Abstract. In this note we study logarithmic transformations in the sense of differential topology on two fibers of the Hopf surface. It is known that such transformations are susceptible to yield exotic smooth structures on four-manifolds. We will show here that this is not the case for the Hopf surface, all integer homology Hopf surfaces we obtain are diffeomorphic to the standard Hopf surface.

1. Introduction

The (standard) Hopf surface $S^1 \times S^3$ fibres over the 2-sphere S^2 via the map obtained by composing the Hopf fibration $S^3 \to S^2$ with the projection on the second factor. Any fibre is diffeomorphic to the torus T^2 and there are no singular fibres, because this map is a submersion. It is a natural problem to study the effect of logarithmic transformations on two fibres in this case. Indeed, this operation was successfully used in the case of the K3 surface to construct exotic K3 surfaces, as well as on other elliptic fibrations. These results have been obtained using gauge theoretical methods, which only apply for manifolds with $b_2^+ \geq 1$ [DK] [FM] [K] [OV]. Note that all K3-surfaces are diffeomorphic four-manifolds, and there exist complex K3-surfaces which are elliptic fibrations. In the case of the K3-surface the resulting manifolds depend only on the multiplicities of the logarithmic transformations, but in our considerations they depend on some additional parameters as well.

For four-manifolds with the rational homology of a Hopf surface the existing gauge theoretical methods do not apply. On the other hand it is a fundamental and open problem whether four-manifolds with small second Betti-number (especially the four-sphere and the Hopf surface) do admit exotic structures. The four-manifold with smallest second Betti number admitting exotic smooth structures which is known at present is $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$ [PSS]. In the complex geometric framework, exotic Hopf surfaces do not exist, for by a result of Kodaira [Ko] every complex surface which is homeomorphic to $S^1 \times S^3$ is a primary Hopf surface, so it is diffeomorphic to $S^1 \times S^3$. Complex surfaces which are rational homology Hopf surfaces have been classified in [EO] using logarithmic transformations. Further results about elliptic surfaces in the class of complex surfaces can be found in [FM].

Our situation here, however, is purely topological in nature, and the logarithmic transformations considered are more general than the complex-geometric ones. In particular, logarithmic transformations with multiplicity zero do not arise in the complex geometric setting, and may even result in manifolds not admitting any complex structure at all [G].

We will first calculate the fundamental group of the manifold obtained by two logarithmic transformations. As it turns out in many cases, including multiplicity 0, the resulting manifold will have the same fundamental group as the Hopf surface. Since the Euler characteristic is invariant under logarithmic transformations, we will obtain a manifold having the same (integer) homology as the Hopf surface. We will then describe a procedure to construct all these manifolds by gluing two copies of $T^2 \times D^2$ via a diffeomorphism between their boundaries. Using diffeomorphisms of $T^2 \times S^1$ which extend over $T^2 \times D^2$, we will be able to show that manifolds given by different gluing diffeomorphisms may still be diffeomorphic. Using this observation, we will find a certain standard form for every homology Hopf surface obtained by this gluing method. The possible standard forms are determined by elements in $\mathrm{Sl}(2, \mathbb{Z})$. Finally, using a handlebody-theoretical argument [LP], we prove that this parameter does not affect the diffeomorphism type.

Acknowledgements

I am grateful to Peter Kronheimer for helpful conversations on this and related topics. I am also indebted to my advisor Andrei Teleman for proof-reading the paper and related suggestions, as well as for the encouragement to write this paper. Furthermore I am thankful to the referee for some useful comments. Finally, I wish to thank Amy Ellingson for proof-reading the English.

2. Logarithmic transformations applied to Hopf surfaces and resulting fundamental group

Definition 2.1. *Let $\pi : X \to \Sigma$ be an elliptic fibration. We say that a four-manifold X' is obtained from X by logarithmic transformation on a regular fibre F of π if X' is obtained from X through the following construction: We cut out a regular neighbourhood νF of F and we glue in a $T^2 \times D^2$ via an arbitrary orientation-reversing diffeomorphism $\varphi : T^2 \times S^1 \to \partial \nu F$. The absolute value of the degree of $\pi|_{\partial \nu F} \circ \varphi|_{pt \times S^1}$ is called the multiplicity of the logarithmic transformation* [G].

The diffeomorphism φ is determined, up to isotopy, by its induced isomorphism of fundamental groups, which is itself, after the choice of some bases, is determined by a matrix in $\mathrm{Gl}(3, \mathbb{Z})$. Alternatively, we fix one such diffeomorphism, which can be used to identify $\partial \nu F$ with $T^2 \times S^1$. Any other diffeomorphism is determined by a self-diffeomorphism of $T^2 \times S^1$, and these diffeomorphisms are given, up to isotopy, by elements in $\mathrm{Sl}(3, \mathbb{Z})$.

We will first give a gluing description of the Hopf-surface $X = S^1 \times S^3$ which will turn out useful. For this we shall first describe S^3 as two solid tori $S^1 \times D^2$ glued together. The two closed discs D^2 will turn out to be the northern and southern hemisphere, respectively, under the Hopf fibration $S^3 \to S^2$. Indeed, S^3 can be seen as the following set:

$$S^3 = \left\{ (z, w) \in \mathbb{C}^2 \middle| \quad |z|^2 + |w|^2 = 2 \right\}$$

The Hopf fibration is then given by the map

$$S^3 \to \mathbb{CP}^1 \quad \text{given by} \quad (z, w) \mapsto [z : w] \ ,$$

and $\mathbb{CP}^1$ is diffeomorphic to S^2. Define S^3_+ to be the set of elements (z, w) such that $0 \leq |w|^2 \leq 1$, and S^3_- to be the set of elements (z, w) with $0 \leq |z|^2 \leq 1$. Then there are diffeomorphisms

$$S^3_+ \xrightarrow{f_+} S^1 \times D^2, \quad \text{given by} \quad (z, w) \mapsto \left(\frac{z}{|z|}, \frac{w}{z} \right), \quad \text{and}$$

$$S^3_- \xrightarrow{f_-} S^1 \times D^2, \quad \text{given by} \quad (z, w) \mapsto \left(\frac{w}{|w|}, \frac{z}{w} \right).$$

When we restrict $f_+ \circ f_-^{-1}$ to the boundary, then the map $\partial(S^1 \times D^2) \to \partial(S^1 \times D^2)$ is given by the formula

$$f_+ \circ f_-^{-1}(u, \xi) = (u\xi, \overline{\xi}) \ .$$

We extend this latter map to the trivial S^1 factor by the identity, so that we get a map $\zeta : T^2 \times \partial D^2 \to T^2 \times \partial D^2$, and the description of the Hopf surface as a gluing

$$X = (T^2 \times D^2) \cup_\zeta (T^2 \times D^2) \ . \tag{1}$$

Now let us consider the manifold X' obtained from the Hopf surface when performing logarithmic transformations on two fibres, say on the fibre F_+ over the north pole $x_+ := [1 : 0]$ and the fibre F_- over the south pole $x_- = [0 : 1]$, associated with diffeomorphisms $\varphi_\pm$. There are natural identifications of $\partial(X - \nu F_\pm)$ with the "inner" boundary of $T^2 \times (D^2 - \mathring{D}^2_{1/2})$ according to the decomposition (1). Therefore the orientation-reversing diffeomorphisms $\varphi_\pm$ can be seen as an orientation-preserving diffeomorphism of $T^2 \times S^1$, because the above "inner" boundary is with opposite orientation to the "outer" boundary. Let us denote by $X_\pm$ the two manifolds $(T^2 \times (D^2 - \mathring{D}^2_{1/2})) \cup_{\varphi_\pm} (T^2 \times D^2)$. Gluing two manifolds along their boundaries is actually a suitable identification of collar neighbourhoods of the boundaries of the two manifolds. In our case, this description is given as

$$X_\pm = \left(T^2 \times \left(D^2 - D^2_{1/3} \right) \right) \cup_{\Phi_\pm} \left(T^2 \times \mathring{D}^2_{2/3} \right) \ ,$$

where $\Phi_\pm : \left(\frac{1}{3}, \frac{2}{3}\right) \times T^2 \times S^1 \to \left(\frac{1}{3}, \frac{2}{3}\right) \times T^2 \times S^1$ is given by $\Phi_\pm (r, u, v, \xi) := \left(\frac{2}{3} - r, \varphi_\pm (u, v, \xi)\right)$. Let us now fix some paths inside $D^2 \times T^2$, where the disc is thought of as a subset of $\mathbb{C}$, centred at the origin. Fix some base-point $(u_0, v_0, \xi_0) \in T^2 \times D^2$, where $|\xi_0| = \frac{1}{2}$, so that the base point is in the "gluing area". Let us define three paths $\alpha_\pm, \beta_\pm, \gamma_\pm$ by the formulae $\alpha_\pm(t) = (u_0, v_0 e^{it}, \xi_0)$, $\beta_\pm(t) = (u_0 e^{it}, v_0, \xi_0)$, and $\gamma_\pm(t) = (u_0, v_0, \xi_0 e^{it})$. The path $\gamma_\pm$ is then a meridian to the fibre $T^2 \times \{0\}$ over $x_\pm$ - its projection onto the fibre is trivial - whereas $\alpha_\pm$ and $\beta_\pm$ induce a basis of the fundamental group of the fibre. Note that by the same formulae we can define paths $(\alpha'_\pm, \beta'_\pm, \gamma'_\pm)$ inside the pieces $T^2 \times D^2$ to be glued in with $\varphi_\pm$. Then $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ induces a basis of $\pi_1(X - \nu F_\pm)$ and $(\alpha'_\pm, \beta'_\pm, \gamma'_\pm)$ induces a basis of $\partial(T^2 \times D^2)$. The diffeomorphisms $\varphi_\pm$ are then determined by their maps of fundamental groups

$$\varphi_*^+ = \begin{pmatrix} * & * & a \\ * & * & b \\ * & * & p \end{pmatrix} \quad \varphi_*^- = \begin{pmatrix} * & * & c \\ * & * & d \\ * & * & q \end{pmatrix} ,$$

which are elements in $\mathrm{Sl}(3, \mathbb{Z})$. The entries marked as $*$ will not be relevant to the fundamental group, as we shall see. We call $(a, b) \in \mathbb{Z}^2$ the direction of the logarithmic transformation φ_+, and $|p|$ is its multiplicity.

In order to compute the fundamental group of X' we shall first compute the fundamental groups of $X_\pm$ and then glue them together via ζ. X_+ is given as the union of two open sets, namely the sets $X_1 = T^2 \times (D^2 - D^2_{1/3})$ and $X_2 = T^2 \times \mathring{D}^2_{2/3}$, with intersection $X_0 = T^2 \times (\mathring{D}^2_{2/3} - D^2_{1/3})$. The manifold X_0 injects into X_1 via the natural inclusion i, and into X_2 via φ_+. The fundamental group of each piece is

$$\begin{aligned} \pi_1(X_0) &= \langle \alpha_0, \beta_0, \gamma_0 \,|\, [\ ,\ ] = 1 \rangle , \\ \pi_1(X_1) &= \langle \alpha, \beta, \gamma \,|\, [\ ,\ ] = 1 \rangle , \quad \text{and} \\ \pi_1(X_2) &= \langle \alpha', \beta' \,|\, [\ ,\ ] = 1 \rangle . \end{aligned}$$

By [,] we simply mean that all commutator relations are satisfied. The Seifert-van Kampen theorem states that $\pi_1(X_+)$ has as generators both the generators of $\pi_1(X_1)$ and of $\pi_1(X_2)$, as relations all those of $\pi_1(X_1)$ and $\pi_1(X_2)$, and the additional relations

$$\begin{aligned} i(\alpha_0) &= \varphi(\alpha_0) \;\Leftrightarrow\; \alpha' = \varphi(\alpha_0), \\ i(\beta_0) &= \varphi(\beta_0) \;\Leftrightarrow\; \beta' = \varphi(\beta_0), \quad \text{and} \\ i(\gamma_0) &= \varphi(\gamma_0) \;\Leftrightarrow\; 1 = \varphi(\gamma_0) . \end{aligned}$$

The first two relations imply that we can drop the generators α' and β' as well as these two relations. Therefore the fundamental group is

$$\pi_1(X_+) = \langle\, \alpha_+, \beta_+, \gamma_+ \,|\, [\ ,\ ] = 1,\, \alpha_+^a \beta_+^b \gamma_+^p = 1 \,\rangle .$$

Correspondingly, we get

$$\pi_1(X_-) = \langle\, \alpha_-, \beta_-, \gamma_- \,|\, [\ ,\ ] = 1,\, \alpha_-^c \beta_-^d \gamma_-^q = 1 \,\rangle .$$

In order to compute the fundamental group of $X' = X_+ \cup_\zeta X_-$ we proceed in the same way. T^2 times a "middle annulus" injects into X_- via the natural inclusion, whereas it injects into X_+ via ζ. As we have $\zeta_*(\alpha_0) = \alpha_+, \zeta_*(\beta_0) = \beta_+$ and $\zeta_*(\gamma_0) = \alpha_+\gamma_+^{-1}$ we get a final formula:

$$\pi_1(X') = \langle\, \alpha, \beta, \gamma \,|\, [\ ,\ ] = 1, \alpha^a\beta^b(\alpha\gamma^{-1})^p = 1,\ \alpha^c\beta^d\gamma^q = 1\ \rangle\ .$$

By the classification of finitely generated Abelian groups, we find that we have an isomorphism $\pi_1(X') \cong \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$, where μ is the highest common divisor of all the 2-minors of a presentation matrix for this group. It is easy to see that there are various choices possible in which this number equals 1, including cases where one or both of the multiplicities may be zero.

Remark. *If we perform the two logarithmic transformations such that they are trivial on the S^1-factor, then the construction is S^1 times Dehn-surgery on the Hopf-link in S^3. The resulting four-manifold is then S^1 times a lens space; this can be seen using the surgery description of lens spaces* [GS]. *However, we shall point out that if the logarithmic transformations are of type* $(1\ ,0\ ,p)$ *and* $(1\ ,0\ ,q)$*, then the Dehn-surgery description of the Lens space we get is not the surgery description on the Hopf link with surgery coefficients* p *and* q*, with respect to the blackboard framing.*

3. Formulation in terms of gluing two copies of $T^2 \times D^2$

We will denote by $X_\varphi := (T^2\times D^2)\cup_\varphi(T^2\times D^2)$ the four-manifold obtained by gluing $T^2 \times D^2$ to $T^2 \times D^2$ via the orientation-reversing diffeomorphism ψ between their boundaries. Let us denote by A^2 an annulus. There are canonical identifications of the boundary-components of $T^2 \times A^2$ with $T^2 \times S^1$, as before.

Two isotopic diffeomorphisms induce diffeomorphic manifolds, so we are only interested in isotopy classes of diffeomorphisms here. Furthermore we may restrict our attention to orientation-reversing diffeomorphisms. We shall also identify the boundaries of the two copies of $T^2 \times D^2$ with the 3-torus $T^2 \times S^1$, once orientation-preserving, once orientation-reversing, and this once and for all. The diffeomorphism φ is then given by an orientation-*preserving* diffeomorphism of T^3. Finally, in the case of the 3-torus, such a diffeomorphism up to isotopy is determined by its associated automorphism of the fundamental group, and therefore by a matrix in $\mathrm{Sl}(3,\mathbb{Z})$. We will show here that all of the manifolds considered so far can be obtained by gluing just two copies of $T^2 \times D^2$ along their boundaries:

Lemma 3.1. *We have the following diffeomorphism:*

$$X_{\psi\ \circ\ \varphi} \cong (T^2 \times D^2) \cup_\psi (T^2 \times A^2) \cup_\varphi (T^2 \times D^2).$$

Proof: As any diffeomorphism of one boundary-component of $T^2 \times A^2$ extends over the whole of $T^2 \times A^2$ the result follows easily. □

Our next objective is to calculate the fundamental group of X_φ. Let us use the bases $(\alpha_\pm, \beta_\pm, \pm\gamma_\pm^{\pm 1})$ from above (up to "orientation") and suppose that the map φ_*, which is now given by an element of $\mathrm{Sl}(3,\mathbb{Z})$, looks as follows:

$$\varphi_* = \begin{pmatrix} a & c & g \\ b & d & h \\ e & f & k \end{pmatrix} \tag{2}$$

According to the theorem of Seifert-van Kampen a presentation of the fundamental group of X_φ is given by

$$\pi_1(X_\varphi) = \langle \alpha, \beta \mid [\alpha,\beta] = 1,\, (\alpha^{g'}\beta^{h'})^{(g,h)} = 1 \rangle \ .$$

Here (g,h) denotes the greatest common divisor of g and h, and g' andh' are such that $g = (g,h)\, g'$, $h = (g,h)\, h'$. We define $(0,0) := 0$. The fundamental group is therefore isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/(g,h)\mathbb{Z}$:

$$\pi_1(X_\varphi) = \mathbb{Z} \oplus \mathbb{Z}/(g,h)\mathbb{Z}.$$

In particular, X_φ is a homology Hopf surface if and only if $(g,h) = 1$, noting that any X_φ has Euler-characteristic zero.

If now we perform the logarithmic transformations associated with $\varphi_\pm$ on the two fibres $F_\pm$ of the Hopf surface, then the resulting manifold will be given by the following gluing construction

$$(T^2 \times D^2) \cup_{\varphi_+^{-1}} (T^2 \times A^2) \cup_\zeta (T^2 \times A^2) \cup_{\varphi_-} (T^2 \times D^2)$$

which is diffeomorphic, by the above lemma, to

$$X_{\varphi_+^{-1} \circ\, \zeta\, \circ\, \varphi_-} \ .$$

Whether this manifold is a homology Hopf surface can now be deciphered from the automorphism $\left(\varphi_+^{-1} \circ \zeta \circ \varphi_-\right)_*$ of the fundamental group. However, calculating the entity (g,h), which a posteriori depends only on the numbers a, b, p and c, d, q, using this matrix product, is a rather difficult problem.

Theorem 3.2. *Suppose the manifold X_φ, constructed as above, is a homology Hopf surface. Then X_φ is diffeomorphic to the Hopf surface X_ζ.*

Corollary 3.3. *If logarithmic transformations on two fibres yield a homology Hopf surface then this four-manifold is diffeomorphic to the standard Hopf surface $S^1 \times S^3$.*

Proof of the Theorem. Observe first that the two manifolds

$$X_\varphi \ , \ X_{\psi_t^{-1} \circ\, \varphi\, \circ\, \psi_b}$$

are diffeomorphic as soon as the diffeomorphisms ψ_t and ψ_b of $T^2 \times S^1$ extend over $T^2 \times D^2$ as diffeomorphisms. A diffeomorphism ψ extends if and only if the associated matrix has the form

$$\psi_* = \begin{pmatrix} r & t & 0 \\ s & u & 0 \\ v & w & 1 \end{pmatrix} . \tag{3}$$

Indeed, it is easy to construct explicitly extensions of these diffeomorphisms; on the other hand, if ψ extends to a diffeomorphismΨ, then the first two entries in the third column of the corresponding matrix must be zero. This can be seen using the commutative diagram

$$\begin{array}{ccc} H_1(T^2 \times S^1) & \xrightarrow{\psi_*} & H_1(T^2 \times S^1) \\ \downarrow & & \downarrow \\ H_1(T^2 \times D^2) & \xrightarrow{\Psi_*} & H_1(T^2 \times D^2). \end{array}$$

This observation can be used to perform certain line operations on φ_* by left-multiplication with matrices induced by extending diffeomorphisms, as well as to perform certain column operations by right-multiplication with these matrices, without changing the diffeomorphism type.

Suppose now that X_φ is a homology Hopf surface with associated matrix φ_* as in (2) above. In particular, the greatest common divisor of g and h is one: $(g, h) = 1$. By left-multiplying with a matrix $u \in \mathrm{Sl}(2, \mathbb{Z}) \subseteq \mathrm{Sl}(3, \mathbb{Z})$, where the inclusion is into the upper left part in the 3×3 matrix, we may assume that $g = 1, h = 0$ in (2). Such a matrix u is of type (3). Now there is a matrix L of type (3) such that left-multiplication of the new matrix φ_* by L adds $-(k-1)$ times the first line of φ_* to its last line. Therefore we may suppose that $k = 1$. Now there is a matrix R of the type (3) such that right-multiplication of the newest φ_* by R will add appropriate multiples of the third column of φ_* to its first and second, so that we may assume $e = f = 0$ because $k = 1$. φ_* in (2) may therefore be supposed to have the form

$$\varphi_* = \begin{pmatrix} a & c & 1 \\ b & d & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4}$$

A corresponding diffeomorphism is given by $\varphi(u, v, z) = (u^a v^c z, u^b v^d, z)$. We can't simplify much further in order to obtain the matrix ζ_*, where ζ is inducing the standard Hopf surface as above. However, the attachment of $T^2 \times D^2$ to the upper $T^2 \times D^2$, which we shall denote by X_+, may be done by attaching first a 2-handle, then two 3-handles, and eventually a 4-handle. To be more precise, decompose the torus T^2 in the obvious way into a 0-handle Σ_0, two 1-handles Σ_{11} and Σ_{12}, and a 2-handle Σ_2. Then the attachment, via φ, of $\Sigma_0 \times D^2$ to X_+ is done along $\Sigma_0 \times \partial D^2$, thus we attach a 2-handle and get $X^{(2)} := X_+ \cup (\Sigma_0 \times \partial D^2)$. It is now easily verified that $\Sigma_{11} \times D^2$ and $\Sigma_{12} \times D^2$ are attached to $X^{(2)}$ along a thickened 2-sphere $S^2 \times D^1$, corresponding to 3-handle attachment. Finally $\Sigma_2 \times D^2$ is glued to the resulting manifold along a 3-sphere, a 4-handle attachment. The union of the two 3- and the one 4-handle is diffeomorphic to a boundary sum $S^1 \times D^3 \natural S^1 \times D^3$, which is the gluing of two pieces of $S^1 \times D^3$ via a diffeomorphism between two discs in their boundaries. The boundary of this manifold is $S^1 \times S^2 \# S^1 \times S^2$. It is known [LP] that any diffeomorphism of $S^1 \times S^2 \# S^1 \times S^2$ extends over the whole boundary sum. Therefore only the 2-handle-attachment is relevant for determining the diffeomorphism type of the closed four-manifold.

On the other hand, the attaching of $\Sigma_0 \times \partial D$ is determined, up to isotopy, by the attachment of the attaching sphere $\{0\} \times S^1$ as well as the isomorphism of normal bundles $\nu_{\Sigma_0 \times S^1}(\{0\} \times S^1) \to \nu_{T^3}(\varphi(\{0\} \times S^1))$ induced by the derivative $d\varphi$. We shall denote by L_φ this bundle isomorphism. After identification of Σ_0 with a ball centred in the origin in $\mathbb{R}^2$ we get a canonical isomorphism $\nu_{\Sigma_0 \times S^1}(\{0\} \times S^1) \cong S^1 \times \mathbb{R}^2$. By a framing f of $\varphi(\{0\} \times S^1)$ we understand a fixed isomorphism of the normal bundle $\nu_{\Sigma_0 \times S^1}(\{0\} \times S^1)$ with $S^1 \times \mathbb{R}^2$. We say that a framing f is isotopic to the framing f' if they are homotopic through bundle isomorphisms. By replacing L_φ with f^{-1} we see that the 2-handle attachment is determined by $(\varphi(\{0\} \times S^1), f)$, the embedding with a given framing of the attaching sphere. Thus, framings and the isomorphisms L_φ are equivalent notions. Up to isotopy, the attachment depends only on the framing up to isotopy. If we fix one framing, we see that all possible isomorphisms of normal bundles are given by bundle automorphisms of $S^1 \times \mathbb{R}^2$.

For the above choice of φ the attachment of the attaching sphere does not depend on the specific entries in φ_*. We identify the normal bundle of $\varphi(\{0\} \times S^1)$ with orthogonal complement to its tangent bundle within $T(T^3)$, and get an identification with $S^1 \times \mathbb{R}^2$ by specifying two constant orthonormal sections of that bundle, $e_1 = (1\,,\,0\,,\,-1)$ and $e_2 = (0\,,\,1\,,\,0)$. The isomorphism L_φ is then given by the *constant* matrix

$$L_\varphi = \begin{pmatrix} a & c \\ b & d \end{pmatrix} .$$

Because this matrix is in $\mathrm{Sl}(2, \mathbb{Z})$ we see that there is an isotopy of bundle automorphisms taking one automorphism into the other. In other words, the corresponding framings are isotopic. □

References

[DK] S. Donaldson, P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Mathematical Monographs (1990).

[EO] W. Ebeling, C. Okonek, *Homology Hopf surfaces*, Compositio Math. 91 (1994), 277-304.

[FM] R. Friedman, J. Morgan, *Smooth Four-Manifolds and Complex Surfaces*, Erg. d. Mathematik u. i. G. (1991)

[G] R. Gompf, *Nuclei of elliptic surfaces*, Topology 30 (1991), 479-511.

[GS] R. Gompf, A. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics, AMS (1999).

[K] D. Kotschick, *On manifolds homeomorphic to* $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$ Invent. Math. 95 (1989), 591-600.

[Ko] K. Kodaira, *Complex structures on $S^1 \times S^3$*, Proc. Nat. Acad. Sci. USA, 55, (1966) 240-243.

[LP] F. Laudenbach, V. Poenaru, *A note on 4-dimensional handlebodies*, Bull. Soc. math. France 100 (1972), 337-344.

[OV] C. Okonek, A. van de Ven, *Stable bundles and differentiable structures on certain elliptic surfaces*, Invent. Math. 86 (1986), 357-370.

[PSS] J.Park, A. Stipsicz, S. Szabo, *Exotic smooth structures on* $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$, preprint (2004), math.GT/0412216

Laboratoire d'Analyse, de Topologie, et de Probabilités, Université de Provence, 13013 Marseille

E-mail address: zentner@cmi.univ-mrs.fr